Singular Stresses and Nonsmooth Boundaries in Continuum Mechanics

Alfredo Marzocchi

Dipartimento di Matematica e Fisica "Niccolò Tartaglia"

Università Cattolica del S. Cuore, Brescia, Italy

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1. Preliminaries in Geometric Measure Theory

In this section we review very briefly some notions that we will need in the next sections. We will also try to focus on some nice examples of "not so regular" sets.

1.1. Hausdorff measure

Definizione 1.1. *Let $E \subseteq \mathbb{R}^n$ and $m \in \mathbb{N}$, $m \geqslant 1$. We set*

$$\widetilde{\mathscr{H}^m}(E) = \lim_{\delta \to 0^+} \inf \left\{ \sum_{j=1}^{\infty} (\operatorname{diam} E_j)^m : E = \bigcup_{j=1}^{\infty} E_j, \operatorname{diam} E_j < \delta \right\}.$$

It is possible to show that $0 < \widetilde{\mathscr{H}^m}([0,1]^m) < +\infty$. Then we define

$$\mathscr{H}^m(E) = \frac{\widetilde{\mathscr{H}^m}(E)}{\widetilde{\mathscr{H}^m}([0,1]^m)}$$

and call it the *m-dimensional Hausdorff measure* of E. If $m = n$, the n-dimensional Hausdorff measure $\mathscr{H}^n$ coincides with the n-dimensional Lebesgue measure $\mathscr{L}^n$. The Hausdorff measure is an isometry-invariant positive outer measure([1]), such that if E, F are nonempty subsets of $\mathbb{R}^n$ with $\inf\{|x-y| : x \in E, y \in F\} > 0$, then $\mathscr{H}^m(E \cup F) = \mathscr{H}^m(E) + \mathscr{H}^m(F)$.

Finally, if for a subset N of $\mathbb{R}^n$ one has that $\mathscr{H}^m(N) < +\infty$, then $\mathscr{H}^p(N) = +\infty$ if $p < m$ and $\mathscr{H}^p(N) = 0$ if $p > m$.

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[1] An outer measure on $\mathbb{R}^n$ is a function m defined on subsets of $\mathbb{R}^n$ with values in $[0, +\infty]$ such that
(1) $m(\varnothing) = 0$;
(2) for all $E, F \subseteq \mathbb{R}^n : E \subseteq F \Rightarrow m(E) \leqslant m(F)$;
(3) if $E_j \subseteq \mathbb{R}^n$ $(j \in \mathbb{N})$, then $m(\bigcup_j E_j) \leqslant \sum_j m(E_j)$.

1.2. Measure-theoretic interior

Definizione 1.2. *Let $M \subseteq \mathbb{R}^n$ be a Lebesgue measurable subset of $\mathbb{R}^n$. We set* ([2])

$$M_* = \left\{ x \in \mathbb{R}^n : \lim_{r \to 0^+} \frac{\mathscr{L}^n(M \cap B_r(x))}{\mathscr{L}^n(B_r(x))} = 1 \right\}.$$

We will call M_* measure-theoretic internal part *of M. Its points are said to be* essentially internal *to M.*

In the language of measure theory, M_* is the set of all points with density 1 with respect to Lebesgue measure.

Moreover, M_* is a Borel subset of $\mathbb{R}^n$, even if M is not ([3]).

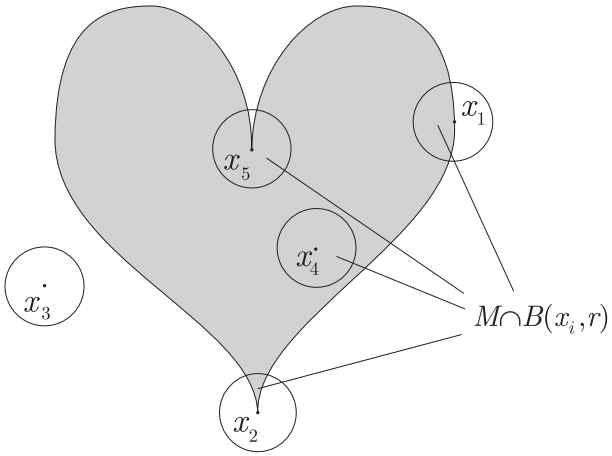

FIGURE 1
The set M_*.

In Figure 1, if M is open in the sense of the topology of $\mathbb{R}^2$, the points x_1, x_2, x_3 do not belong to M_*, while x_4, x_5 belong to M_*. In particular, x_5 does not belong to M but belongs to M_*. In general, M_* is neither greater nor smaller than M. If $N = M \cup \{x_2\} \setminus \{x_5\}$, then N_* must contain x_5 but not x_2.

Definizione 1.3. *Let $M \subseteq \mathbb{R}^n$. We set*

$$\partial_* M = \mathbb{R}^n \setminus \big(M_* \cup (\mathbb{R}^n \setminus M)_* \big).$$

We will call $\partial_* M$ the measure-theoretic boundary *of M.*

This means that the points of the measure-theoretical boundary are exactly those point which are neither essentially internal to M nor essentially internal to the complement of M. It is clear that $\partial_* M$ is a Borel set. With reference to Figure 2, x_2 and x_5, which are respectively essentially internal to $\mathbb{R}^n \setminus M$ and M, do not belong to $\partial_* M$.

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[2] We will denote by $B_r(x)$ the open unit ball in $\mathbb{R}^n$.

[3] The collection of all Borel subsets of a set E will be denoted by $\mathfrak{B}(E)$.

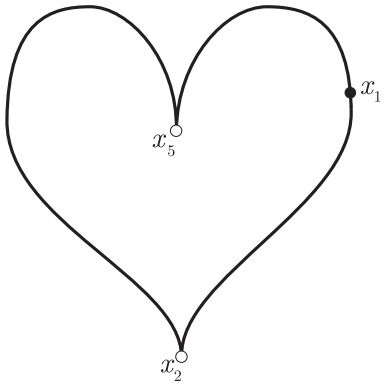


FIGURE 2
$\partial_* M$.

But, if the "heart" in Figure 1 had not been "cusped" in x_2, then x_2 would have belonged to $\partial_* M$, as Figure 3 shows.

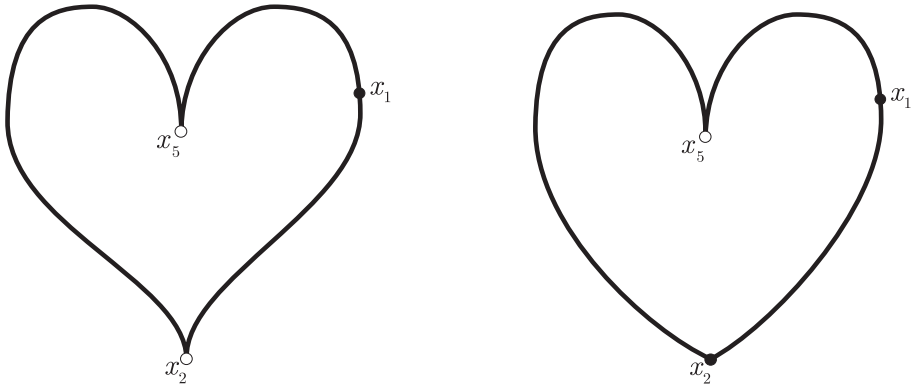
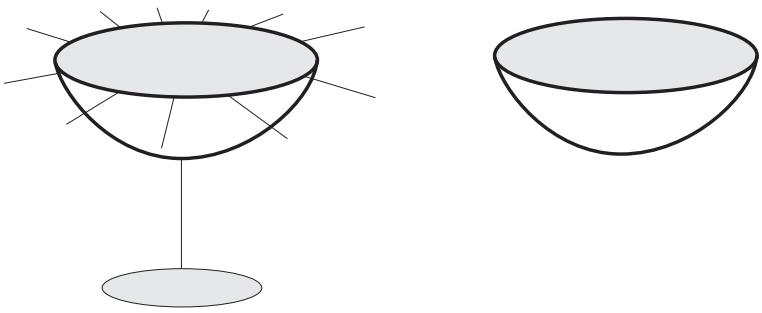


FIGURE 3

Definizione 1.4. *We will say that* $M \subseteq \mathbb{R}^n$ *is* normalized *if* $M_* = M$.

Thus, normalized means "measure-theoretically open". This definition is also similar to that of "regularly open set", that is, a set which coincides with the internal part of its closure. This latter definition eliminates sets with some missing isolated points or with tiny "moustaches".



not normalized normalized

FIGURE 4

But it is worth noticing that the "heart" of Figure 1 is not normalized since $x_5 \notin M_*$, while it is regularly open. We also set $M^* = M \cup \partial_* M$, the *measure-theoretic closure* of M. This name comes from the fact that the complement of a measure-theoretic closed set is measure-theoretic open, *i.e.* normalized.

Proposition 1.1. *If $M \subseteq \mathbb{R}^n$ is normalized, then $(\mathbb{R}^n \setminus M)^* = \mathbb{R}^n \setminus M$,* i.e. *the complement is measure-theoretically closed .*

Proof. Since all points of M have density 1, it must be $\partial_* M \subseteq \mathbb{R}^n \setminus M$. Moreover, by definition of measure-theoretic boundary, $\partial_* M = \partial_*(\mathbb{R}^n \setminus M)$. Hence, since M is normalized,

$$(\mathbb{R}^n \setminus M)^* = (\mathbb{R}^n \setminus M) \cup \partial_*(\mathbb{R}^n \setminus M) = (\mathbb{R}^n \setminus M) \cup \partial_* M = (\mathbb{R}^n \setminus M). \blacksquare$$

The following proposition states some properties of the concepts just introduced and it is left as an exercise.

Proposition 1.2. *Let $M \subseteq \mathbb{R}^n$. Then the following properties hold:*

(a) $(M_*)_* = M_*$;
(b) $M_*, M^*, \partial_* M \in \mathfrak{B}(\mathbb{R}^n)$;
(c) $\operatorname{int} M \subseteq M_* \subseteq M^* \subseteq \operatorname{cl} M$;
(d) *M is $\mathscr{L}^n$-measurable* $\iff \mathscr{L}^n(M \triangle M_*) = 0 \iff \mathscr{L}^n(\partial_* M) = 0$ ([4]).

1.3. Sets with finite perimeter and exterior normal

Definizione 1.5. *We say that a set $M \subseteq \mathbb{R}^n$ has* finite perimeter*, if $\mathscr{H}^{n-1}(\partial_* M) < +\infty$.*

Finite perimeter sets are also called *Caccioppoli sets*. They may be of course smooth sets, but some of them may be quite nasty, and therefore interesting, although it happens that they are $\mathscr{L}^n$-measurable. This follows immediately from the fact that if the perimeter is finite, then the Lebesgue measure of the same set must be zero from the remark we made at the end of subsection 1.1.

Example 1.3. [15] Take $x \in \mathbb{R}^n$, $\alpha, r > 0$ and consider the set $D = \{y \in \mathbb{R}^n : |x - y| \leqslant r\}$. Let $\{x_k : k \in \mathbb{N}\} \subseteq D$ an enumeration of all points in D with rational components. Put $\omega_n = \mathscr{L}^n(B_1(0))$ and consider, for $0 < \alpha < \omega_n r^n$, the decreasing sequence

$$r_k = \sqrt[n]{\frac{\omega_n r^n - \alpha}{2\omega_n}}\, 2^{-k/n}$$

Now set

$$E = D \setminus \bigcup_{k \in \mathbb{N}} B_{r_k}(x_k), \qquad C = E_*.$$

Then $C \subseteq D$ is bounded and normalized since $C_* = (E_*)_* = E_* = C$. Moreover, the set E does not contain any rational point, hence $\operatorname{int} E = \varnothing$; But E is a closed subset of $\mathbb{R}^n$, and

[4] We denote by $A \triangle B$ the symmetric difference of A and B, that is $(A \cup B) \setminus (A \cap B)$.

$C \subseteq E$ by virtue of (c) of Proposition 1.2. Hence, $\operatorname{int} C = \varnothing$. Let's see that this set has finite perimeter. First of all, $\partial_* C = \partial_* E$ and

$$\begin{aligned}\mathscr{H}^{n-1}(\partial_* E) &\leqslant \mathscr{H}^{n-1}(\operatorname{bdry} D) + \sum_{k\in\mathbb{N}} \mathscr{H}^{n-1}(\operatorname{bdry} B_{r_k}(x_k)) \\ &= \left(r^{n-1} + \left(\frac{\omega_n r^n - \alpha}{2\omega_n} \right)^{(n-1)/n} \sum_{k\in\mathbb{N}} 2^{-k(n-1)/n} \right) \mathscr{H}^{n-1}(\operatorname{bdry} B_1(0)).\end{aligned}$$

Let's see finally that $\mathscr{L}^n(C) \geqslant \alpha$. Indeed, by (d) of Proposition 1.2, $\mathscr{L}^n(C) = \mathscr{L}^n(E)$ and

$$\mathscr{L}^n(E) \geqslant \mathscr{L}^n(D) - \mathscr{L}^n\left(\bigcup_{k\in\mathbb{N}} B_{r_k}(x_k) \right) \geqslant \omega_n r^n - \omega_n \sum_{k\in\mathbb{N}} r_k^n = \alpha.$$

A picture of the set C may be as follows:

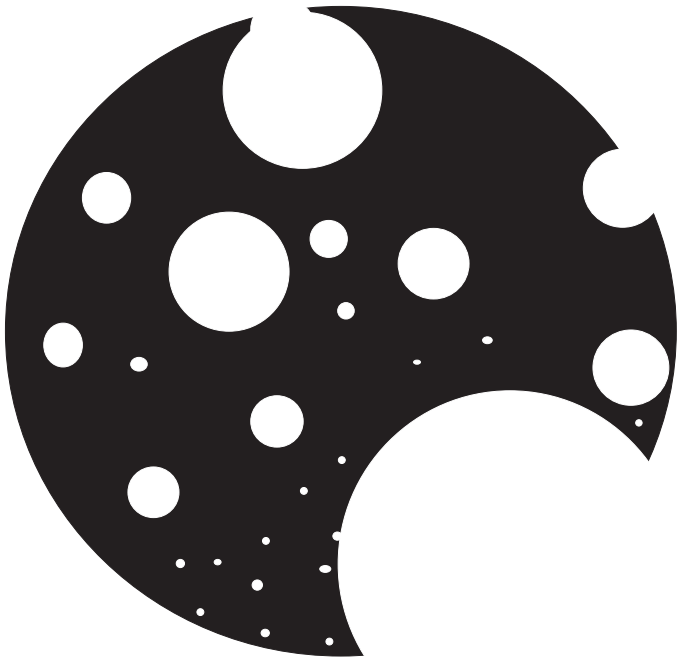


FIGURE 5
The cheese set C.

It doesn't seem that C has empty interior from the figure, but indeed: if we magnify the set, we see more and more small holes. But, since our sight has a finite magnification power, we don't see the smaller holes and the figure appears, as it is, to have finite Lebesgue measure.

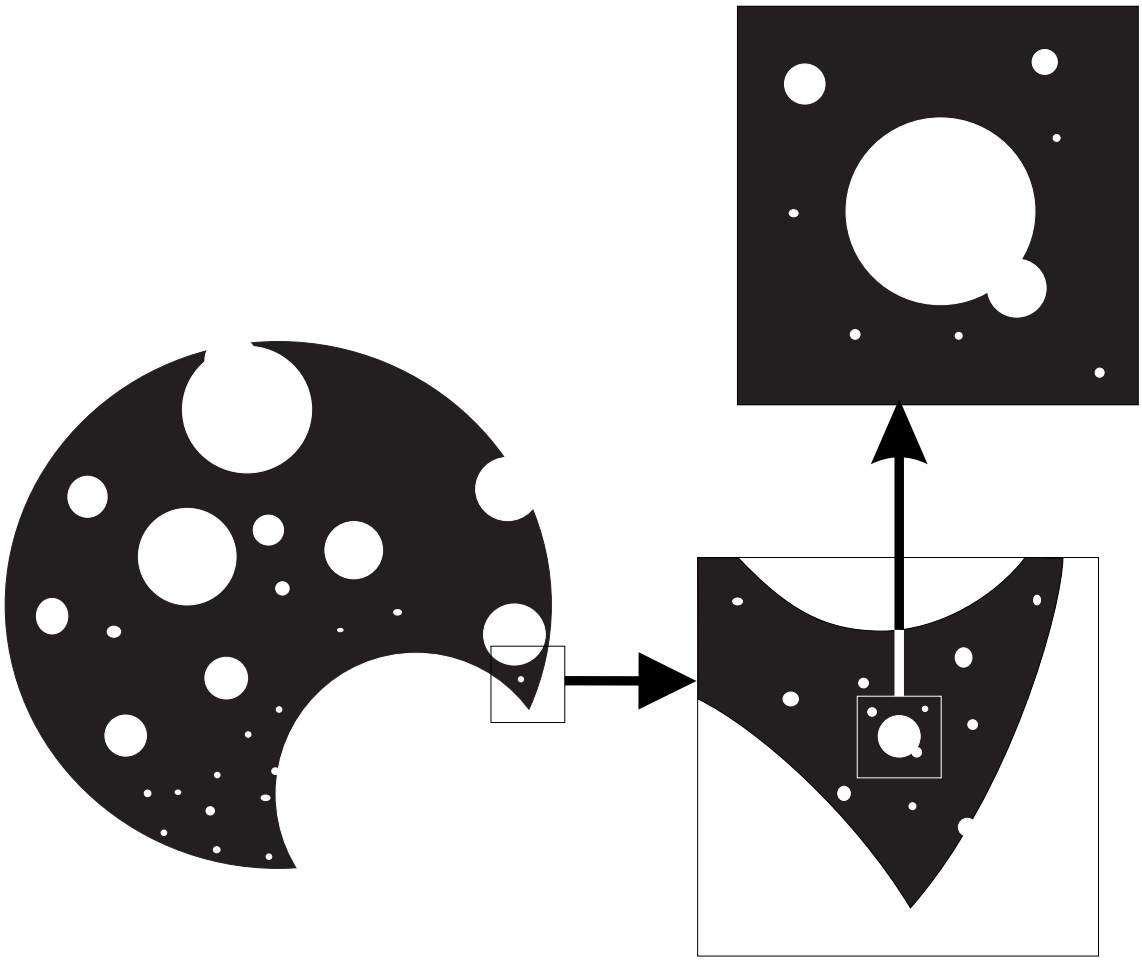


FIGURE 6
The set C and its magnifications.

Example 1.4. Another interesting example of finite perimeter set is the following. Take the interval $I_0 = [0, 1]$, divide it into three equal parts and raise the middle part by $1/4$. Call this set I_1: its $\mathscr{H}^1$ measure, that is, its length, is $1 + 1/2$. Then repeat the above procedure for each of the five subintervals just obtained, but now displace the middle parts by $1/40$. It's clear that the gain in length is twice the amount of the displacement of the five middle parts, that is $10 \cdot 1/40 = 1/4$. So I_2 has length $1 + 1/2 + 1/4$. Repeat the procedure with displacement $1/400$, and, since there are now 25 middle parts, the gain in length will be $50/400 = 1/8$. Thus the set I_3 will have length $1 + 1/2 + 1/4 + 1/8$. If we call I the limit set of this procedure, it will have length 2, but it will be nowhere smooth. People who know fractal sets will see here a variation of the construction of Von Koch's curve, which we will encounter later.

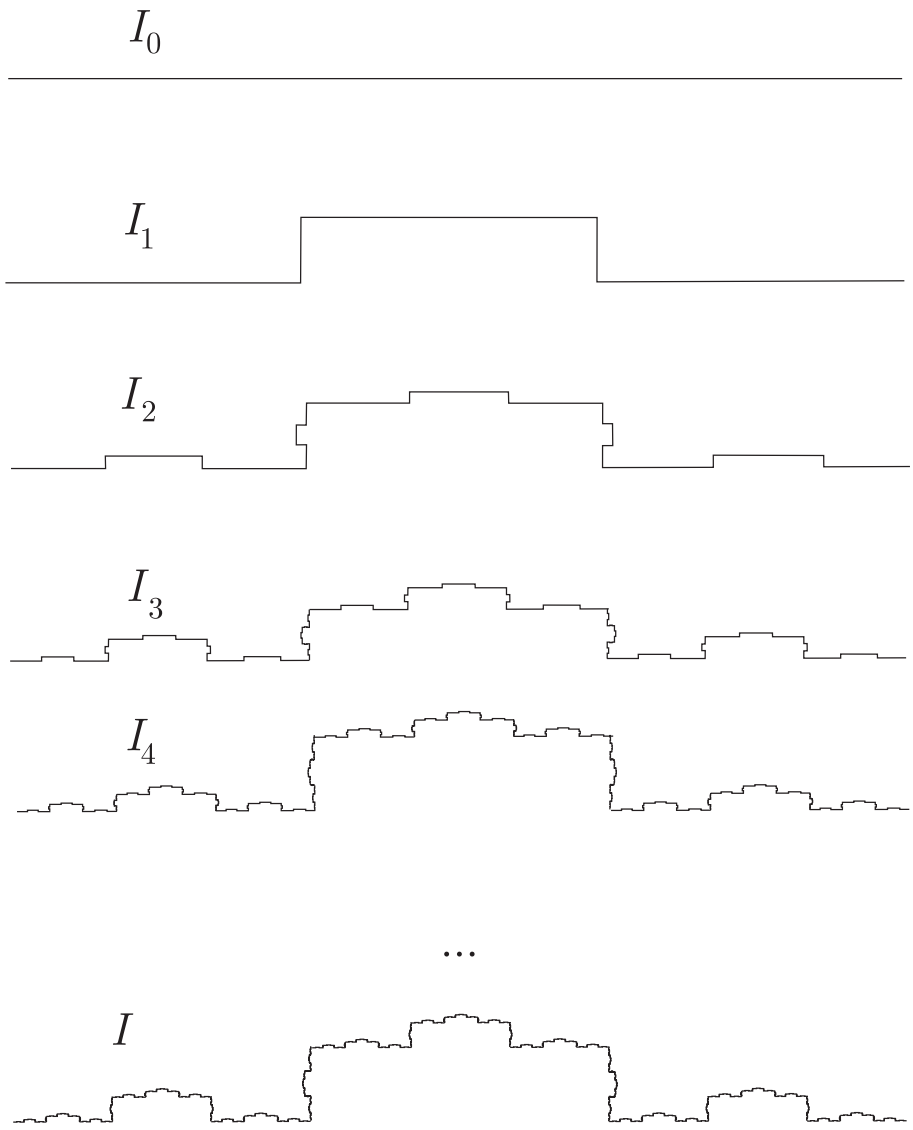


FIGURE 7
The construction of the set I.

We now introduce the concept of *unit outer normal* to the measure-theoretic boundary of a set.

Definizione 1.6. *Let $M \subseteq \mathbb{R}^n$ and $x \in \partial_* M$. We denote by $\boldsymbol{n}^M(x)$ a unit vector such that*

$$\begin{aligned} \lim_{r\to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in B_r(x) \cap M : (\xi - x)\cdot \boldsymbol{n}^M(x) > 0\}\right)}{r^n} &= 0, \\ \lim_{r\to 0^+} \frac{\mathscr{L}^n\left(\{\xi \in B_r(x) \setminus M : (\xi - x)\cdot \boldsymbol{n}^M(x) < 0\}\right)}{r^n} &= 0. \end{aligned} \tag{1.1}$$

This means that the direction of the unit outer normal must be such that the volume of the black parts in Figure 9 must decrease very rapidly when $r \to 0^+$. This is not the case on a corner, as the same figure shows.

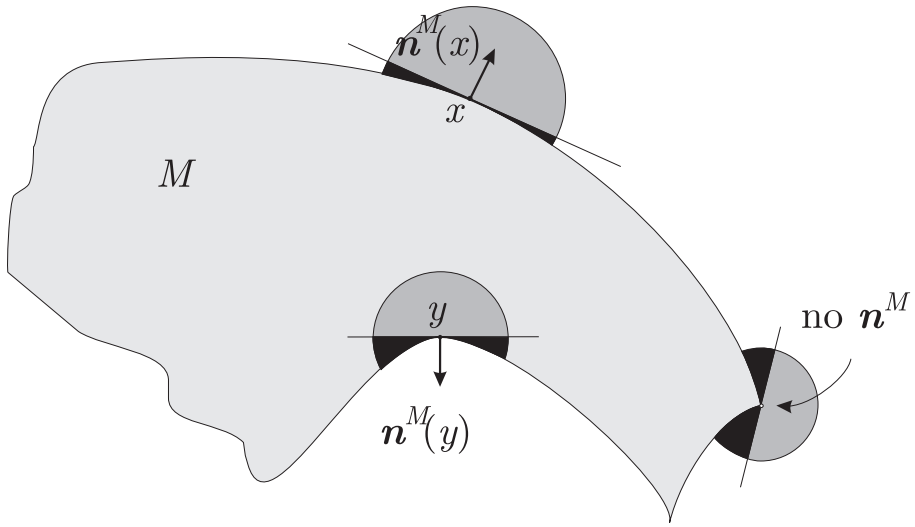


FIGURE 8
The outer unit normal to a set M.

No more than one such vector can exist. We can define a map, always denoted by $\boldsymbol{n}^M$, setting $\boldsymbol{n}^M(x) = \mathbf{0}$ if the limits do not both obtain. The bounded map $\boldsymbol{n}^M : \partial_* M \to V$ is called *the unit outer normal* ([5]) to M. It turns out that $\boldsymbol{n}^M$ is a Borel map, that is, $(\boldsymbol{n}^M)^{-1}(A) \in \mathfrak{B}(\partial_* M)$ for any open subset $A \subseteq V$. Furthermore, if $\boldsymbol{n}^M(x) \neq \mathbf{0}$, then $x \in \partial_* M$.

There is another nice definition of unit outer normal [16], in which only one condition occurs. Namely, if we set

$$H^+_{\boldsymbol{n}}(x) = \{\xi \in \mathbb{R}^n : (\xi - x) \cdot \boldsymbol{n} > 0\}$$

i.e. the half space towards which the normal points, then $x \in \partial_* M$ admits a unit normal vector $\boldsymbol{n}$ if and only if

$$x \in (M \triangle H^+_{\boldsymbol{n}}(x))_*.$$

In fact, if $H^-_{\boldsymbol{n}}(x)$ is the the half-space opposite to $H^+_{\boldsymbol{n}}(x)$, since

$$(M \triangle H^+_{\boldsymbol{n}}(x))_* = (M \cup H^+_{\boldsymbol{n}}(x))_* \cap ((\mathbb{R}^n \setminus M) \cup H^-_{\boldsymbol{n}}(x))_*$$

it follows that $x \in (M \triangle H^+_{\boldsymbol{n}}(x))_*$ if and only if the two limits in (1.1) hold.

A very important property of sets with finite perimeter is that the outer normal exists $\mathscr{H}^{n-1}$-almost everywhere. Thus, this holds also for the sets above introduced! For example, if we take as M a subset of the plane with I as a part of its boundary, the highest point has a vertical outer normal. This can be seen trying to "put a hat" on this set, as Figure 10 shows: the black part decreases as $r \to 0^+$ much faster than any sector. In the same way, all other middle points of the middle parts that have been displaced admit an outer normal and many more.

[5] although the vector $\boldsymbol{n}^M$ has only $\mathscr{H}^{n-1}$-a.e. length one: but this is irrelevant when expressions are integrated with respect to $\mathscr{H}^{n-1}$.

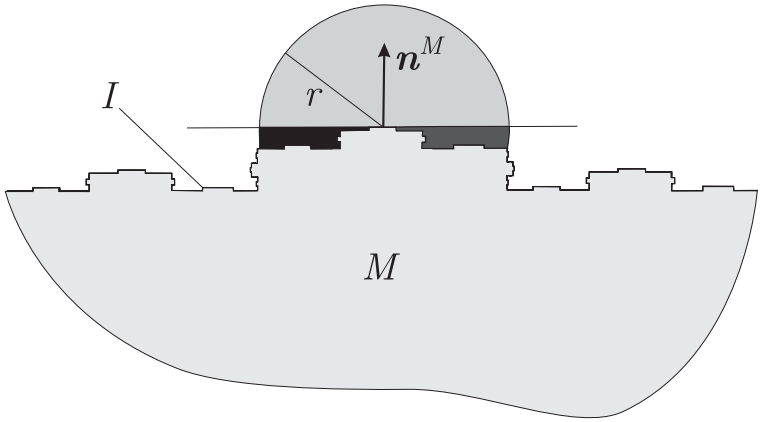


FIGURE 9
The outer normal to M at a point of I.

1.4. The divergence theorem for sets with finite perimeter

One of the main features of sets with finite perimeter is the Gauss-Green theorem.

Teorema 1.7 (De Giorgi). *If $M \subseteq \mathbb{R}^n$ is a set with finite perimeter, then $\left|\boldsymbol{n}^M(x)\right| = 1$ for $\mathscr{H}^{n-1}$-almost every $x \in \partial_* M$ and the formula*

$$\int_M \mathsf{T}\nabla f \, d\mathscr{L}^n = \int_{\partial_* M} f \, \mathsf{T}\boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \operatorname{div} \mathsf{T} \, d\mathscr{L}^n \tag{1.2}$$

holds whenever $f : \mathbb{R}^n \to \mathbb{R}$ and $\mathsf{T} : \mathbb{R}^n \to \operatorname{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ are Lipschitz continuous with compact support.

(Notice that, since f is Lipschitz continuous, it is $\mathscr{L}^n$-almost everywhere differentiable, so that the left-hand side makes sense.)

From this, the divergence theorem for a bounded set M with finite perimeter

$$\int_M \operatorname{div} \mathsf{T} \, d\mathscr{L}^n = \int_{\partial_* M} \mathsf{T}\boldsymbol{n}^M \, d\mathscr{H}^{n-1}$$

and its scalar version

$$\int_M \operatorname{div} \boldsymbol{v} \, d\mathscr{L}^n = \int_{\partial_* M} \boldsymbol{v} \cdot \boldsymbol{n}^M \, d\mathscr{H}^{n-1}$$

follow taking a suitable big ball containing M and taking $f \equiv 1$ on M and $\mathsf{T} = \boldsymbol{a} \otimes \boldsymbol{v}$, where $\boldsymbol{a}$ is a constant vector field.

2. Forces

Now we come to something more clearly related to Continuum Mechanics.

2.1. The continuous body

The union of two normalized sets is in general *not* normalized. This happens since it may be a surface which separates the two sets which is made by points essentially internal to the union, as for example in the case of two adjacent open rectangles.

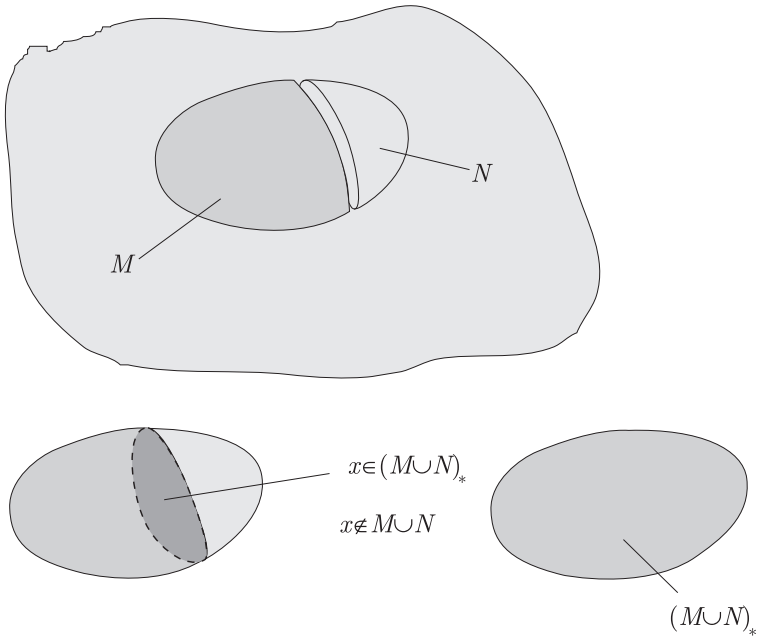


FIGURE 10
Union and normalized union.

But normalized sets are useful since they have no isolated points or low dimensional pieces, so they are good candidates to be considered as subbodies.

Also the difference of normalized sets is not normalized (as it happens with open sets), while the intersection of normalized sets is normalized.

On the other hand, if we want to deal with surface integrals, we are forced, for the time being, to choose sets with finite perimeter. This allows us to use the divergence theorem and Gauss-Green formula.

Finally, we restrict ourselves to bounded bodies: this avoids us to add perhaps complicated conditions at infinity which would hide the real problem. As we shall see, bounded bodies are complicated enough for our purposes.

Definizione 2.1. *Let B be a bounded normalized subset of $\mathbb{R}^n$ of finite perimeter and let $\mathscr{P}$ be a collection of normalized subsets of B of finite perimeter. We say that $\mathscr{P}$ is* a system of parts *(or* a system of subbodies*) of B, if the following conditions are satisfied:*

(a) $\varnothing, B \in \mathcal{P}$;
(b) *if* $M, N \in \mathcal{P}$, *then* $M \vee N := (M \cup N)_*$, $M \wedge N := M \cap N$, $M - N := (M \setminus N)_* \in \mathcal{P}$;

The elements of $\mathcal{P}$ are called parts *(or* subbodies*) of B and the pair $(B, \mathcal{P})$ is called* a continuous body.

From what we said above, the collection $\mathscr{B}$ of all normalized subsets of B with finite perimeter is a system of parts of B, as well as the collection of all normalized subsets M

of B such that $\mathrm{cl}(M) \subseteq \mathrm{int}(B)$. This class is of a certain interest, so we use for it a special symbol:

$$\mathscr{M}^\circ = \{M \in \mathscr{B} : \mathrm{cl}(M) \subseteq \mathrm{int}(B)\}.$$

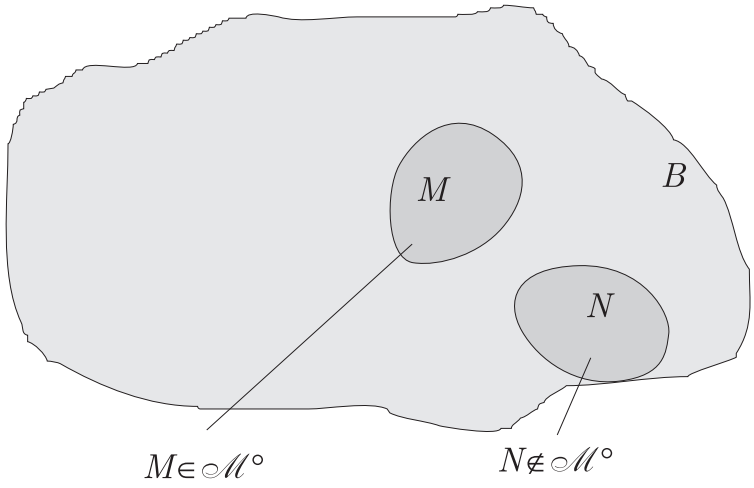


FIGURE 11
The collection $\mathscr{M}^\circ$.

2.2. Divergence measure tensor fields

The classical (local) form of the balance equation of Continuum Mechanics reads

$$\mathrm{div}\,\mathsf{T} + \boldsymbol{f} = \boldsymbol{0} \tag{2.1}$$

where $\boldsymbol{f}$ contains both distributed density of bulk (volume) forces, such as weight or gravitational forces, and inertial forces, i.e. density times acceleration. Typically, one has $\boldsymbol{f} = \rho(\boldsymbol{b} - \boldsymbol{a})$. Suppose now that $\boldsymbol{b}$ is in some sense singular, for example a measure not absolutely continuous with respect to volume measure like Dirac's delta distribution. Or suppose that the velocity has a discontinuity along a surface, as in the case of a shock wave. Since

$$\boldsymbol{a} = \frac{\partial \boldsymbol{v}}{\partial t} + (\nabla \boldsymbol{v})\boldsymbol{v},$$

the acceleration has a singularity of Dirac's delta type on the surface. In both these cases, the balance equation (2.1) implies that $\mathrm{div}\,\mathsf{T}$ *must* be a measure.

Of course, it may well happen that $\boldsymbol{f}$ is a general distribution, and in this case we would know very few about the stresses that it induces. But in this case it is also possible that the continuous body itself does not admit such a simple local description, as in the case of second-gradient materials, and the whole matter should be reframed with the appropriate force functionals, as we will discuss few later on.

The other way around, one can say: if B is a bounded normalized set of finite perimeter (the "body") and if T is a $L^1_{\mathrm{loc}}(\mathrm{int}\,B)$ tensor field, then $\mathrm{div}\,\mathsf{T}$ is, by the theory of distributions, the linear functional on $C^\infty_{\mathrm{c}}(\mathrm{int}\,B)$

$$\langle \mathrm{div}\,\mathsf{T}, f\rangle = -\int_{\mathrm{int}\,B} \mathsf{T}\nabla f \, d\mathscr{L}^n$$

and it is in general a distribution of order one. But if it happens to be of order zero, then it is canonically associated with a measure and there is a hope of giving (2.1) a sense. This motivates the following

Definizione 2.2. *Let* $\mathsf{T} \in L^1_{\mathrm{loc}}(\operatorname{int} B; \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^n))$. *We say that* T has divergence measure *if for every compact set* $K \subseteq \operatorname{int} B$ *there exists* $c_K \geqslant 0$ *such that*

$$\left| \int_{\operatorname{int} B} \mathsf{T} \nabla f \, d\mathscr{L}^n \right| \leqslant c_K \max_K f$$

whenever $f \in C^\infty_{\mathrm{c}}(\operatorname{int} B)$ *and* $\operatorname{supt} f \subseteq K$.

In this case we know from the theory of distribution that there exist a uniquely determined $\eta \in \mathfrak{M}(\operatorname{int} B)$ and a uniquely determined $\mu-$almost everywhere Borel map $\boldsymbol{u} : \operatorname{int} B \to \mathbb{R}^n$ such that $|\boldsymbol{u}(x)| = 1$ for η-a.e. $x \in \operatorname{int} B$ and

$$- \int_{\operatorname{int} B} \mathsf{T} \, \nabla f \, d\mathscr{L}^n = \int_{\operatorname{int} B} f \, \boldsymbol{u} \, d\eta$$

for any Lipschitz function $f : \operatorname{int} B \to \mathbb{R}$ with compact support. We set

$$\int_M \boldsymbol{v} \cdot \operatorname{div} \mathsf{T} = \int_M \boldsymbol{v} \cdot \boldsymbol{u} \, d\eta$$

for any $\boldsymbol{v} \in C^\infty_{\mathrm{c}}(\operatorname{int} B; \mathbb{R}^n)$. Roughly speaking, we have "$\boldsymbol{u}\, d\eta = \operatorname{div} \mathsf{T}$".

2.3. (Almost all) subbodies

Now we come to an important point. Suppose T is a C^1 tensor field. Then the divergence theorem says that

$$\int_{\partial_* M} \mathsf{T} \boldsymbol{n}^M \, d\mathscr{H}^{n-1} = \int_M \operatorname{div} \mathsf{T} \, d\mathscr{L}^n.$$

Let's denote for a moment by $\operatorname{div} \mathsf{T}$ the measure $(\operatorname{div} \mathsf{T})\mathscr{L}^n$. Since M is $\mathscr{L}^n$-measurable, we already noticed that $\mathscr{L}^n(\partial_* M) = 0$ and therefore $\operatorname{div} \mathsf{T}(\partial_* M) = 0$. Since we want to deal with tensor fields with divergence measure and keep the divergence theorem in the form

$$\int_{\partial_* M} \mathsf{T} \boldsymbol{n}^M \, d\mathscr{H}^{n-1} = \int_M \operatorname{div} \mathsf{T} = \operatorname{div} \mathsf{T}(M) \tag{2.2}$$

we see that M must be $\operatorname{div} \mathsf{T}$-measurable, so we impose that there exists a positive measure η such that $\eta(\partial_* M) = 0$. This η will turn then out to be just $|\operatorname{div} \mathsf{T}|$, the total variation of the measure $\operatorname{div} \mathsf{T}$ ([6]).

Of course, it may happen that there are subbodies for which this is not satisfied, but, since η is finite on compact sets, there are not so many Borel subsets with finite area and positive η measure.

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[6] We won't use in the sequel the definition of total variation of a measure. The reader not familiar with it can find it in [5], p 98.

On the other hand, the left-hand side of (2.2) is an integral of an L^1_{loc} function, and this must also be finite. Therefore we require that there exists a positive locally integrable function h on $\operatorname{int} B$ ([7]) such that

$$\int_{\partial_* M} h \, d\mathscr{H}^{n-1} < +\infty.$$

This h will turn out to be $|\mathsf{T}\boldsymbol{n}^M|$. This motivates the following

Definizione 2.3. *Let $(B, \mathscr{B})$ be a continuous body. For every $h \in L^1_+(\operatorname{int} B)$ and $\eta \in \mathfrak{M}(\operatorname{int} B)$ we set*

$$\mathscr{M}^\circ_{h\eta} = \left\{ M \in \mathscr{M}^\circ : \ \int_{\partial_* M} h \, d\mathscr{H}^{n-1} < +\infty \text{ and } \eta(\partial_* M) = 0 \right\},$$

Then, given $\mathscr{M} \subseteq \mathscr{M}^\circ$,

(*i*) *we say that $\mathscr{M}$* contains almost all of *$\mathscr{M}^\circ$, if $\mathscr{M}_{h\eta} \subseteq \mathscr{M}$ for some $h \in L^1_+(\operatorname{int} B)$ and $\eta \in \mathfrak{M}(\operatorname{int} B)$;*

(*ii*) *given a property π, we say that π* holds almost everywhere in *$\mathscr{M}$, if the set*

$$\{M \in \mathscr{M}^\circ : \pi(M) \text{ is defined and } \pi(M) \text{ holds}\}$$

contains almost all of $\mathscr{M}^\circ$.

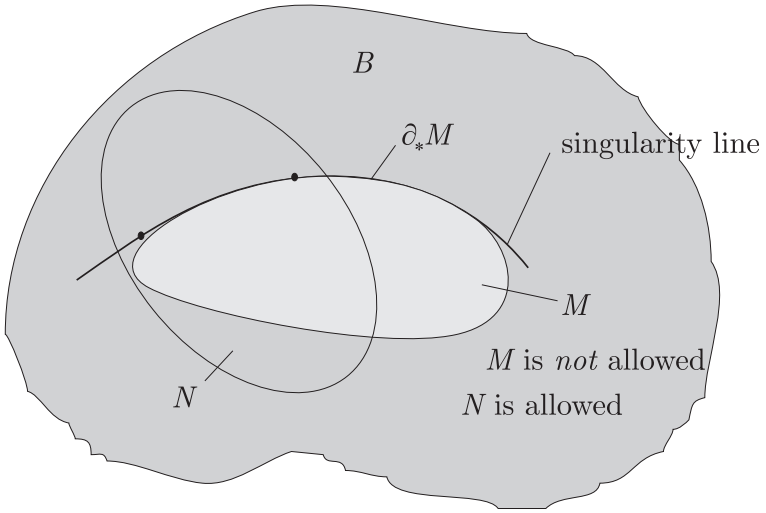

FIGURE 12
Almost all subbodies

Figure 13 illustrates what may happen: suppose there is a line with a singularity, say, of the stress tensor. Then it may happen that the flux of $\mathsf{T}\boldsymbol{n}^M$ is not L^1 on some M because its surface $\partial_* M$ "intersects too much" the singularity line.

With respect to the concepts introduced above, in [2] Degiovanni, Marzocchi and Musesti proved the following theorem:

[7] We will use the notation $h \in L^1_{+,\mathrm{loc}}$ for this. Actually, h is not an equivalence class of functions.

Teorema 2.4. *Let* T *be a tensor field with divergence measure. Then there exist* $h \in L^1_{\mathrm{loc}}(\operatorname{int} B)$ *and* $\eta \in \mathfrak{M}(\operatorname{int} B)$ *such that for every* $M \in \mathscr{M}_{h\eta}$ *and for every locally Lipschitz function* $f : \operatorname{int} B \to \mathbb{R}$*, the function* T *is* $\mathscr{H}^{n-1}$*-summable on* $\partial_* M$ *and*

$$\int_M \mathsf{T}\nabla f \, d\mathscr{L}^n = \int_{\partial_* M} f \, \mathsf{T}\boldsymbol{n}^M \, d\mathscr{H}^{n-1} - \int_M f \operatorname{div} \mathsf{T} \tag{2.3}$$

Notice the difference between (1.2) and (2.3): in the first the last integral is taken with respect to the volume measure, while here we have the measure $\operatorname{div} \mathsf{T}$.

2.4. Force and power

Now we come to the main definition of this course.

Definizione 2.5. *Let* $\mathscr{M} \subseteq \mathscr{M}^\circ$ *a collection of subbodies which contains almost all of* $\mathscr{M}^\circ$*. A function*

$$F : \mathscr{M} \times C^\infty(\operatorname{int} B; \mathbb{R}^n) \to \mathbb{R}$$
$$F : (M, \boldsymbol{v}) \mapsto P(M, \boldsymbol{v})$$

such that

(a) *for every* $M \in \mathscr{M}$*,* $P(M, \cdot)$ *is linear on* $C^\infty(\operatorname{int} B, \mathbb{R}^n)$*;*
(b) *for every* $\boldsymbol{v} \in C^\infty(\operatorname{int} B, \mathbb{R}^n)$*, if* $M \wedge N = \varnothing$*, then*

$$P(M \vee N, \boldsymbol{v}) = P(M, \boldsymbol{v}) + P(N, \boldsymbol{v});$$

(c) *there exist* $\mu_0, \mu_1 \in \mathfrak{M}(\operatorname{int} B)$ *such that* $\forall M \in \mathscr{M}$, $\forall \boldsymbol{v} \in C^\infty(\operatorname{int} B, \mathbb{R}^n)$

$$|P(M, v)| \leqslant \int_M ||\boldsymbol{v}|| \, d\mu_0 + \int_M ||\nabla \boldsymbol{v}|| \, d\mu_1, \tag{2.4}$$

will be called a (first-order gradient) force functional*. Its value* $P(M, \boldsymbol{v})$ *will be called (first-order gradient)* power *or* working *expended on the subbody* M *by the force functional on the velocity field* $\boldsymbol{v}$*. Finally, a set function satisfying* (b) *will said to be* $*$-additive*.*

Several remarks here are mandatory.

Remark 2.1. We adopt here the idea that the force is a mapping which associates to a virtual velocity field $\boldsymbol{v}$ the working (or power) expended on $\boldsymbol{v}$. This is more general than saying that the force is a vector field, what happens in the presence of a scalar product. But we will scarcely make use of the symbol F, since we are more interested on its value $P(M, \boldsymbol{v})$. This idea, which goes up to D'Alembert, has been put to attention in Continuum Mechanics by Ruggeri [18] and Germain ([6], [7]).

Remark 2.2. The choice of $\boldsymbol{v} \in \mathbb{R}^n$ is just for physical reasons. One could even consider virtual velocity fields with values in $\mathbb{R}^N$, $N \geqslant 1$. If $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_N\}$ denotes the canonical basis in $\mathbb{R}^N$, for every $i = 1, \dots, N$ it is possible to define $P_i(M, v) = P(M, v\,\boldsymbol{e}_i)$, which is a force functional on scalar velocity fields. On the other hand, if $\boldsymbol{v} = (v_1, \dots, v_N)$, then by linearity

$$P(M, \boldsymbol{v}) = \sum_{i=1}^N P_i(M, \boldsymbol{v}_i)\,.$$

For this reason, in proofs we sometimes treat the case of scalar velocity fields, as the results for vector velocity fields can be easily deduced from the corresponding ones in the scalar case.

Remark 2.3. The choice of first-gradient force functionals comes from simplicity reasons. One could also consider force functionals where (c) of definition 2.5 is replaced by

(c) there exist $\mu_0, \dots, \mu_k \in \mathfrak{M}(\operatorname{int} B)$ such that $\forall M \in \mathscr{M}$, $\forall \boldsymbol{v} \in C_c^\infty(\operatorname{int} B; \mathbb{R}^n)$

$$|P(M, \boldsymbol{v})| \leqslant \sum_{j=0}^{k} \int_M ||\nabla^{(j)} \boldsymbol{v}(x)||\, d\mu_j(x).$$

Of particular interest are second-gradient force functionals, which may describe edge effects, and third-gradient force functionals, which may describe vertex effects. Since the study of these features would go beyond the scope of the course, we will restrict to the first-order case. The treatment of this cases can be found in [14] and [4].

Remark 2.4. The fact that the power expended is bounded by measures allows to include singular measures such as Dirac delta functions which are not absolutely continuous with respect to Lebesgue measure. If equality holds in (2.4) and μ_0, μ_1 are absolutely continuous with respect to Lebesgue measure, then the power assumes the classical form

$$P(M, \boldsymbol{v}) = \int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mathscr{L}^n + \int_M \mathsf{B} : \nabla \boldsymbol{v}\, d\mathscr{L}^n$$

for appropriate vector- and tensor-valued fields $\boldsymbol{a}$ and B.

The given properties of the force functional allow to prove a representation theorem.

Teorema 2.6. *There exist two bounded Borel maps* $\boldsymbol{a} : \operatorname{int} B \to \mathbb{R}^n$, $\mathsf{B} : \operatorname{int} B \to \operatorname{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ *such that*

$$\forall M \in \mathscr{M}, \forall \boldsymbol{v} \in C_c^\infty(\operatorname{int} B): \quad P(M, \boldsymbol{v}) = \int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mu_0 + \int_M \mathsf{B} : \nabla \boldsymbol{v}\, d\mu_1$$

where μ_0, μ_1 *are given by* (c) *of Definition 2.5. Moreover,* $\boldsymbol{a}$ *and* B *are uniquely determined resp.* μ_0, μ_1*-a.e.*

Sketch of the proof. The main ideas are the following: first, introduce the concept of a *full grid*. This is constructed as follows. Take the real axis and take away a set with zero 1-Lebesgue measure (e.g. the rationals). Call then $\widehat{G}$ the remaining set and add to this an origin and an orthonormal, positively oriented basis in $\mathbb{R}^n$. Then, an n-interval whose projection has endpoints in $\widehat{G}$ is called a G-interval. Finally, denote with $\mathcal{M}_G$ the set of all G-intervals contained in $\operatorname{int} B$.

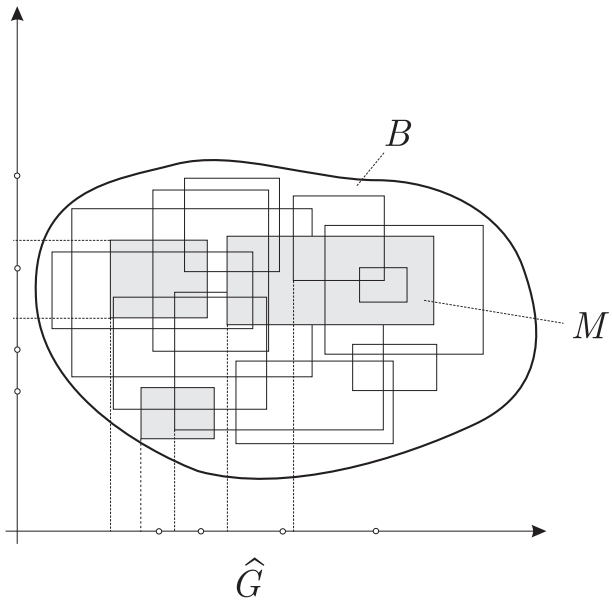

FIGURE 13
G-interval.

Now the following is proved in [3].

Theorem 2.5. *Let X be a finite-dimensional normed space and $\lambda : \mathcal{M}_G \times X \to \mathbb{R}$ a function such that*

(1) *for every $M \in \mathcal{M}_G$, $\lambda(M, \cdot\,)$ is linear;*
(2) *for every $y \in X$, $\lambda(\cdot\,, y)$ is $*$-additive;*
(3) *there exists $\mu \in \mathfrak{M}(\Omega)$ such that*

$$\forall M \in \mathcal{M}_G, \forall y \in X: \quad |\lambda(M, y)| \leqslant |y| \mu(M)\,.$$

Then there exists a bounded Borel map $A : \Omega \to X^*$ such that

$$\forall M \in \mathcal{M}_G, \forall y \in X: \quad \lambda(M, y) = \int_M \langle A(x), y\rangle \, d\mu(x)\,.$$

Moreover, A is uniquely determined μ-a.e.

Second, for every $M \in \mathcal{M}_G$ and for every $\boldsymbol{v}_0 \in \mathbb{R}^n$, $\mathsf{F} \in \mathrm{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ one can find a test velocity field $\boldsymbol{v}$ such that $\boldsymbol{v}_{|M} = \boldsymbol{v}_0 + \mathsf{F}\boldsymbol{x}$ on M.

One has then

$$\forall M \in \mathcal{M}_G,\ \forall \boldsymbol{v}_0 \in \mathbb{R}^n:\ |P(M, \boldsymbol{v}_0)| \leqslant \int_M |\boldsymbol{v}_0| \, d\mu_0\,,$$

hence there exists a bounded Borel map $\boldsymbol{a} : \Omega \to \mathbb{R}^n$ such that

$$\forall M \in \mathcal{M}_G,\ \forall \boldsymbol{v}_0 \in \mathbb{R}^n:\ P(M, \boldsymbol{v}_0) = \int_M \boldsymbol{a} \cdot \boldsymbol{v}_0 \, d\mu_0 \tag{2.5}$$

and $|\boldsymbol{a}| \leqslant 1$ μ_0-a.e.

Given $\mathsf{F} \in \mathrm{Lin}$ and the vector field $\mathsf{F}x$, we now claim that

$$\left| P(M, \mathsf{F}x) - \int_M \boldsymbol{a} \cdot \mathsf{F}x \, d\mu_0 \right| \leqslant \int_M |\mathsf{F}| \, d\mu_1 \tag{2.6}$$

for every $M \in \mathcal{M}_G$.

We need at this point the following

Lemma 2.6. *Let $\lambda : \mathcal{M}_G \to \mathbb{R}$ be $*$-additive and suppose that there exists $\mu \in \mathfrak{M}(\Omega)$ such that $\lambda \ll \mu$.*

Then $\lambda \leqslant 0$ if and only if for μ-a.e. $x_0 \in \Omega$ one has

$$\lim_{i\to\infty} \frac{\lambda(M_i)}{\mu(M_i)} \leqslant 0$$

whenever $(M_i) \subseteq \mathcal{M}_G$ is a sequence of open n-cubes with $x_0 \in M_i$ and $\operatorname{diam} M_i \to 0$ as $i \to \infty$.

Consider then the function

$$\lambda_1(M) = P(M, \mathsf{F}x) - \int_M \boldsymbol{a} \cdot \mathsf{F}x \, d\mu_0 - \int_M |\mathsf{F}| \, d\mu_1 \, .$$

Then λ_1 is $*$-additive and $\lambda_1 \ll \mu$, where $\mu = \mu_0 + \mu_1$. For every $x_0 \in \Omega$ it can be easily checked that

$$\lambda_1(M) = P(M, \mathsf{F}x - \mathsf{F}x_0) - \int_M \boldsymbol{a} \cdot (\mathsf{F}x - \mathsf{F}x_0) \, d\mu_0 - \int_M |\mathsf{F}| \, d\mu_1 \, .$$

Let $x_0 \in \Omega$ and (M_i) a sequence as in Lemma 2.6; we have

$$\lim_{i\to\infty} \frac{\lambda_1(M_i)}{\mu(M_i)} \leqslant \lim_{i\to\infty} \frac{1}{\mu(M_i)} \int_{M_i} |\mathsf{F}(x - x_0)|(1 + |\boldsymbol{a}|) \, d\mu_0 = 0$$

Now a big theorem comes.

Theorem 2.7 (Lebesgue-Besicovitch Differentiation Theorem). *Let $\mu \in \mathfrak{M}(\Omega)$ and $f \in L^1_{loc}(\Omega; \mu)$. Then for μ-a.e. $x_0 \in \Omega$*

$$\lim_{i\to\infty} \frac{1}{\mu(M_i)} \int_{M_i} f \, d\mu = f(x_0) \tag{2.7}$$

whenever $(M_i) \subseteq \mathcal{M}_G$ is a sequence of open n-cubes with $x_0 \in M_i$ and $\operatorname{diam} M_i \to 0$ as $i \to \infty$.

By this theorem, the right-hand side vanishes of the previous equation for μ-a.e. $x_0 \in \Omega$, hence $\lambda_1 \leqslant 0$. In the same way, setting

$$\lambda_2(M) = P(M, \mathsf{F}x) - \int_M \boldsymbol{a} \cdot \mathsf{F}x \, d\mu_0 + \int_M |\mathsf{F}| \, d\mu_1 \, ,$$

one can prove that $\lambda_2 \geqslant 0$ and (2.6) follows.

Then, by Corollary 2.5 there exists a bounded Borel map $\mathsf{A} : \Omega \to \mathrm{Lin}$ such that

$$\forall M \in \mathcal{M}_G, \, \forall \mathsf{F} \in \mathrm{Lin} : \; P(M, \mathsf{F}x) = \int_M \boldsymbol{a} \cdot \mathsf{F}x \, d\mu_0 + \int_M \mathsf{A} \cdot \mathsf{F} \, d\mu_1 \tag{2.8}$$

and $|\mathsf{A}| \leqslant 1$ μ_1-a.e. Equation (2.8), combined with (2.5), proves the claim for all G-intervals.

Finally, since we know the good candidates for the representation of the force functional, we set $P : \mathcal{B}_\eta \times C^\infty(\Omega; \mathbb{R}^n) \to \mathbb{R}$ defined as

$$P(M, \boldsymbol{v}) = \int_M \boldsymbol{a}(x) \cdot \boldsymbol{v}(x) \, d\mu_0(x) + \int_M \mathsf{A}(x) \cdot \nabla \boldsymbol{v}(x) \, d\mu_1(x)$$

As for uniqueness, consider a generic function $\tilde{P} : \mathcal{B}_\eta \times C^\infty(\Omega;\mathbb{R}^n) \to \mathbb{R}$ which satisfies the properties of a force functional on all its domain with $\eta = \mu_0 + \mu_1$. Then in particular it is a force of order one and there exist two bounded Borel maps $\boldsymbol{a}, \mathsf{A}$ as before. Let $\boldsymbol{v} \in C^1(\Omega;\mathbb{R}^n)$ and set

$$\forall M \in \mathcal{B}_\eta : \quad \lambda(M) = \tilde{P}(M, \boldsymbol{v}) - \int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mu_0 - \int_M \mathsf{A} \cdot \nabla \boldsymbol{v}\, d\mu_1 \,.$$

Let $x_0 \in \Omega$, (M_i) be a sequence as in Theorem 2.6 and $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{v}(x_0) - \nabla\boldsymbol{v}(x_0)(x - x_0)$. It is not difficult to see that

$$\lambda(M) = \tilde{P}(M, \boldsymbol{w}) - \int_M \boldsymbol{a} \cdot \boldsymbol{w}\, d\mu_0 - \int_M \mathsf{A} \cdot \nabla \boldsymbol{w}\, d\mu_1 \,.$$

Since $\boldsymbol{w}(x_0) = 0$ and $\nabla\boldsymbol{w}(x_0) = 0$, taking into account Theorem 2.7 and (c) of the definition of force one has

$$\lim_{i\to\infty} \frac{\lambda(M_i)}{\eta(M_i)} = 0 \,,$$

for η-a.e. $x_0 \in \Omega$, hence $\lambda = 0$. This completes the proof. ■

2.5. Balance of forces

Equilibrium in Mechanics is obtained imposing a balance of forces. (If, among all possible forces, also inertial forces are considered, one gets also the dynamical equations of motion, but we don't want to go further into this matter). But in Continuum Mechanics there is no simple way (or even complicated, up to today) to derive the classical balance laws imposing the equilibrium condition on all possible points of a continuous body; it's definitely better to impose equilibrium on all subbodies. Therefore we may state our equilibrium condition as follows:

Axiom 2.8 (d'Alembert's Principle). *Let $\mathscr{B}$ a continuous body. We impose that $\mathscr{B}$ is in equilibrium if and only if the sum of all force functionals defined on it is zero at every instant of time.*

Since subbodies and virtual velocity fields are the variables of the force functional, the axiom just stated is equivalent to say that at every instant of time, for any subbody and for any virtual velocity field, the sum of all powers expended on the virtual velocity field on the subbody is zero:

$$\forall t \in I, \forall M \in \mathscr{M}, \forall \boldsymbol{v} \in C^\infty(\operatorname{int} B) : \qquad \sum_i P_i(M, \boldsymbol{v}) = 0.$$

It is clear that this definition of equilibrium depends on the choice of the chosen family if subbodies and, possibly, on the class of admissible virtual velocity fields.

2.6. Internal forces

In the classical framework it is usual to distinguish between *internal* forces ([8]) and external ones; in particular, internal forces have to be restricted by the Principle of Material Frame Indifference.

Axiom 2.9 (**Power of internal forces).** *The power expended by internal forces on* rigid *virtual velocity fields must be zero at any instant of time.*

We will indicate by $P^{(i)}(M, \boldsymbol{v})$ the power expended by internal forces on a virtual velocity field. Then we can prove the following

Teorema 2.7. *Let $\mathscr{M}$ be a family of subbodies containing almost all of $\mathscr{M}^\circ$ and let $P^{(i)}(M, \boldsymbol{v})$ the power expended by internal forces on a virtual velocity field $\boldsymbol{v}$. Then*

$$P^{(i)}(M, \boldsymbol{v}) = -\int_M \mathsf{T} \cdot \mathsf{D}\, d\mu_1$$

for every $M \in \mathscr{M}$ and $\boldsymbol{v} \in C^1(\operatorname{int} B)$, where T is a symmetric tensor field.

Notice that we do not yet require that T must have divergence measure. To prove this, we need a preliminary result.

Theorem 2.10. *Let $\mu \in \mathfrak{M}(\Omega)$ and $f \in L^1_{loc}(\Omega; \mu)$ be such that*

$$\forall M \in \mathcal{M}_G : \quad \int_M f\, d\mu = 0\,.$$

Then $f(x) = 0$ for μ-a.e. $x \in \Omega$.

Proof. It is an easy consequence of the Lebesgue-Besicovitch theorem, because the left-hand side of (2.7) vanishes μ-a.e.

Proof of theorem 2.7. Remember that a velocity field is rigid if and only if the symmetric part of the gradient vanishes identically ([8], p. 69) . Consider a vector field $\boldsymbol{v}_0 \in \mathbb{R}^n$. Since

$$\forall M \in \mathcal{M}_G : \quad P_{(i)}(M, \boldsymbol{v}_0) = \int_M \boldsymbol{a} \cdot \boldsymbol{v}_0\, d\mu_0$$

and the left-hand side vanishes in view of Axiom 2.9, the arbitrariness of $\boldsymbol{v}_0$ implies

$$\forall M \in \mathcal{M}_G : \quad \int_M \boldsymbol{a}\, d\mu_0 = 0\,.$$

Hence by Corollary 2.10 one has $\boldsymbol{a} = 0$ μ_0-a.e. and

$$\forall M \in \mathcal{M}_G : \quad P_{(i)}(M, \boldsymbol{v}) = \int_M \mathsf{A} \cdot \nabla \boldsymbol{v}\, d\mu_1\,. \tag{2.9}$$

Now consider a skew matrix $\mathsf{F} \in \operatorname{Lin}(\mathbb{R}^n, \mathbb{R}^n)$ and the vector field $\boldsymbol{v}(x) = \mathsf{F}x$. From (2.9) and Axiom 2.9 it follows that

$$\forall M \in \mathcal{M}_G : \quad \int_M \mathsf{A} \cdot \mathsf{F}\, d\mu_1 = \int_M \mathsf{A}^\omega \cdot \mathsf{F}\, d\mu_1 = 0$$

and the arbitrariness of F yields $\mathsf{A}^\omega = 0$ μ_1-a.e. ■

[8] This means that a choice of what degrees of freedom are to be considered "internal" has been made, as well as having neglected undesired coupling between them. One could also simply define internal forces by axiom 2.8.

2.7. Distance, Contact and Cauchy forces, Cauchy fluxes

We just saw that a force is a first-gradient linear functional. On the other hand, frequently external forces are just vector-valued distributions on the body, the so-called distance forces (or bulk or volume forces), whose expended power depends only on the virtual velocity field and not on its derivatives. Therefore we put the following

Definizione 2.8. *A force functional with order zero will be called a* distance *force functional.*

Therefore, for a distance force, theorem (2.6) reduces to the representation

$$P^{(b)}(M, \boldsymbol{v}) = \int_M \boldsymbol{a} \cdot \boldsymbol{v} \, d\mu_0.$$

It may of course also happen that a force acts only as the result of a contact of subbodies. In this case, a subbody experiences an expending of power which concentrates on its surface: no matter what the velocity field is at the interior, the power expended by contact forces is the same. In view of the linearity, we may state the following

Definizione 2.9. *A force functional such that*

$\forall M \in \mathscr{M}$, $\forall \boldsymbol{v} \in C^\infty(\operatorname{int} B; \mathbb{R}^n)$: $\quad \operatorname{supt} \boldsymbol{v} \subseteq M \;\Rightarrow\; P^{(c)}(M, \boldsymbol{v}) = 0\,.$

is said to be a contact force *functional.*

The following theorem characterizes contact force functionals.

Teorema 2.10. *A force functional (of order one) is a contact force functional if and only if*

$$\boldsymbol{a}\mu_0 = \operatorname{div}(\mathsf{B}\mu_1) \tag{2.10}$$

on $\operatorname{int} B$ *in the sense of distributions.*

Proof. Consider $\boldsymbol{v} \in C^\infty_{\mathrm{c}}(\operatorname{int} B; \mathbb{R}^n)$ and let $M \in \mathcal{M}_G$ be such that $\operatorname{supt} \boldsymbol{v} \subseteq M$. One has

$$\langle \boldsymbol{a}\mu_0, \boldsymbol{v} \rangle = \int_{\operatorname{int} B} \boldsymbol{a} \cdot \boldsymbol{v} \, d\mu_0 = \int_M \boldsymbol{a} \cdot \boldsymbol{v} \, d\mu_0$$

and, by definition of distributional divergence,

$$\langle \operatorname{div}(\mathsf{B}\mu_1), \boldsymbol{v} \rangle = -\int_\Omega \mathsf{B} \cdot \nabla \boldsymbol{v} \, d\mu_1 = -\int_M \mathsf{B} \cdot \nabla \boldsymbol{v} \, d\mu_1\,.$$

Then it is also clear that P is a contact force functional if and only if the left-hand sides coincide.

Notice that this implies that B is a tensor field with divergence measure.

One would expect that a contact force admits a representation as an integral over the surface of the subbody M. For, in the classical case, where $\mu_0 = \mu_1 = \mathscr{L}^n$, (2.10) becomes just $\boldsymbol{a} = \operatorname{div} \mathsf{B}$, and thus by the (classical) divergence theorem

$$P^{(c)}(M, \boldsymbol{v}) = \int_M (\operatorname{div} B \cdot \boldsymbol{v} + \mathsf{B} : \nabla \boldsymbol{v}) \; d\mathscr{L}^n = \int_M \operatorname{div}(\mathsf{B}^{\mathsf{T}} \boldsymbol{v}) \, d\mathscr{L}^n = \int_{\partial_* M} \mathsf{B}\boldsymbol{n}^M \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}.$$

But up to this point we don't have any relationship between μ_0 and μ_1, and we cannot expect the divergence theorem to be true. Therefore we introduce a special class of contact forces, the Cauchy ones.

Definizione 2.11. *A force functional such that there exists $h \in L^1_{+,\mathrm{loc}}(\operatorname{int} B)$ such that*

$$\forall M \in \mathscr{M},\ \forall \boldsymbol{v} \in C^\infty_{\mathrm{c}}(\operatorname{int} B; \mathbb{R}^n): \qquad \left|P^{(C)}(M, \boldsymbol{v})\right| \leqslant \int_{\partial_* M} |\boldsymbol{v}|\, h\, d\mathscr{H}^{n-1}$$

is said to be a Cauchy force *functional.*

It is apparent that a Cauchy force functional is also a contact force functional. The converse is not always true.

Cauchy forces are important because they are related with Cauchy fluxes. We briefly recall their definition and the definition of material surface.

Definizione 2.12. *A* material surface *is a pair of Borel maps $(\widehat{S}, \boldsymbol{n}_S)$ such that there exists $M \in \mathscr{M}^\circ$ with $\widehat{S} \subseteq \partial_* M$ and $\boldsymbol{n}_S = \boldsymbol{n}^M_{|S}$. Denote with $\mathscr{S}^\circ$ the collection of all material surfaces in B. For given $h \in L^1_{+,\mathrm{loc}}$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$, the set*

$$\mathscr{S}^\circ_{h\nu} = \{S \in \mathscr{S}^\circ : S \text{ is subordinated to some } M \in \mathscr{M}^\circ_{h\eta}\}$$

defines almost all material surfaces *in B in the sense that a property π will be said to hold on almost all material surfaces if the set*

$$\{S \in \mathscr{S}^\circ : \pi(S) \text{ is defined and } \pi(S) \text{ holds}\}$$

is contained in $S^\circ_{h\eta}$ for some $h \in L^1_{+,\mathrm{loc}}$ and $\nu \in \mathfrak{M}(\operatorname{int} B)$.

Definizione 2.13. *Let $\mathscr{R} \subseteq \mathscr{S}^\circ$ be a set containing almost all of $\mathscr{S}^\circ$ and consider $Q : \mathscr{R} \to \mathbb{R}^n$. We say that a function Q is* a Cauchy flux *on B, if the following properties hold:*

(a) *if $S, T \in \mathscr{R}$ are compatible and disjoint with $S \cup T \in \mathscr{R}$, then*

$$Q(S \cup T) = Q(S) + Q(T);$$

(b) *there exists $h \in L^1_{+,\mathrm{loc}}(\operatorname{int} B)$ such that the inequality*

$$|Q(S)| \leqslant \int_S h\, d\mathscr{H}^{n-1}$$

holds on almost all of $\mathscr{S}^\circ$.

Finally, a Cauchy flux is said to be equilibrated, *if*

$$Q(-S) = -Q(S)$$

for almost all material surfaces.

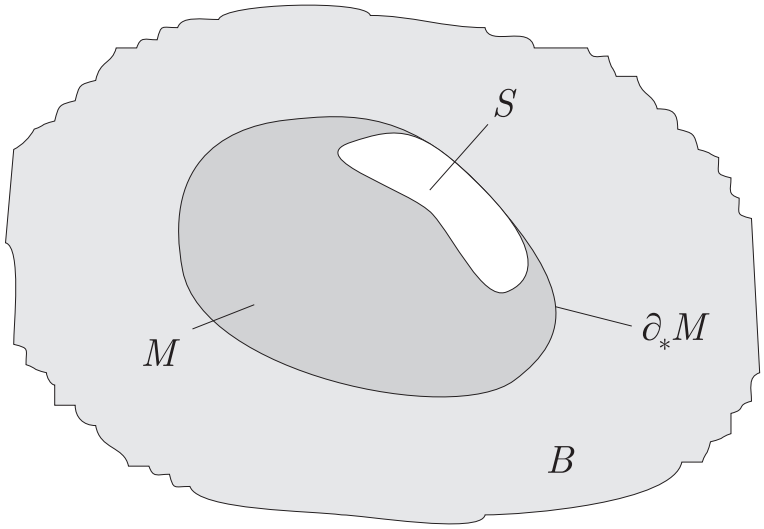

FIGURE 14

Cauchy fluxes have been in the last years the object of deep studies ([9], [23], [19], [20], [21], [22], [17]). In fact, as one would expect, they can be obtained in an essentially unique way by a Cauchy force.

Teorema 2.14. *For every Cauchy force functional there exists an almost everywhere uniquely defined equilibrated Cauchy flux Q such that*

$$P^{(C)}(M, \boldsymbol{v}) = \int_M \boldsymbol{t}_{P,M} \cdot \boldsymbol{v} \, d\mathscr{H}^{n-1}$$

and

$$Q(S) = \int_S \boldsymbol{t}_{P,M} \, d\mathscr{H}^{n-1}$$

for almost every subbody and material surface.

For the proof of this result, see [14], thms. 3.10 and 3.17.

2.8. Weak balance

Now we introduce a notion which is related to the equilibrium condition given in 2.8.

Definizione 2.15. *A force functional of order one is said to be* weakly balanced*, if there exists $\nu \in \mathfrak{M}(\operatorname{int} B)$ such that*

$$\forall M \in \mathscr{M}, \, \forall \boldsymbol{v} \in C^\infty_{\mathrm{c}}(M; \mathbb{R}^n) : \qquad |P(M, \boldsymbol{v})| \leqslant \int_M ||\boldsymbol{v}|| \, d\nu \, .$$

Compare this with the definition of contact force functional. This is *not* as saying that the force is of order zero, since there the power could be expended on virtual velocity fields with compact support in $\operatorname{int} B$, while here only velocity fields with compact support in M are considered. Clearly, a contact force is always balanced with $\nu = 0$.

The meaning of the definition of weak balance is the following. In the classical case, one states the balance of a quantity by requesting that the total amount of a quantity in a subbody is related to the total in- or outflux of a suitable flux vector through the surface of the subbody itself; in other words, for example in the case of linear momentum, the formula may read as follows:

$$\int_M \rho \boldsymbol{a}\, d\mathscr{H}^{n-1} + \int_{\partial_* M} \boldsymbol{t}\, d\mathscr{H}^{n-1} = \int_M \rho \boldsymbol{b}\, d\mathscr{L}^n.$$

If we recall the link between Cauchy flux and Cauchy force, in our language this becomes a particular case of a law of the type

$$\int_M \rho \boldsymbol{a} \cdot \boldsymbol{v}\, d\mathscr{L}^n + \int_{\partial_* M} \boldsymbol{t} \cdot \boldsymbol{v}\, d\mathscr{H}^{n-1} = \int_M \rho \boldsymbol{b} \cdot \boldsymbol{v}\, d\mathscr{L}^n.$$

that is

$$P^{(a)}(M, \boldsymbol{v}) + P^{(C)}(M, \boldsymbol{v}) = P^{(e)}(M, \boldsymbol{v})$$

where the suffix 'a' stands for "acceleration" and 'e' stands for "external". Moreover, if $\rho \boldsymbol{b}$ is bounded on M and $\boldsymbol{v}$ has compact support in M, then

$$\left|P^{(a)}(M, \boldsymbol{u}) + P^{(C)}(M, \boldsymbol{v})\right| \leqslant \sup_M ||\rho \boldsymbol{b}|| \int_M ||\boldsymbol{v}||\, d\mathscr{L}^n$$

which we recognize as a particular case of weak balance. The term "weak" stands of course for the possibility for the measure ν to be not absolutely continuous with respect to the volume measure.

A very nice feature of weakly balanced forces is that they admit an unique decomposition into a contact and a distance part.

Teorema 2.16 (Bulk-contact decomposition). *For every weakly balanced force functional there exist two uniquely determined distance and contact force functionals such that*

$$P(M, \boldsymbol{v}) = P^{(b)}(M, \boldsymbol{v}) + P^{(c)}(M, \boldsymbol{v})$$

for almost every M and every $\boldsymbol{v} \in C^\infty(\operatorname{int} B; \mathbb{R}^n)$.

Proof. If $M \in \mathcal{M}_G$ and $\boldsymbol{v} \in C^\infty_{\mathrm{c}}(M; \mathbb{R}^n)$, we have

$$\left|\int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mu_0 + \int_M \mathsf{B} \cdot \nabla \boldsymbol{v}\, d\mu_1\right| = |P(M, \boldsymbol{v})| \leqslant \int_M |\boldsymbol{v}|\, d\nu\,.$$

Since $\mathcal{M}_G$ is an open cover of Ω, this is sufficient (as in the proof of the representation theorem) in order to have for every $\boldsymbol{v} \in C^\infty_{\mathrm{c}}(M; \mathbb{R}^n)$

$$\int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mu_0 + \int_M \mathsf{B} \cdot \nabla \boldsymbol{v}\, d\mu_1 = \int_M \boldsymbol{b} \cdot \boldsymbol{v}\, d\nu$$

where $\boldsymbol{b}$ is uniquely determined ν-almost everywhere. We put then

$$P^{(b)}(M, \boldsymbol{v}) = \int_M \boldsymbol{b} \cdot \boldsymbol{v}\, d\nu$$

(i.e. not only for velocity fields with compact support in M) and

$$P^{(c)}(M, \boldsymbol{v}) = P(M, \boldsymbol{v}) - P^{(b)}(M, \boldsymbol{v}).$$

It is now apparent that if $\operatorname{supt} \boldsymbol{v} \subseteq M$, then $P^{(c)}(M, \boldsymbol{v}) = 0$. Uniqueness follows by construction. ■

2.9. Balance laws and Cauchy Stress Theorem

Now we are in position to apply our results to the equilibrium of a (first gradient) continuous body. We suppose that *three* types of forces are present on the body: an internal force, whose density we will suppose absolutely continuous with respect to volume measure, a Cauchy force and a distance force. Denote by $P(M, \boldsymbol{v})$ the total power expended by these forces on a virtual velocity field on the subbody M. Then, by the results shown above, we have

$$P(M, \boldsymbol{v}) = -\int_M \mathsf{T} : \mathsf{D}\, d\mathscr{L}^n + \int_{\partial_* M} \boldsymbol{t} \cdot \boldsymbol{v}\, d\mathscr{H}^{n-1} + \int_M \boldsymbol{a} \cdot \boldsymbol{v}\, d\mu_0.$$

Now we use the divergence theorem for tensor fields with divergence measure

$$\int_M \mathsf{T} : \mathsf{D}\, d\mathscr{L}^n = \int_{\partial_* M} \mathsf{T}\boldsymbol{n}^M \cdot \boldsymbol{v}\, d\mathscr{H}^{n-1} - \int_M \boldsymbol{v} \cdot \operatorname{div} \mathsf{T}$$

and therefore get

$$P(M, \boldsymbol{v}) = \int_M \boldsymbol{v} \cdot (\operatorname{div} \mathsf{T} + \boldsymbol{a}\mu_0) + \int_{\partial_* M} (\boldsymbol{t} - \mathsf{T}\boldsymbol{n}^M) \cdot \boldsymbol{v}\, d\mathscr{H}^{n-1}.$$

Now we apply the equilibrium axiom and require that $P(M, \boldsymbol{v}) = 0$ for almost every M and every $\boldsymbol{v} \in C^\infty(\operatorname{int} B)$. Since 0 is clearly a weakly balanced force, the distance-contact decomposition theorem applies and we have

$$\int_M \boldsymbol{v} \cdot (\operatorname{div} \mathsf{T} + \boldsymbol{a}\mu_0) = 0, \qquad \int_{\partial_* M} (\boldsymbol{t} - \mathsf{T}\boldsymbol{n}^M) \cdot \boldsymbol{v}\, d\mathscr{H}^{n-1} = 0 \tag{2.11}$$

for almost every subbody M and every $\boldsymbol{v} \in C^\infty(\operatorname{int} B)$. The first equation simply asserts that

$$\operatorname{div} \mathsf{T} + \boldsymbol{a}\mu_0 = 0 \qquad \text{(in the sense of measures)} \tag{2.12}$$

while the second, being a surface integral, by the arbitrariness of M and $\boldsymbol{v}$ we get almost everywhere

$$\boldsymbol{t} = \mathsf{T}\boldsymbol{n}^M$$

which is Cauchy's Stress Theorem.

Remark 2.11. We could have made the whole matter a bit simpler requiring equilibrium by weak balance; but the distance-contact decomposition theorem is nice, as it is nice the framework "à la d'Alembert" of the vanishing of the power. One can of course suitably redefine the forces acting on the body and simply require weak balance.

Remark 2.12. Our way to get the balance law is of course not just a complication of the classical way: in (2.12) there is a general measure, although to obtain the result on the boundary we needed something on the internal power.

Remark 2.13. In spite of its innocent form, eq. (2.12) is in weak form (its extended form is just (2.11)). From that we see that virtual velocity fields become just the test functions that we know from PDEs. But to get the equations we *didn't* first localize the integral form of the balance law (which needs regularity) and then weaken the classical equation,

as sometimes it happens to see. This derivation does not imply anything more on the (possible) solutions of the field equations than what we required.

3. Extensions of the notion of force

At this point we could be happy with our extensions to divergence measure stress fields and sets with finite perimeter. Indeed, this seems to be the most general theory when one wants to keep normals and surface integrals and allow concentrated stresses or fluxes. But in nature there are objects which have as *simplest* modelization a set which does *not* have finite perimeter, like a fractal. And they also raise physical problems. For example, which is the flux of oxygen flowing from the lungs to blood, or of air through the leafage of a tree?

For this reasons we want now to explore the possibility of extending our definitions to more irregular objects. Of course, the notion of outer normal will have to be replaced by something else, which is surprisingly still intiuitive. By "explore" we mean here that from now on we will not be very rigorous, nor that we will give all the definitions in great detail. The interested reader can refer to the quoted references.

3.1. Fractal domains

A not trivial thing about fractals is to define what a fractal is. The ones we are accustomed to see are not rectifiable sets, i.e. set without finite perimeter. But it isn't enough: the curve given in polar coordinates by $\rho = 1/\vartheta \quad (\vartheta \geqslant \pi/2)$ is C^∞ and bounded but has infinite perimeter, and none of us would call it a fractal. Conversely, our set I of sec. 1 is more likely to be called a fractal. Many fractals are self-similar: i.e. if we cut some special parts of them but no matter how small, we have that the part is similar to the whole set. A typical case of this is the well-known von Koch's curve, depicted in Figure 16. (But notice: also smooth curves similar to $\rho = 1/\vartheta$ may have some self-similar parts).

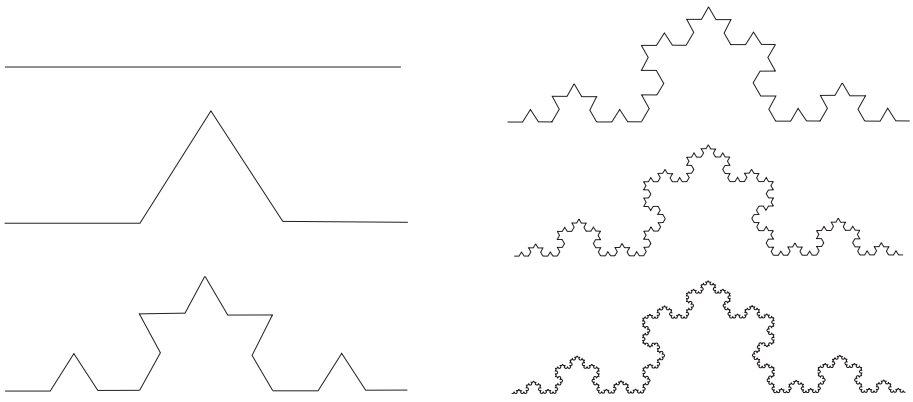

FIGURE 15
The construction of the von Koch's curve K.

(Notice that not all connected subsets of the von Koch's curve are similar to the whole set, although every connected subset of the curve contains a copy of the entire set)

If we glue together three von Koch curves, we find a closed curve whose interior is the von Koch's island, or "snowflake" S.

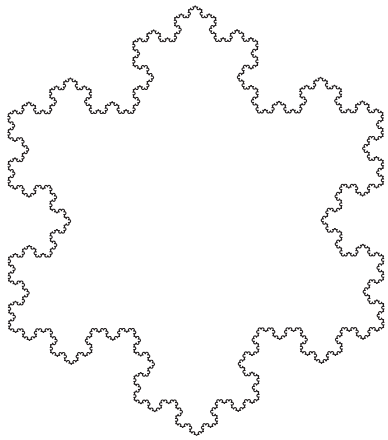

FIGURE 16
Von Koch's snowflake S.

Another celebrated fractal is the Cantor set, whose construction procedure is illustrated in Figure 16.

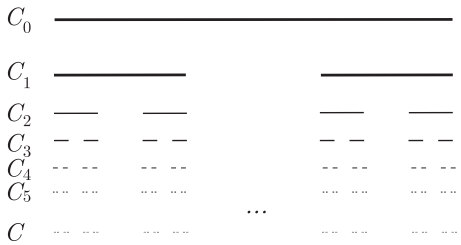

FIGURE 17
The construction of the Cantor set.

Anyway, many fractals are not self-similar, such as the Costiera and many other sets that are encountered in nature.

FIGURE 18
What's this? (Solution at the end of the References)

They often happen to be *statistically self-similar*: although we don't give a precise definition of this, we illustrate this with an example. We take a random variable (possibly

multidimensional) and start as when we want to construct the von Koch's curve. But this time (some of) the values that we use for the next step are no longer fixed, but distributed as the chosen random variable. This implies that at the step K_1 there is no more an equilateral triangle, and so on. The resulting of an iteration of the process may be as shown by fig. 19.

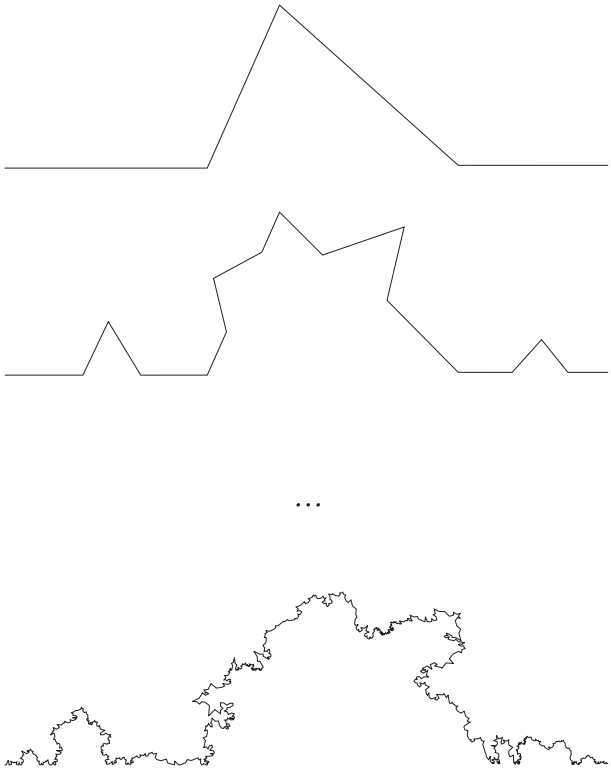

FIGURE 19
The construction of a statistically self-similar fractal.

It is well-known that the Hausdorff dimension of fractal sets may not be an integer, and commonly this is not the case. Another definitions of dimension have been proposed for fractals, which may differ from Hausdorff's, and it is an interesting point in Mechanics whether these numbers are related to physical properties of surfaces or lines that the fractals are going to model.

Fractals may be very nasty, too. For example, Harrison [10] constructed a self-similar Jordan curve in $\mathbb{R}^3$ that has a Hausdorff dimension strictly greater than two, and that is not contained in any surface of finite area. Let's see how it works: one starts with a square in $\mathbb{R}^3$, takes away the middle third of every side and joins some extra polygonal sides as illustrated in Figure 17. Then it suffices to repeat the procedure.

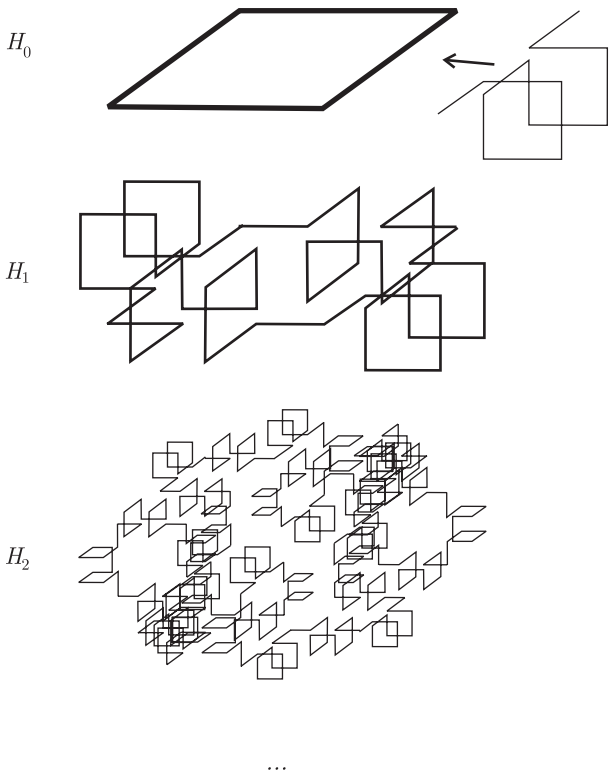

FIGURE 20
The construction of the Harrison curve H.

I let you figure out what kind of "curve" comes out. Indeed, a self-similar fractal has a dimension which is equal to $\log M/\log N$, where M is the number of the equally long pieces substituted in place of the old pieces (for example, $M = 4$ and $N = 3$ for the von Koch's curve). In this case the pieces are not equally long since the curve must not overlap itself, so the dimension of the set H will not exactly given by this formula. In our case $M = 11$ and $N = 3$ and since $\log 11/\log 3 \approx 2.18 > 2$, the curve is indeed as nasty as claimed.

We won't go very deep into the details of fractals ([9]), since we are more interested with mechanical objects. Moreover, the concept that we will introduce contain some of our fractals, but also other irregular objects.

3.2. Force on a fractal object

Since many fractals do not have locally finite perimeter (and they have parts with infinite perimeter bounded by arbitrarily small disks), there may be problems in defining quantities with surface integrals. For example, even in the case of a constant quantity, its surface integral over arbitrarily small subsets will be infinite.

[9] Also because they are many many many many many...

But it is still possible to hope: let's see an interesting example (see also [21]) concerning the flux across von Koch's snowflake.

We recall that there exist approximations S_h of the snowflake which are piecewise constant (they are exactly those illustrated by the procedure in fig. 15), so that they admit a parametrization $\boldsymbol{x}_h : [0,1] \to \mathbb{R}^2$ with $\boldsymbol{x}(0) = O$, where O is a fixed point on S (not arbitrary, it must belong also to every S_h). These parametrizations converge uniformly to a continuous function $\boldsymbol{x} : [0,1] \to \mathbb{R}^2$ which is a parametrization of S. It is not difficult to see that $|\dot{\boldsymbol{x}}_h| = (4/3)^h$, so that $\boldsymbol{x}$ is continuous but nowhere differentiable and hence the normal cannot be defined. Indeed the outer normal exist on every affine piece of S_h, and we can think of it as $\boldsymbol{n}_h = \dot{\boldsymbol{x}}_h^\perp(t)/\left|\dot{\boldsymbol{x}}_h^\perp(t)\right|$, where $\boldsymbol{u}^\perp$ stands for the $\pi/2$-clockwise rotation of the vector $\boldsymbol{u}$.

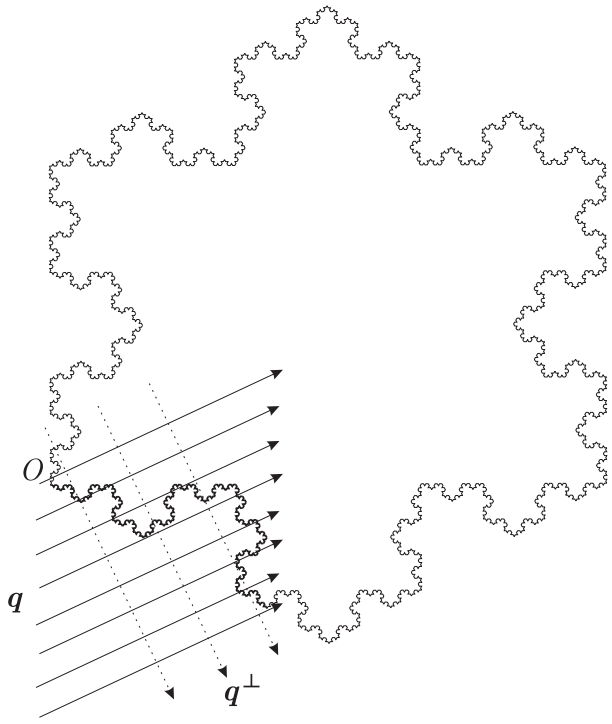

FIGURE 21
Flux across von Koch's snowflake.

Let now a constant flux vector $\boldsymbol{q}$ be given on the plane and take an "arc" a on S as depicted in Figure 21 which is the image of a certain $[0,T]$. Then, since $\boldsymbol{q} \cdot \boldsymbol{x}^\perp = \boldsymbol{q}^\perp \cdot \boldsymbol{x}$, the flux through an approximation a_h of the arc is

$$\Phi_h(T) = \int_{a_h} \boldsymbol{q} \cdot \boldsymbol{n}_h(t)\, d\ell = \int_0^T \boldsymbol{q}^\perp \cdot \dot{\boldsymbol{x}}_h(t)\, dt = \boldsymbol{q}^\perp \cdot \boldsymbol{x}_h(T)$$

since $\boldsymbol{x}_h(0) = O$. Therefore $\Phi_h(T)$ tends as $h \to +\infty$ to

$$\Phi(T) = \boldsymbol{q}^\perp \cdot \boldsymbol{x}(T)$$

which is natural to interpret as the total flux of the heat flowing through the arc a. So the flux is defined also when the length of the curve is infinite.

The reason for this is that, due to the fact that the normals on S_h change very fast from one affine piece to its neighbors, the most part of the heat which enters S_h leaves it almost immediately.

Indeed, if it is true that $\Phi_h(T) \to \Phi(T) \in \mathbb{R}$, it is also apparent that

$$\int_{a_h} |\boldsymbol{q} \cdot \boldsymbol{n}_h| \, d\ell \to +\infty$$

since the integrand is positive and the length of a_h diverges. Therefore, the total variation of $\boldsymbol{q}$ is infinite.

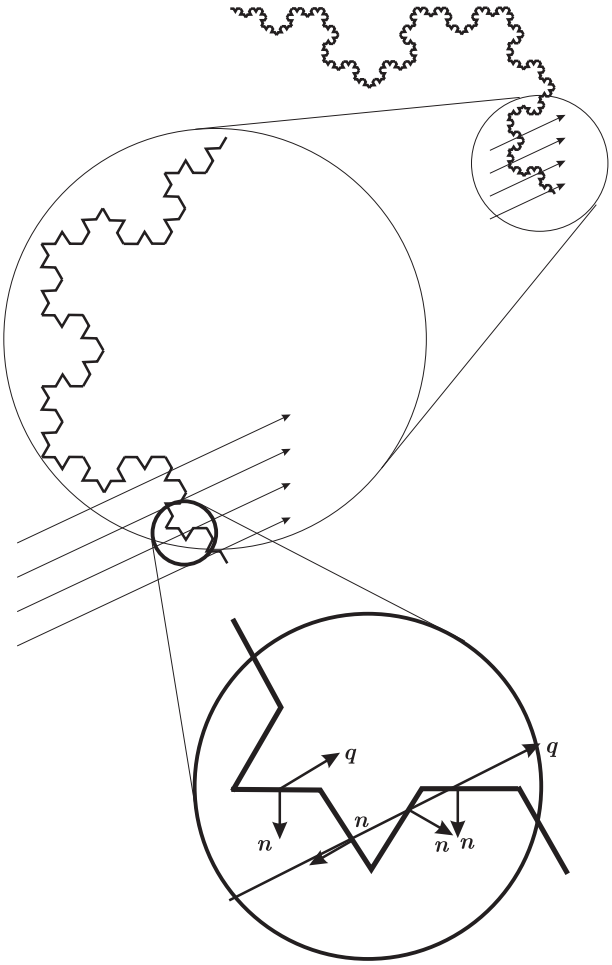

FIGURE 22
Details of the flux across von Koch's snowflake.

The case of forces (scalar, in this example) is just obtained substituting $\boldsymbol{q}$ with $v\boldsymbol{q}$, where v is a C^∞ function.

We notice here that the fact that S can be approximated by a polygon has helped us a lot in doing our calculations, and it seems deeply related to the subject. Indeed, this turns out to be a crucial point when one wants to extend the theory of forces (or fluxes) to more complicated objects. For this reason we make now a (as short as possible) digression in polyhedral approximations.

3.3. Polyhedral chains

Definizione 3.1. *A nonempty bounded subset of the euclidean space which is an intersection of a finite number of half spaces is called a* cell. *The* plane *of the cell is the smallest affine subspace containing the cell and the* dimension *of the cell is the dimension of this plane.*

Polyhedra are cells of dimension 3, polygons are cells of dimension 2, segments are cells of dimension 1, while spheres, circles and curves are not cells.

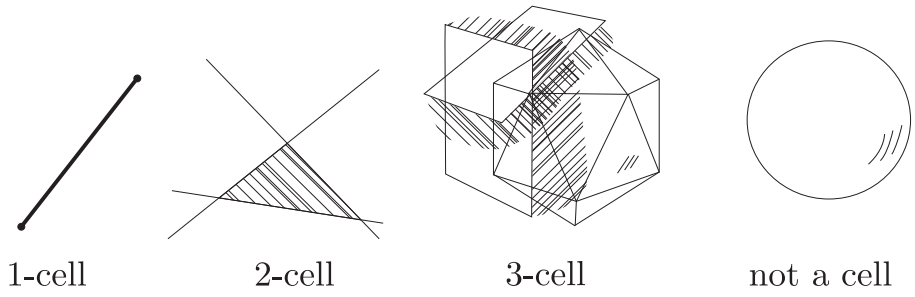

FIGURE 23
Cells.

Since planes (of dimension $r \geqslant 1$) are orientable in exactly two ways, every cell inherits two possible orientations, and become an *oriented* r-cell. The two orientations of the same cell are indicated by σ and $-\sigma$. It is possible also to orient 0-cells in the familiar way (see [12], p. 5).

Definizione 3.2. *A* polyhedral r-chain *is an element of the vector space spanned by formal linear combinations of r-cells (therefore associative etc.) with coefficients in $\mathbb{Z}$ with the properties that*

(1) $1\sigma = \sigma$;
(2) $(-1)\sigma = -\sigma$;
(3) *If an oriented cell σ is subdivided into a sum of smaller cells $\sigma_1, \ldots, \sigma_m$, then the two chains σ and $\sigma_1 + \ldots + \sigma_m$ coincide.*

By this definition one sees that polyhedral chains are in fact classes of equivalence, but we don't want to go into these details. Notice the difference with set union: 2σ is not $\sigma \cup \sigma$. It's like soccer or Pokémon carda: you may have "doubles"

Exactly as for simplexes, there is a notion of boundary for an r-cell. We don't write a formal definition, but say just this: the boundary of an r-cell σ is a sum of $(r-1)$-cells σ_i with the orientation given in a way that $r-1$ vectors in the plane of σ_i plus a transversal vector pointing towards the exterior of σ are oriented as σ.

For a polyhedral chain, the boundary is defined by linearity, that is, if $A = \sum a_i \sigma_i$, then $\partial A = \sum a_i \partial \sigma_i$, and this is a linear operator from the space $\mathcal{P}_r$ of all polyhedral r-chains to the space $\mathcal{P}_{r-1}$ of all polyhedral $(r-1)$-chains. All usual properties of the boundary, like $\partial\partial = 0$ of course hold.

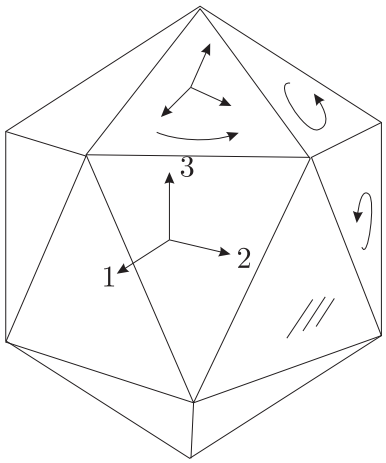

FIGURE 24
Boundary of an r-cell.

Polyhedral chains are too simple (if they look already complicated, then you can start to worry about). They are very good for integration since they are not affected by subdivision, but we would of course have also curved domains like spheres, surfaces and curves, so that we want to pass to the limit. But how? That's the point.

3.4. Flat chains

If we want to define solids starting with polyhedra, a way to do this is consider sequences of polyhedra of increasing volume (better if nested one into the next one) such that the volume of the difference goes to zero. This is just the completion of the space of all polyhedra with respect to a certain norm.

The clever intuition of Whitney was that to introduce a new norm, a very strange one, which is smaller than the Lebesgue norm. Thus, sequences converging in the Lebesgue norm will converge also in the new one, but there may be converging sequences which are not Lebesgue convergent.

The *mass* of a polyhedral r-chain given by $A = \sum a_i \sigma_i$ is defined by

$$|A| = \sum |a_i| \, \mathscr{H}^r(\sigma_i).$$

Here there is nothing new: if the cells are disjoint and counted one time, the mass is simply the r-dimensional volume.

Definizione 3.3. *Let A be a polyhedral r-chain in $\mathbb{E}^n$. We define*

$$|A|^\flat = \inf_{C \in \mathcal{P}_{r+1}} \{|A - \partial C| + |C|\}$$

and call it the flat norm *of A.*

(The minus sign is *not* set difference, but just the reversed orientation!)

Notice first of all the $r + 1$ under the inf. This means that in order to compute the flat norm, also the mass of polyhedral chains of one dimension more are important. In

a certain sense, the minimizing sequences of chains which converge to the norm must optimally embed the given polyhedral chain.

An alternative expression is the following:

$$|A|^\flat = \inf\{|B| + |C| : A = B + \partial C, B \in \mathcal{P}_r, C \in \mathcal{P}_{r+1}\}.$$

The fact that $|\cdot|^\flat$ is a norm is not trivial, nor it is trivial how to compute it. One thing is clear: taking $C = 0$, one finds $|A|^\flat \leqslant |A|$.

In order to understand how complicated the calculation of the flat norm can be, let's see the case when A is given by two perpendicular segments of length a and b as shown in Figure 25.

It may well happen that the infimum in the flat norm is not achieved on a polyhedral chain. For example, if the infimum is attained at a curve which is the graph of a function u (say, defined on $[0, c]$), then one must count the shaded area and the length of the arc (since it must disappear in the decomposition), and finally the two remaining parts of the segments (all of which is indicated with B).

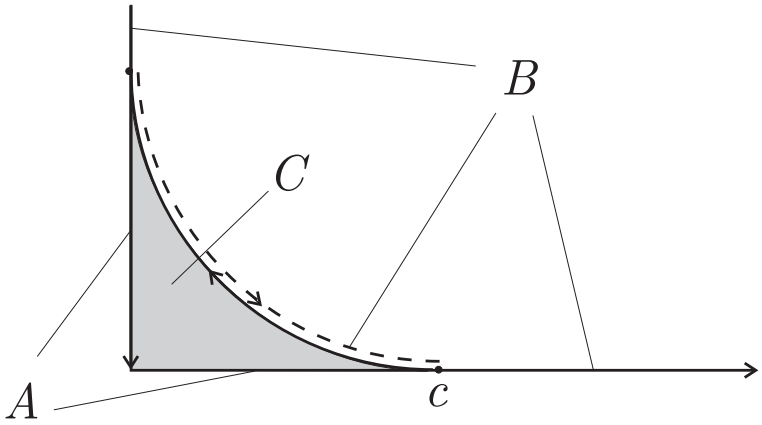

FIGURE 25
Calculation of the flat norm.

If one seeks an infimum for the flat norm, then he can look for the infimum of the following functional

$$\mathscr{F}[u, c] = \int_0^c (\sqrt{1 + u'^2} + u)\, dx + a - u(0) + b - c = \inf.$$

with the condition $u(c) = 0$. This variational problem involves also c and the infimum may depend on a and b. In some simple cases (e.g. $a = b$) this can be solved and furnishes a better estimate of the flat norm of A than the sum of the masses of the two segments. But it is still unknown if this is really the infimum, since we don't know that the curve must be a graph.

It's also known that the flat norm of a k-cell is the mass of the cell. That is, what makes the flat norm different from mass is making chains.

Another interesting property of the flat norm is that it is more sensitive to translation than mass. For, if we take a polyhedral chain made by two disjont oppositely oriented segments both with mass equal to one, the mass of the chain will be 2. If we translate a segment onto the other, the mass of the chain remains equal to 2 and suddenly becomes zero when the two supports of the segments coincide. It isn't so for the flat norm, since

it is certainly smaller than the area of the parallelogram enveloping the segments plus the two segments, which goes to zero as they overlap. Thus, the flat norm is more sensitive to translation than mass (in this case, the volume). One reason for that is apparent: the flat norm takes into account the orientation of the segments.

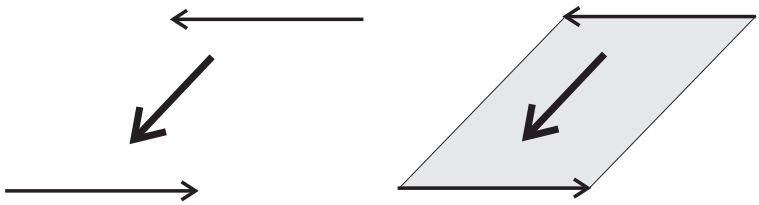

FIGURE 26
Continuity of the flat norm.

Let's examine also the important case of two oppositely oriented segments forming two opposite edges of a square. Let's call d the length of the edge and D the couple of the segments (i.e. the polyhedral chain). Finally, suppose, for reasons that will be clear, that d is very small. In this case, we know that the area of a closed chain is less than its perimeter. Suppose then that, in the decomposition $A = B + \partial C$, the part B contains a complete boundary of a chain. In this case it will be more convenient to put this boundary into the boundary of C, so that we may think that B does not contain closed chains. Moreover, B cannot be empty since A is not a cycle (i.e. a boundary). If C is empty, then B must have length $2d$. If C is not empty, then only two cases are possible: the first is when C joins the two segments, the second when it does not. But in the first case some sides of C going from one segment to the other must be compensated by B, otherwise A would be a boundary, and these have length at least d (probably $2d$). In the second case, there must be parts of the boundary of C which must be compensated in B and that would cause $|B|$ be greater than $2d$. Therefore the flat norm of such chain (called, as we shall see, a *dipole*), is proportional to d when $d \to 0$.

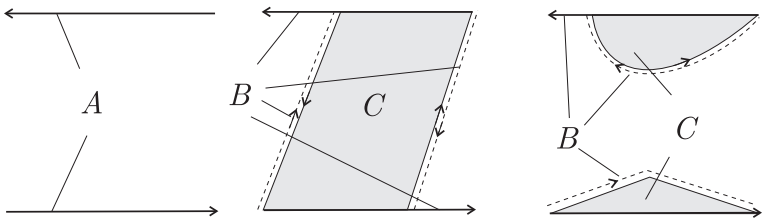

FIGURE 27
Flat norm of a dipole.

Are our fractals particular cases of flat chains? Many of them are. Let's see, for example, that von Koch's curve is one of them. To do this, we see the approximations K_h of the curve as a particular sum of 1-cells with the same orientation such that the middle parts of the segments vanish. It's definitely better to see it with a figure:

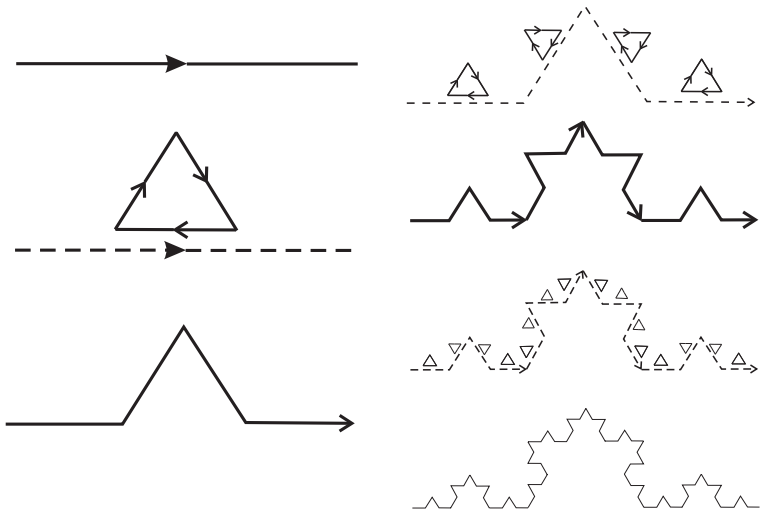

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FIGURE 28
Approximation of the von Koch's curve with polyhedral chains.

There are 4^h 1-cells σ_{ih} at step h, each of them having side length 3^{-h} and area $3^{-2h+1/2}/4$. Moreover $S_{1h} + \ldots + S_{kh}$ is the boundary of $\sigma_{1h} + \ldots + \sigma_{kh}$. Hence it follows that

$$|S_{1h} + \ldots + S_{kh}|^\flat \leqslant |\sigma_{1h}| + \ldots + |\sigma_{kh}| = \frac{\sqrt{3}}{4}\left(\frac{4}{9}\right)^k$$

which tends to zero as $k, h \to +\infty$, so that the sequence (S_k) is a Cauchy sequence with respect to the flat norm.

It is clear that also the statistically self-similar von Koch's curve has this property.

The Harrison curve, however, is *not* a flat chain. For this, we will need something still more accurate.

3.5. The troubles of integration

Integration on polyhedral chains is a simple matter, since they are just sums of r-cells. If A is a polyhedral r-chain, and ω a, say, C^1 k-form, then

$$\int_A \omega = \sum_i \int_{\sigma_i} \omega$$

and it is clear what the integrals on a r-cell is. Furthermore, the sum is finite so that there's no problem at all.

The question becomes more delicate if one wants to integrate on flat chains, since they may be viewed as infinite sums of cells. But first let's ask: what we want to integrate? It is clear that whatever a flat chain may help, no reasonable definition of outer normal will appear. Then the idea is the following: in $\mathbb{R}^n$, the normal to a $(n-1)$-surface can also be

given with a $(n-1)$-differential form. Then, if Stokes' theorem in the form

$$\int_{\partial A} \omega = \int_A d\omega$$

hold, it would be possible to generalize Stokes' formula (curl theorem) and divergence theorem also to domains without smooth boundary.

Let's see how it works in a very intuitive manner, not rigorous. In $\mathbb{R}^3$, let $\boldsymbol{v}$ be the vector field whose flux across a surface S we want to compute. Let ω the form canonically associated to $\boldsymbol{v}$ (that is, with the same components) and let $\star\omega$ be given in coordinates by

$$\star\omega = v_1\, dx_2 \wedge dx_3 - v_2\, dx_1 \wedge dx_3 + v_3\, dx_1 \wedge dx_2.$$

Now, if we admit for the differentiation operator d the usual properties known from differential geometry

(1) $d(\alpha+\omega) = d\alpha + d\omega$;
(2) $\omega \wedge \alpha = -\alpha \wedge \omega$, whatever α and ω may be;
(3) If f is a function of the coordinates x_i then $df = \sum_i \frac{\partial f}{\partial x_i} dx_i$;
(4) $d(\alpha \wedge \omega) = d\alpha \wedge \omega + \alpha \wedge d\omega$;
(5) $d(d\omega) = 0$

then

$$d\star\omega = \frac{\partial v_1}{\partial x_1}\, dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial v_2}{\partial x_2}\, dx_1 \wedge dx_2 \wedge dx_3 + \frac{\partial v_3}{\partial x_3}\, dx_1 \wedge dx_2 \wedge dx_3$$

and Stokes' formula holds

$$\int_{\partial A} v_3\, dx_1 \wedge dx_2 + v_1\, dx_2 \wedge dx_3 + v_3\, dx_1 \wedge dx_2 = \int_A \operatorname{div} \boldsymbol{v}\, dx_1 \wedge dx_2 \wedge dx_3.$$

If A is a regular surface, then $dx_1 \wedge dx_2 \wedge dx_3$ corresponds to volume measure and therefore the left-hand side represents the flux through ∂A, with integrand $\boldsymbol{v} \cdot \boldsymbol{n}\, d\mathscr{H}^2$. But, as said before, if $\boldsymbol{n}$ fails to exist, it would be possible to integrate the associated flux form. The $\star$ operator is called the *Hodge* star operator.

The next question is: what kind of forms do we want to integrate? It is reasonable to expect a kind of balancing in the regularity of the forms that we integrate and the irregularity of the domains on which we integrate. Thus, if we want the integral to be defined with the expected properties on flat chains, we need to impose more regularity on the integrands.

It turns out that the good forms to be integrated on a flat chain are the so-called *flat forms* (what else?). They are bounded measurable differential forms characterized by the fact that there exists a positive constant C such that for all k-cells σ and for all $(k+1)$-cells τ the inequalities

$$\left|\int_\sigma \omega\right| \leqslant C\, M(\sigma), \qquad \left|\int_{\partial\tau} \omega\right| \leqslant C\, M(\tau) \tag{3.1}$$

hold.

But at this point a first trouble appears. Let's see a simple example.

Example 3.1. The form $\omega_0 = H(x+y)(dx+dy)$, where H is the Heaviside function, is flat. This can be readily seen as follows: first, since ω_0 is bounded,

$$\left|\int_\sigma \omega\right| \leqslant \sup|\omega|\, M(\sigma);$$

Second, by Stokes' theorem

$$\int_{\partial\tau}\omega_0=\int_\tau d\omega=\int_\tau\left(\frac{\partial H}{\partial x}-\frac{\partial H}{\partial y}\right)\,dx\wedge dy=\int_\tau(\delta_0-\delta_0)\,dx\wedge dy=0$$

and so ω_0 is flat. Let now ω_ε a perturbation of ω obtained with a "rotation" by an angle ε of ω_0, i.e. $\omega_\varepsilon=H(x+y)\mathsf{R}_\varepsilon(dx+dy)$ ([10]). ω_ε is still bounded, so the first of the inequalities (3.1) holds, but, after a very simple calculation

$$d\omega_\varepsilon=2\sin\varepsilon\delta_0$$

and so the integral $\int_\tau d\omega$ is singular and cannot be bounded by the area of τ.

Namely, flat forms are not closed with respect to small perturbations like those "rotations". This is not very nice, since mathematical problems are often modelizations and simplifications of the so-called physical world, and therefore it is important to be able to say that if "reality doesn't differ very much from the model, then also results will not differ that much".

But the greatest trouble with the flat norm is another, related to the calculation of fluxes, which is the most important thing for us. The flux over a surface is indeed an integration over the surface, but, in the context of integration of forms, the object on which a flux is performed is more complicated than a surface, since it must contain information on the "normal direction". This object is not well defined in the context of flat norms, since it may happen that it is not a flat chain when the original domain really is.

3.6. Chainlets

Chainlets ([11]) are more general objects that flat chains, but their definition is in principle analogous: we define an appropriate norm on the space of polyhedral chains and complete the space with respect to this norm. What really makes the difference is the right choice of the norm. Let's begin with a definition.

Definizione 3.4. *Let σ be a cell in $\mathbb{R}^n$ and let $\boldsymbol{v}$ be a vector in $\mathbb{R}^n$. A k-dimensional 1-*dipole *(or 1-difference k-cell) is defined by*

$$\sigma^1=T_{\boldsymbol{v}}\sigma-\sigma$$

where

$$T_{\boldsymbol{v}}\sigma=\{x\in\mathbb{R}^n:x=y+\boldsymbol{v},y\in\sigma\}$$

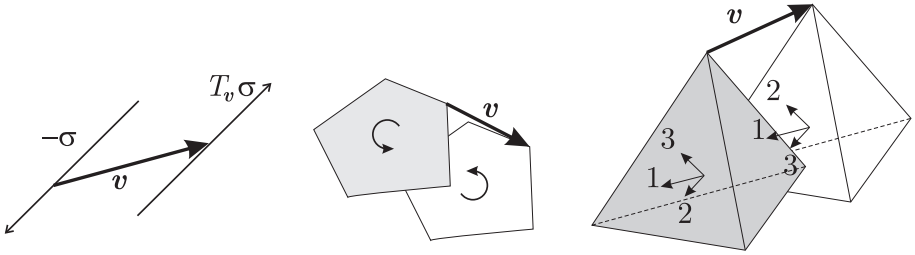
is the translation of σ by the vector $\boldsymbol{v}$.

A 1-dipole is just the sum of oppositely oriented k-cells, no matter how far or near they are.

---

[10] We don't call this a rotation of ω, since rotations are defined in a different way using pushforwards.

[11] These objects have been first introduced by Harrison in 1998 ([11]) and extended and generalized in subsequent papers. In this section, we borrow most of the ideas from [12], the most up-to-date paper on the subject, to which the reader is referred for proofs or details.



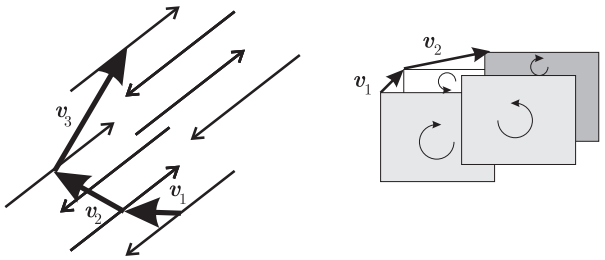
1-dim. 1-dipole 2-dim. 1-dipole 3-dim 1-dipole

FIGURE 29
Dipoles.

This induces a "difference operator" $\Delta_{\boldsymbol{v}} = T_{\boldsymbol{v}} - \mathrm{Id}$. Then, a j-dipole is defined recursively as the Δ of a $(j-1)$-dipole.

Definizione 3.5. *Let $(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j) \in \mathbb{R}^n \times \ldots \times \mathbb{R}^n$. Let a k-dimensional 0-dipole be a k-cell in $\mathbb{R}^n$ and let a k-dimensional j-dipole defined by*

$$\sigma^j = \Delta_{\boldsymbol{v}_j} \sigma^{j-1}.$$



1-dimensional 3-dipole 2-dimensional 2-dipole

FIGURE 30
j-dipoles.

The vectors of the translation contribute to the definition of the "mass" of a j-dipole, since the *j-dipole mass* is defined as

$$||\sigma^j||_j = M(\sigma)||\boldsymbol{v}_1|| \cdot \cdots \cdot ||\boldsymbol{v}_j||.$$

The space of all k-dimensional j-dipoles is denoted by $\mathcal{D}_k^j$. The *k-natural norm* of a polyhedral k-chain is defined as follows.

Definizione 3.6. *Let $P \in \mathcal{P}_k$ a polyhedral k-chain. The* r-natural norm $(r \geqslant 1)$ *of P_k is given by*

$$||P||^{\natural_r} = \inf \left\{ \sum_{j=0}^{r} ||D^j||_j + ||C||^{\natural_{r-1}} : P = \sum_{j=0}^{r} D^j + \partial C, D^j \in \mathcal{B}_k^j, C \in \mathcal{P}_{k+1} \right\},$$

provided that $||P||^{\natural_0} = M(P)$.

It seems quite frightening, so let's see what it means in the simplest case, i.e. $r = 1$. Moreover, let's rewrite it for a 1-cell, i.e. $k = 1$. (Let's also write $D^0 = B$ in agreement with the definition of flat norm).

$$||P||^{\natural_1} = \inf \left\{ M(B) + ||D^1||_1 + M(C) : P = B + D^1 + \partial C, D^1 \in \mathcal{D}_1^1, C \in \mathcal{P}_2 \right\}.$$

Thus, in this case, with respect to the flat norm, not only the complete boundaries and the polyhedral parts are to be taken into account, but also all dipoles that may appear in the decomposition. They can be very many, if the polyhedral chain has many elementary cells translated. Recall also that if a polyhedral chain is subdivided, it remains the same by our property 3.2-(3)

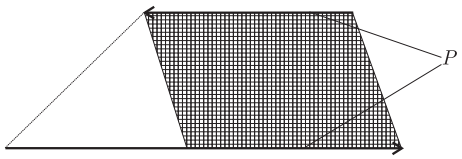


FIGURE 31
Calculation of the 1-natural norm.

The 1-natural norm is equivalent to Whitney's *sharp norm*, which we won't define here.

Let's see an interesting example of a sequence of chains which converges in the 1-norm but not in the flat norm, due to Rodnay and Segev [17]. It's been depicted in Figure 32 below. The sets R_k obviously do not converge in the mass norm, since it is always equal to one. The differences $S_k = R_{k+1} - R_k$ have a flat norm which go to zero as 2^{-k}, but since there are 2^k of them, the sequence (R_k) does not converge in the flat norm. But those differences have a 1-norm which is smaller than the area of the dipoles, which is 2^{-2k} each, and thus

$$||R_{k+1} - R_k||^\natural_1 \leqslant 2^{-k} \to 0.$$

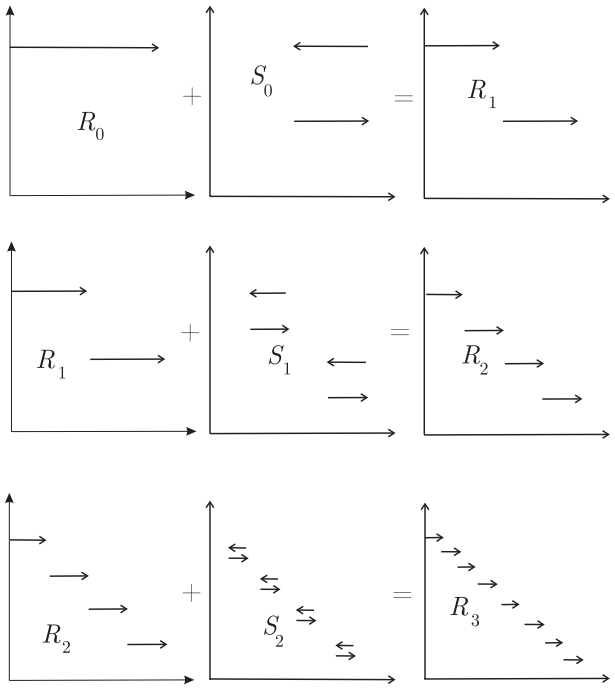


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FIGURE 32
The "staircase strainer" of Rodnay and Segev.

Also in this case, like it happened for the flat norm, it is not trivial to show that the natural norms are in fact norms.

Another interesting example of this convergence is the graph of an L^1-function.

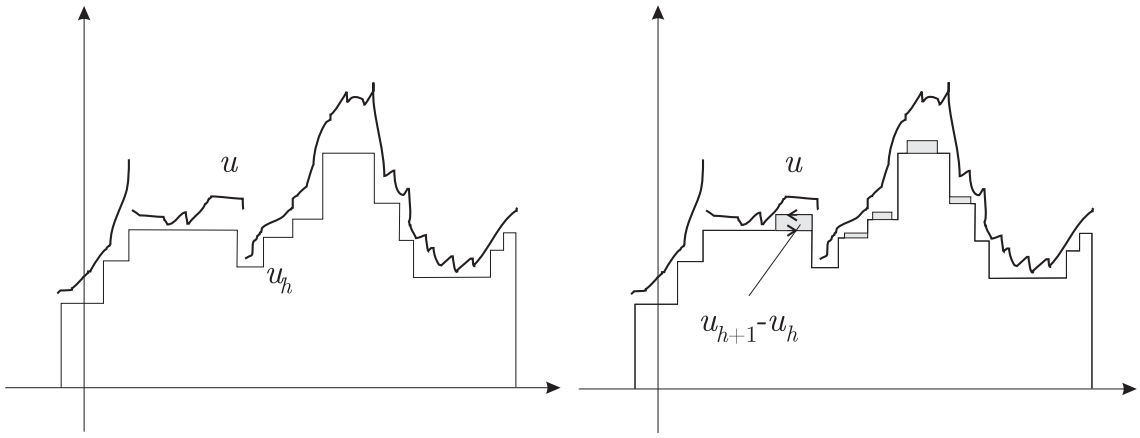


FIGURE 33
Approximation of the graph of an L^1-function.

If $(u_k)_{k\in\mathbb{N}}$ is a sequence of step functions converging to an L^1 function $u : \mathbb{R} \to \mathbb{R}$, then consider the graph of u_k as a polyhedral chain (for simplicity we still denote it by u_k, with the orientation induced by that of the real axis. Then $u_{k+1} - u_k$ as a polyhedral chain can be seen as a sum of dipoles, whose 1-norm converges. It's worth while remembering that the graph of an L^1 function can be *very* irregular (as for example the "Devil's Staircase" ([12])).

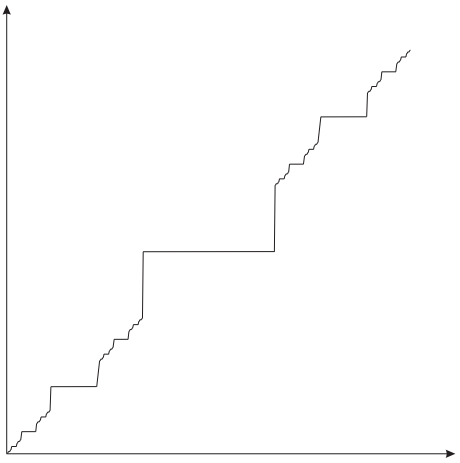


FIGURE 34
The "Devil's staircase".

Let's see a last example, but of great conceptual importance.

[12] Known also as *Vitali's function*. This is constructed as follows. It is known that the Cantor set C is the set of all points in $[0, 1]$ which admit a decimal expansion in base 3 containing no 1's. Then the staircase D is defined as follows: if $x \in C$, then $D(x)$ is given by the replacement of all 2's with 1's in the decimal expansion of x, then interpreted as an expansion in base 2; if $x \notin C$, then $D(x)$ is constant on all the connected component containing x which has been put away in the construction of C. It is a continuous monotone function with derivative a.e. equal to zero.

Let $x \in \mathbb{R}$ and let Q_k an oriented 1-cell centered at x and of length 2^{-k}, $k \in \mathbb{N}$. Let then $D_k = 2^k Q_k$. We claim that the sequence D_k converges to "something" in the 1-norm (notice that the mass of D_k is always one). In fact, when we compute $D_{k+1} - D_k$, we can subdivide any of the 2^k cells of D_k into four 1-cells of length $2^{-(k+2)}$ and the 2^{k+1} cells of D_{k+1} into 1-cells of length $2^{-(k+2)}$. We notice that exactly 2^k of these cells cancel, and what remains are 2^k dipoles of length $2^{-(k+2)}$ and translation vector length $2^{-(k+2)}$. This is enough to be able to say that

$$||D_{k+1} - D_k||_1 \leqslant 2^k \cdot 2^{-(k+2)} \cdot 2^{-(k+2)} = 2^{-k-4} \to 0.$$

Therefore the sequence (D_k) is convergent and its limit is a 1-chainlet. This is called *Dirac* δ.

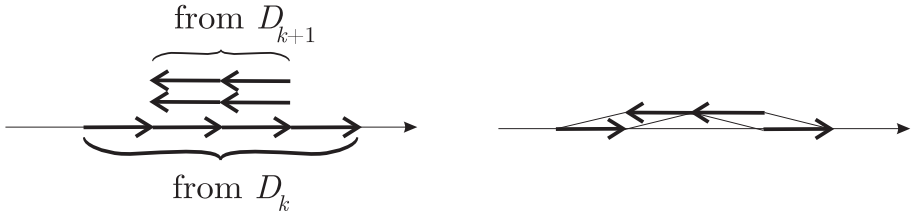


FIGURE 35
The approximating sequence of the δ chainlet.

Suppose now to be able to integrate a 1-form like $\varphi\, dx$ with continuous φ on the limit D. Then, any reasonable definition of integral should give, using the mean value theorem,

$$\int_D \varphi\, dx = \lim_{k\to+\infty} \int_{D_k} \varphi\, dx = \lim_{k\to+\infty} \int_{-2^{-k-1}}^{2^{-k-1}} 2^k \varphi(x)\, dx = \varphi(0)$$

as expected.

Why don't we stop with the convergence in the 1-norm? It's complicated enough, but there's a big problem with it, namely, the boundary of such an object (a "sharp chain"), although it's defined, may not be a sharp chain. Therefore we must go further.

We are now in position to introduce chainlets.

Definizione 3.7. *The completion of the space of k-polyhedral cells with respect to the r-natural norm is the space $\mathcal{N}_r^k$ of* k-chainlets *of class $\mathcal{N}^r$.*

k-chainlets are more irregular than flat chains, since the norm on them is smaller.

At this point we can state some recent results of Harrison (see for instance [12]), which allow us to perform the customary operations on chainlets.

Teorema 3.8. *If J is a k-chainlet of class $\mathcal{N}^r (r \geqslant 0)$, then ∂J is a $(k-1)$-chainlet of class $\mathcal{N}^{r+1}$ and*

(1) $||\partial J||^{\natural_{r+1}} \leqslant ||J||^{\natural_r}$
(2) $||J - T_{\boldsymbol{v}} J||^{\natural_{r+1}} \leqslant ||\boldsymbol{v}||\,||J||^{\natural_r}$.

Before stating Stokes' Theorem, we just need the definition of degree of regularity of a k-form defined in an open set $U \subseteq \mathbb{R}^n$. We define

$$\begin{aligned}
||\omega||_0 &= \sup\left\{\frac{\int_\sigma \omega}{M(\sigma)} : \sigma \text{ is a } k\text{-cell with} \operatorname{supt}\omega \subseteq \sigma\right\},\\
||\omega||_{r+1} &= \sup\left\{\frac{||\omega - T_{\boldsymbol{v}}\omega||_r}{||\boldsymbol{v}||} : \operatorname{supt}(\omega - T_{\boldsymbol{v}}\omega) \subseteq U\right\},\\
||\omega||'_0 &= \sup\left\{\frac{\int_\tau \omega}{M(\tau)} : \sigma \text{ is a } (k+1)\text{-cell with} \operatorname{supt}\omega \subseteq \sigma\right\},\\
||\omega||'_{r+1} &= \sup\left\{\frac{||\omega - T_{\boldsymbol{v}}\omega||'_r}{||\boldsymbol{v}||} : \operatorname{supt}(\omega - T_{\boldsymbol{v}}\omega) \subseteq U\right\},
\end{aligned} \tag{3.2}$$

and finally

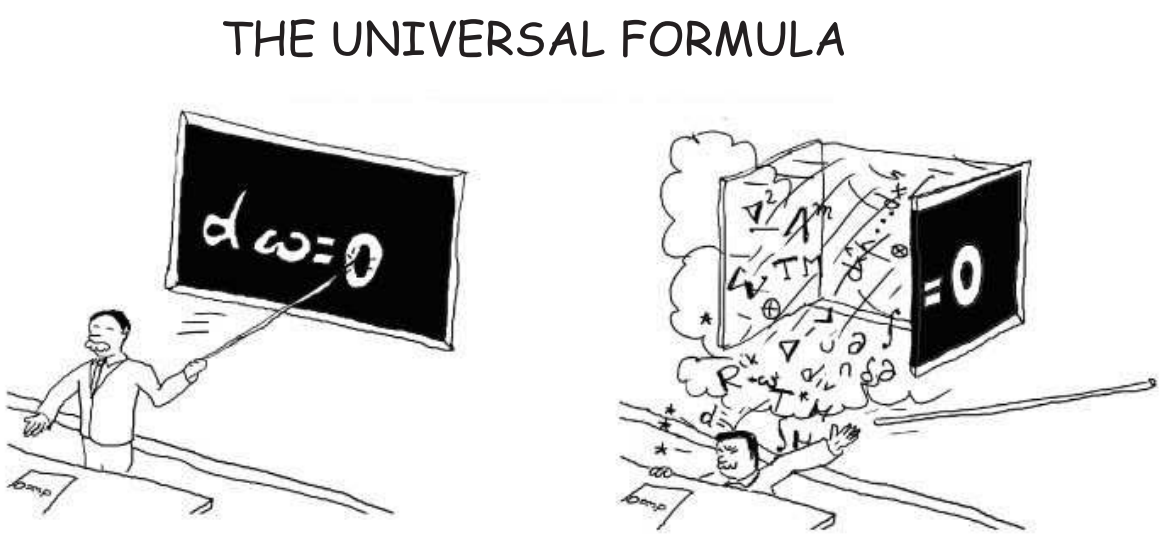
$$|\omega|_0 = ||\omega||_0, |\omega|_r = \sup\{||\omega||_0, \ldots, ||\omega||_r, ||\omega||'_0, \ldots, ||\omega||'_{r-1}\}.$$

If ω verifies $|\omega|_r < +\infty$, it is said to be *of class* $\mathcal{B}^r$.

Teorema 3.9 (Generalized Stokes' Theorem). *If J is a k-chainlet of class $\mathcal{N}^r(r \geqslant 0)$ supported in a certain open set U and if ω is a $k-1$-form of class B^{r+1} with values in U, then*

$$\int_{\partial J} \omega = \int_J d\omega.$$

This wonderful result hides, as many important formulas do, a great deal of information for the advantage of conciseness and formal simplicity. Nevertheless, it is a great result.



The last point we have to face with in this framework is the concept of flux. We already noticed that the support of a chainlet is not sufficient to assign it. Exactly as one can assign on a surface both a vector field and integrate it and a vector field and calculate its flux, here we must distinguish between simple integration on forms or fluxes (or circulations, or still something else, but let's restrict to the case of fluxes).

In the smooth case the flux can be defined assigning a normal vector but here we can't do it. Hence the idea is to introduce a chainlet $\perp J$ which accounts for the information on the normal necessary to compute the flux. J and $\perp J$ will have the same support, but will be different.

We won't give here an explicit definition of $\perp J$; a detailed introduction can be found in Harrison [12]. The idea, however, can be given: one must construct a polyhedral approximation to $\perp J$ using polyhedra "complementary" to J, as the following figure shows.

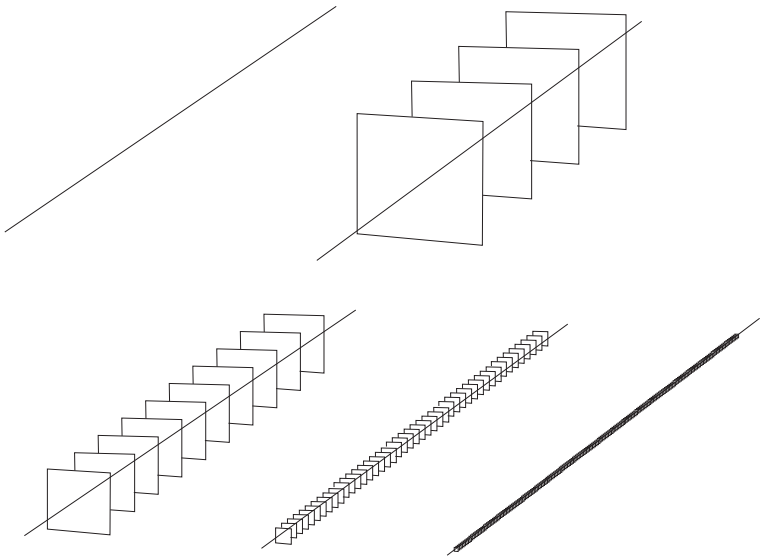


FIGURE 36
$\perp J$ when J is a 1-cell.

It's clear that the dipole norm will play an important rôle in this convergence.

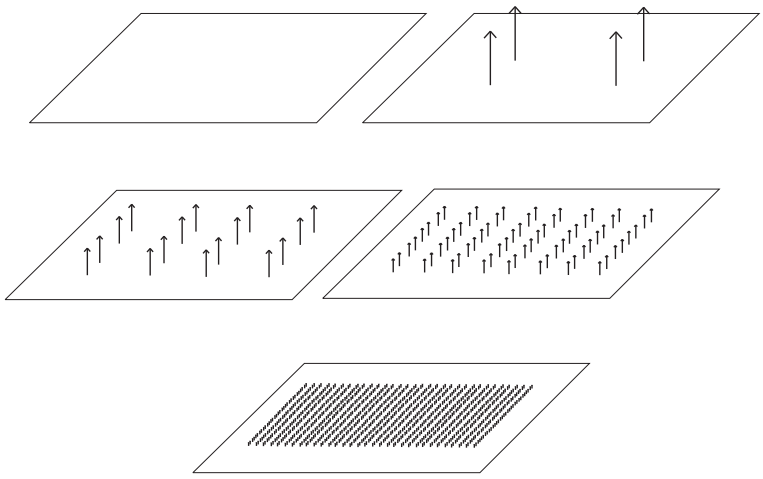


FIGURE 37
$\perp J$ when J is a 2-cell.

The main theorem in this subject is related to the Hodge star operator $\star$, which transforms a k-form into a $(n-k)$-form by an operation which is also related to "complementing". For further details, see Harrison [12].

Teorema 3.10 (**Hodge Star Theorem**). . *For all k-chainlets of class $\mathcal{N}^r$, $\perp J$ is a $(n-k)$-chainlet of class $\mathcal{N}^r$ and it holds*

$$\int_{\perp J} \omega = \int_J \star\omega$$

for all $(n-k)$-forms of class $\mathcal{B}^r$ defined in a neighborhood of $\operatorname{supt} J$.

We already saw that $d\star\omega$ is related to divergence. Let's see what the two last theorems imply.

Teorema 3.11 (**Generalized Divergence Theorem**). *Let J be a k-chainlet of class $\mathcal{N}^r$ and ω a differential $(n-k+1)$-form of class $\mathcal{B}^{r+1}$ defined in a neighborhood of* $\operatorname{supt} J$. *Then*

$$\int_J d\star\omega = \int_{\partial J} \star\omega = \int_{\perp\partial J} \omega.$$

Thus we interpret the integral on $\perp \partial J$ as the flux.

In our case, we will have $k = n$, since we want to integrate over a domain in $\mathbb{R}^n$. Then ω will be a 1-form and $\star\omega$ a $(n-1)$-form. Then ∂J will be a $(n-1)$-chainlet, and $\perp \partial J$ a 1-chainlet (the "field of normal directions"). In this case Figure 38 applies.

The power of the theorems and concepts introduced up to here allow also many more intermediate cases. For example, the well-known curl theorem

Teorema 3.12 (**Generalized Curl Theorem**). *Let J be a k-chainlet of class $\mathcal{N}^r$ and ω a differential $(k-1)$-form of class $\mathcal{B}^r$ defined in a neighborhood of* $\operatorname{supt} J$. *Then*

$$\int_{\perp J} \star d\omega = \int_{\partial J} \omega.$$

Here, we will have $k = 2$ and so ω will be a 1-form. Then the integral on the left is the flux of the curl of ω, while the integral on the right will be the circulation over the boundary of J. In this case Figure 37 applies.

4. Applications

At last, we come to our desired applications. This section will be not formal, but it will show one of the possible goals in using chainlet theory in Continuum Mechanics. Also, it is at present very short, but I think it will be important and interesting to expand it.

4.1. Forces on chainlets

The point to start with is to give a definition of force which makes use of chainlets. We will not describe for the moment a particular family of chainlets and we will denote with $\mathscr{J}$ a given collection of them, but with support contained in the set B which represents our body.

Let's also define forces on chainlets directly, not making use of formal definition like 2.5. It's an interesting question whether definition 2.5 extended to chainlets produces a representation theorem like 2.6.

Definizione 4.1. *A function* $\mathscr{J} \times C^\infty(\operatorname{int} B, \mathbb{R}^n)$ *given by*

$$P^{(b)}(J, \boldsymbol{v}) = \int_J (\boldsymbol{a} \cdot \boldsymbol{v})\, d\mathcal{L}^n + \int_J (\mathsf{B} : \nabla \boldsymbol{v})\, d\mathcal{L}^n$$

where $d\mathcal{L}^n$ *denotes the volume form* $(= dx^1 \wedge \cdots \wedge dx^n)$*, and where* $\boldsymbol{f} : \operatorname{int} B \to \mathbb{R}^n$ *and* $\mathsf{B} : \operatorname{int} B \to \operatorname{Lin}(\mathbb{R}^n; \mathbb{R}^n)$*, will be called a* force.

We notice that linearity in $\boldsymbol{v}$ is preserved. Additivity in J is on the contrary another matter, since chainlets look more like currents than domains. Moreover, we don't require explicit regularity assumptions on $\boldsymbol{f}, \mathsf{B}$ since they depend on the chosen class of chainlets. Clearly, the more irregular the chainlets, the more regular will have to be forms, hence also their vector and tensor coefficients.

It seems also possible to take measure-valued forms.

If $\mathsf{B} = 0$, we will speak as before of a *distance force*.

The analogous of theorem 2.7 should be easy to prove also in this framework, since it is always possible to take test functions $\boldsymbol{v}$ which are constant or affine on $\operatorname{supt} J$. Therefore we are led to suppose that an internal force will be represented by

$$P^{(i)}(J, \boldsymbol{v}) = -\int_J (\mathsf{T} : \mathsf{D})\, d\mathcal{L}^n.$$

Clearly, we can suppose T to be symmetric.

Now we come to contact power. Here we need a map which associates to $\boldsymbol{v}$ a $(n-1)$-differential form on ∂J, that is a $(0, n)$-tensor Q, alternating in the last $(n-1)$ variables:

$$P^{(c)}(J, \boldsymbol{v}) = \int_{\partial J} \mathsf{Q}\boldsymbol{v}.$$

Here Q is the correspondent of the vector field $\boldsymbol{t}$, since it is something like

$$\sum_k t_k\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n$$

where $\widehat{\ }$ stands for a missing component. Notice that we are not supposing $P^{(c)}$ to be a flux (we have J and not $\perp J$).

Finally, we require the balance of power

$$P^{(b)}(J, \boldsymbol{v}) + P^{(i)}(J, \boldsymbol{v}) + P^{(c)}(J, \boldsymbol{v}) = 0$$

for all $J \in \mathscr{J}$ and $\boldsymbol{v} \in C^\infty(\operatorname{int} B; \mathbb{R}^n)$.

If $\boldsymbol{v}$ is a vector field on $\operatorname{int} B$, let us denote by a tilde $\tilde{\boldsymbol{v}}$ the 1-form having the same components. We already know that $d(\star\tilde{\boldsymbol{v}})$ corresponds to $\operatorname{div} \boldsymbol{v}\, dx^1 \wedge \cdots \wedge dx^n$, therefore for symmetric T

$$d(\star\widetilde{\mathsf{T}\boldsymbol{v}}) = (\operatorname{div} \mathsf{T} \cdot \boldsymbol{v} + \mathsf{T} : \mathsf{D})\, d\mathcal{L}^n.$$

so that, using the generalized Gauss-Green theorem,

$$\int_J (\operatorname{div} \mathsf{T} + \boldsymbol{f}) \cdot \boldsymbol{v}\, d\mathcal{L}^n + \int_{\partial J} \mathsf{Q}\boldsymbol{v} - \int_{\perp \partial J} \widetilde{\mathsf{T}\boldsymbol{v}} = 0$$

for all $J \in \mathscr{J}$ and $\boldsymbol{v} \in C^\infty(\operatorname{int} B; \mathbb{R}^n)$. Now we make use of the Hodge Star theorem and find

$$\int_J (\operatorname{div} \mathsf{T} + \boldsymbol{f}) \cdot \boldsymbol{v}\, d\mathcal{L}^n + \int_{\partial J} (\mathsf{Q} - \star\widetilde{\mathsf{T}})\boldsymbol{v} = 0 \tag{4.1}$$

where

$$\star\tilde{T} = \sum_i T_{ik}\, dx^1 \wedge \cdots \wedge \widehat{dx^k} \wedge \cdots \wedge dx^n.$$

If we take now $\boldsymbol{v}$ with support contained into the support of $\mathsf{Q} - \star\widetilde{\mathsf{T}}$, we get that the second integral is zero and then

$$\int_J (\operatorname{div}\mathsf{T} + \boldsymbol{f}) \cdot \boldsymbol{v}\, d\mathcal{L}^n = 0.$$

Once we know that, we return to (4.1) and find

$$\int_{\partial J} (\mathsf{Q} - \star\widetilde{\mathsf{T}})\boldsymbol{v} = 0.$$

Now, depending on the family $\mathscr{J}$, this will imply (or not) some consequence on the integrand, or this last equation could be interpreted in a weak sense. In any case, we find Cauchy's Stress Theorem in the form that asserts that Q is represented by a differential form $(\star\widetilde{\mathsf{T}})$.

Remark 4.1. One has to be careful with the choice of $\mathscr{J}$. We have already seen that Dirac's δ is a chainlet. Hence, chainlets are not so related to subsets of $\mathbb{R}^n$ as one may think. If the choice of $\mathscr{J}$ is too large, in particular if $\mathscr{J}$ contains the family $\{\delta_x : x \in \operatorname{int} B\}$, then the wonderful "integral" form

$$\int_J (\operatorname{div}\mathsf{T} + \boldsymbol{f})\, d\mathcal{L}^n = 0$$

could be just

$$\operatorname{div}\mathsf{T}(x) + \boldsymbol{f}(x) = 0$$

i.e. the classical form of the differential law! (In fact, the integrands have to be smooth).

A good question therefore is: is there a choice of a family of chainlets which are representable by subbodies (possibly irregular)? A partial answer has been given by Šilhavý [21], who considers "normal chains", which are objects on which the action of a form can be represented by integration on a subset on $\mathbb{R}^n$. This is done however in the case of Whitney's flat norm.

5. Diffused bodies (back to GMT)

In this last section we describe an alternative approach to the description of an irregular subbody. Namely, we define it to be a "diffused subbody" represented by a function ϑ between 0 and 1. If $\vartheta(x) = 0$, then x does not belong to the subbody and if $\vartheta(x) = 1$ then we interpret this saying that x belongs to the subbody. But, of course, also intermediate cases are possible.

5.1. Forces on diffused objects

Let

$$\Theta(\operatorname{int} B) = \{\vartheta \in C_{\mathrm{c}}(\operatorname{int} B) : 0 \leqslant \vartheta \leqslant 1 \text{ on } \operatorname{int} B\}$$

Definizione 5.1. *A force of order $k \in \mathbb{N}$ is a function $F : \Theta(\operatorname{int} B) \times C^\infty(\operatorname{int} B; \mathbb{R}^n) \to \mathbb{R}$ such that*

(a) $P(\vartheta, v) = P(\vartheta_1, v) + P(\vartheta_2, v)$ *whenever* $\vartheta, \vartheta_1, \vartheta_2 \in \Theta(\operatorname{int} B)$ *are such that* $\vartheta = \vartheta_1 + \vartheta_2$, *and for all* $v \in \mathbb{C}^\infty(\operatorname{int} B)$,
(b) $P(\vartheta, v_1 + v_2) = P(\vartheta, v_1) + P(\vartheta, v_2)$;
(c) *there exist* $\mu_0, \mu_1, \dots, \mu_k \in \mathfrak{M}(\operatorname{int} B)$ *such that*

$$|P(\vartheta, v)| \leqslant \sum_{j=0}^{k} \int_{\operatorname{int} B} \vartheta \left| D^j v \right| \, d\mu_j.$$

The first theorem we give is the counterpart of the representation theorem 2.6.

Teorema 5.2. *For every force of order* k *there exist* k *Borel bounded functions* $A_j : \operatorname{int} B \to \operatorname{Sym}_j$ *such that*

$$P(\vartheta, v) = \sum_{j=0}^{k} \int_{\operatorname{int} B} \vartheta A_j \cdot D^j v \, d\mu_j = \sum_{j=0}^{k} \int_{\operatorname{int} B} \vartheta D^j v \cdot d\boldsymbol{\lambda}_j,$$

and where A_j *is uniquely determined* μ_j*-a.e..*

5.2. Weak balance and contact forces

Definizione 5.3. *A force is said to be a* distance force *if it has order zero.*

Definizione 5.4. *A force is said to be* weakly balanced *if there exists a measure* $\mu \in \mathfrak{M}(\operatorname{int} B)$ *such that*

$$|P(\vartheta, v)| \leqslant \int_{\operatorname{int} B} \vartheta v \, d\mu$$

whenever $(1 - \vartheta)v = 0$ *on* $\operatorname{int} B$.

Notice the difference with our previous notion of balancing. There we had that the inequality held if v had compact support in M. Here, the bound holds only where $\vartheta = 1$.

Definizione 5.5. *A force is said to be a* contact *force if*

$$P(\vartheta, v) = 0$$

whenever $(1 - \vartheta)v = 0$ *on* $\operatorname{int} B$.

Hence, contact powers are weakly balanced and also the decomposition theorem holds.

Teorema 5.6 (Distance-contact decomposition). *For every weakly balanced force there exist two uniquely determined distance and contact forces such that*

$$P(\vartheta, \boldsymbol{v}) = P^{(b)}(\vartheta, \boldsymbol{v}) + P^{(c)}(\vartheta, \boldsymbol{v})$$

for every $\vartheta \in \Theta(\operatorname{int} B)$ *and every* $\boldsymbol{v} \in C^\infty(\operatorname{int} B; \mathbb{R}^n)$.

Up to here, the ϑ's were only continuous. If one wants to say more, a possibility is to suppose ϑ more regular. Let's therefore suppose $\vartheta \in C_c^\infty(\operatorname{int} B)$.

Since the force is weakly balanced, if we denote by B the density of the distance part with respect to μ, then we can write

$$P^{(c)}(\vartheta, v) = P(\vartheta, v) - P^{(b)}(\vartheta, v) = \int_{\operatorname{int} B} \vartheta v (d\lambda_0 - B d\mu) + \sum_{j=1}^{k} \int_{\operatorname{int} B} \vartheta D^j v \cdot d\boldsymbol{\lambda}_j.$$

On the other hand, by weak balance one has (remember that $v = \vartheta v$ in this case)

$$\sum_{j=0}^{k} \int_{\operatorname{int} B} D^j(\vartheta v) \cdot d\boldsymbol{\lambda}_j = \int_{\operatorname{int} B} v B \, d\mu$$

and therefore

$$P^{(c)}(\vartheta, v) = \sum_{j=1}^{k} \int_{\operatorname{int} B} [\vartheta D^j v - D^j(\vartheta v)] \cdot d\boldsymbol{\lambda}_j.$$

It's interesting to notice that this expression vanishes when $\vartheta = 0$ or $\vartheta = 1$, hence it is supported by the "boundary" of the body.

Since in our case $k = 1$, this yields

$$P^{(c)}(\vartheta, v) = -\int_{\operatorname{int} B} (v \nabla \vartheta) \cdot d\boldsymbol{\lambda}_1.$$

It is finally possible to study the case $\vartheta \to \chi_M$. If M is not too bad (e.g. finite perimeter) and $\boldsymbol{\lambda}_1 \ll \mathscr{L}^n$, i.e. $\boldsymbol{\lambda}_1 = \boldsymbol{q}\, d\mathscr{L}^n$ with $\boldsymbol{q}$ continuous, then $\nabla\vartheta$ tends to $\boldsymbol{n}^M$ and one recovers the usual form

$$P^{(c)}(M, v) = \int_{\partial_* M} (v \boldsymbol{n}^M) \cdot \boldsymbol{q} \, d\mathscr{H}^{n-1}.$$

References

[1] Banfi C., Marzocchi A., Musesti A. *On the Principle of Virtual Powers in Continuum Mechanics*, submitted.

[2] M. Degiovanni, A. Marzocchi & A. Musesti, Cauchy fluxes associated with tensor fields having divergence measure, *Arch. Ration. Mech. Anal.* **147** (1999), 197–223.

[3] Degiovanni M., Marzocchi A., Musesti A., *Edge-force densities and second-order powers*, Annali di Matematica Pura ed Applicata, in press.

[4] A. Di Carlo, A. Tatone, (Iper-)tensioni & equi-potenza, 15th *AIMETA Congress of Theoretical and Applied Mechanics*, (2001).

[5] H. Federer, *Geometric Measure Theory*, Die Grundlehren der Mathematischen Wissenschaften, **153**, Springer-Verlag, New York, 1969.

[6] P. Germain, La méthode des puissances virtuelles en mécanique des milieux continus. Première partie. Théorie du second gradient, *J. Mécanique* **12** (1973), 235–274.

[7] P. Germain, The method of virtual power in continuum mechanics. Part 2: Microstructure, *SIAM J. Appl. Math.* **25** (1973), 556–575.

[8] M. E. Gurtin, *An Introduction to Continuum Mechanics*, Academic Press, New York 1981.

[9] M. E. Gurtin, L. C. Martins, Cauchy's theorem in classical physics, *Arch. Ration. Mech. Anal.* **60** (1976), 306–324.

[10] J. Harrison Stokes' theorem for nonsmooth chains, *Bull. Amer. Math. Soc.* **29** (1993), 235–242.

[11] J. Harrison Isomorphisms of differential forms and cochains, *J. Geom. Anal.* **8** (1998), 797–807.

[12] J. Harrison, Lectures on chainlet geometry - new topological methods in geometric measure theory, draft available online at `harrison@math.berkeley.edu`.

[13] A. Marzocchi, A. Musesti, Decomposition and integral representation of Cauchy interactions associated with measures, *Cont. Mech. Thermodyn.* **13** (2001), 149–169.

[14] Marzocchi A., Musesti A., Balanced virtual powers in Continuum Mechanics, *Meccanica* **38** (2003), 369–389.
[15] Marzocchi A., Musesti A., The Cauchy Stress Theorem for bodies with finite perimeter, *Rend. Sem. Mat. Padova* **109** (2003), 1-11.
[16] A. Musesti, Private communication.
[17] G. Rodnay, R, Segev Cauchy's Flux Theorem in light of Geometric Integration Theory, *J. Elasticity* **71** (2003), 183–203.
[18] T. Ruggeri Sulla Meccanica delle Microstrutture, *Atti Ist. Veneto Scienze, Lettere ed Arti*, vol. CXXIX (1971), 257–271.
[19] M. Šilhavý, The existence of the flux vector and the divergence theorem for general Cauchy fluxes, *Arch. Ration. Mech. Anal.* **90** (1985), 195–211.
[20] M. Šilhavý, Cauchy's stress theorem and tensor fields with divergences in L^p, *Arch. Ration. Mech. Anal.* **116** (1991), 223–255.
[21] M. Šilhavý, Geometric integration theory and Cauchy's stress theorem, (Private, March-September 2004) (Notes of the 2004 Ravello Summer School)
[22] M. Šilhavý, Divergence measure fields and Cauchy's stress theorem, *Rend. Sem. Mat. Univ. Padova* **113** (2005), 15–45.
[23] W. P. Ziemer, Cauchy flux and sets of finite perimeter, *Arch. Ration. Mech. Anal.* **84** (1983), 189–201.

Figure 38
Solution to the problem of Figure 18.