`Classroom Notes`
Mechanical Modeling of Muscles at a Glance

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1 Key Words

Mechanics and muscle physiology share many key words with quite different meanings; here we state our usage of some key terms.

- A muscle is *active* when, under the appropriate electro-chemical stimuli, it shortens and/or, exerts a force.
- A muscle *shortens* when, unloaded and unconstrained, is activated.
- *Contraction* is somehow used as synonymous of activation, an electro-chemical notion, or of shortening, a kinematical one. Beware of its meaning.
- A *twitch* is a single electro-chemical stimulus to a muscle, usually lasting a tenth of a second or less. An appropriate sequence of twitches activates a skeletal muscle, a single twitch is necessary for heart muscle.
- The adjectives *rest* or *ground* always refer to mechanics: they label a state at zero elastic energy. Activation changes the ground state of muscles.
- The *slack length* is the length to which a non active muscle will return when unloaded; never confuse slack with ground length: they coincide when muscle is unloaded and inactive. Slack is used for a whole muscle, or for its subunits. A sarcomere slack length is $\sim 2.1\mu m$.
- During an *isometric* exercise a muscle stays at fixed length; when activated, the muscle exerts a force against its constraints. Beware that in gym or physiological jargon, you may hear of 'isometric contraction', a misleading expression in our context: here contraction does not have any kinematical significance.
- During an *isotonic* exercise a muscle is subject to a fixed load; when activated, it shortens (provided the load is below a given threshold).
- During an *auxotonic* exercise a muscle has to deal mostly with inertia forces, that may be very high with respect to the load sustained, especially during acceleration and deceleration. Auxotonic movement is the most common in sport and training; most injuries occur when one is trying to overcome inertia or does not have the eccentric strength to decelerate properly.
- A muscle is in a *tetanized state* when, during isometric activation, the force developed reaches a steady plateau.
- We shall use the term *work* with its mechanical meaning (force times displacement), and *workout* to denote a physical-exercise session. Thus, you can have very intense isometric workout, without any mechanical working.
- A *preload* is a load applied to a muscle prior its activation, thus stretching the muscle with respect to the slack length.
- An *afterload* is a load applied to a muscle after it has been activated. The beating heart operates with both a preload and an afterload.

2 Mechanical Muscles Modeling

Muscles are organized bundles of multi-nuclei, long cells, called myofibers, whose mechanical characteristics depend on both the intrinsic properties of those fibers and their overall architecture. There are three types of muscles: skeletal, heart–both striated–and smooth. Skeletal muscles are the prime mover of locomotion, and are controlled by voluntary nerves; heart muscle functions with single twitches, and once activated, it stays refractory to further electrical stimuli until a certain time lapse. Heart and smooth muscle are not controlled by voluntary nerves.

2.1 Force and Velocity

Browsing physiological literature dealing with muscles, you may found two of the following basic descriptions:

- A muscle is a force generator–emphasis is put on dynamics, that is, on forces;
- A muscle generates motion–emphasis is on kinematics, that is, on displacements.

Those two points of view often receive elusive discussion; actually, a muscle can generate a motion while sustaining some load. Basic experiments of muscle mechanics measure the force exerted by a muscle in isometric conditions, or the shortening velocity under isotonic conditions.

A muscle is typically described in terms of muscle length, mass, myofiber length, pennation angle (the angle between the line of action and the myofiber long axis), and physiological cross-sectional area (PCSA), typically expressed in mm^2. PCSA is an approximation of the overall cross section area of a whole muscle, projected along the muscles line of action; it is calculated as:

$$PCSA = \frac{M\ \cos(\theta)}{\rho\, L_f} \tag{1}$$

with M the muscle mass (in grams, g), ρ the muscle density (g/mm^3), θ the surface pennation angle, and L_f the myofiber length. This formulation provides a good estimate of experimentally measured isometric muscle force output. Here we list the measured values of specific tension (force/PCSA) and shortening velocity for some muscles (from [Schneck and Bronzino, (2003)])

Skeletal Muscle Specific Tension & Shortening Velocity

Species	Musle type	Preparation	Specific tension (kPa)	V_{max}
Human	Slow	single fiber	133	0.86 L/s
Human	Fast	single fiber	166	4.85 L/s
Rat	Soleus	whole muscle	319	18.2 $\mu m/s$
Cat	Soleus	whole muscle	-	13 $\mu m/s$
Mouse	Soleus	whole muscle	-	31.7 $\mu m/s$
Guinea pig	Soleus	whole muscle	154	-

Muscle maximum contraction velocity is primarily dependent on the type and number of sarcomeres in series along the muscle fiber length. This property has been experimentally determined for a number of skeletal muscles.

2.2 Muscle as a Black Box

We present a list of wishful features for mechanical muscles modeling

- Less is more: a model should be as simple as possible.
- A muscle must control both position and force independently of each other; this request rules out any standard elastic or viscoelastic response.
- Models with low dimension, as 0D (system of springs & dashpots) or 1D models (system of bars), should be upgradable to a fully 3D models as smoothly as possible.
- Any kind of muscular workout requires energy; an adequate model should provide some insight into energetics, and should predict energy consumption even in correspondence of null mechanical working.

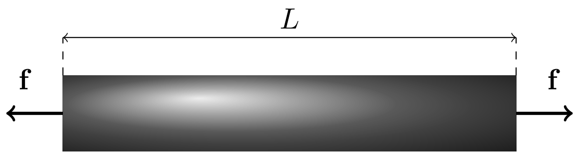

Figure 1: Muscle as a black box, exercising at length L while sustaining a load f.

Let us envision an exercising muscle as a black box, and let us describe the state of the muscle using just two parameter: its visible length $L(\tau)$ and the load $f(\tau)$, a force, it sustains at a given time τ. Many strain measures can be used to represent the length $L(\tau)$ in terms the slack length L_s; among them, we have:

$$\lambda(\tau) = \frac{L(\tau)}{L_s}\,, \quad \varepsilon(\tau) = \frac{L(\tau) - L_s}{L_s} = \lambda(\tau) - 1\,. \tag{2}$$

We define

$$\begin{array}{ll} \mathcal{P}(\dot{L}) = f \cdot \dot{L} & \text{mechanical } \textit{power} \text{ done by external load } f \text{ on muscle velocity } \dot{L}; \\ \mathcal{L} = \int_{\mathscr{T}} f(\tau) \cdot \dot{L}(\tau)\, d\tau & \text{mechanical } \textit{work} \text{ done during a time interval } \mathscr{T}; \end{array} \tag{3}$$

a superposed dot denoting differentiation with respect to time.

2.3 Gym session

Isometric. From $\dot{L} = 0$, it follows

$$\mathcal{P}(\dot{L}) = 0. \tag{4}$$

Isotonic. From $\dot{f} = 0$, it follows

$$\mathcal{L} = f \cdot \int_{\mathscr{T}} \dot{L}(\tau)\, d\tau = f \cdot \Delta L \tag{5}$$

with $\Delta L = L(\tau_1) - L(\tau_o)$, being $\mathscr{T} = (\tau_o, \tau_1)$.

3 Hill's equation for tetanized muscle

Hill's equation, a celebrated equation in muscle mechanics, has been proposed by Hill in 1938 [Hill, (1938)] after experiments on tetanized skeletal muscle from frog sartorius based on accurate calorimetric measurements taken during a sudden change of muscular exercise, from *isometric* to *isotonic* conditions.

At first, a muscle bundle, clamped at one end, is electrically stimulated and mantained at fixed length L_o (isometric conditions) by the appropriate reaction force; eventually, a maximum force F_o is reached. At this tetanized state, the reaction force is suddenly decreased to a constant value $F \leq F_o$ (isotonic conditions): it follows that the muscle contracts. The experiment is repeated with different initial length L_o and/or isotonic loads F, and the resulting mean shortening velocity V and time course of heat production $h(\tau)$ is registered.

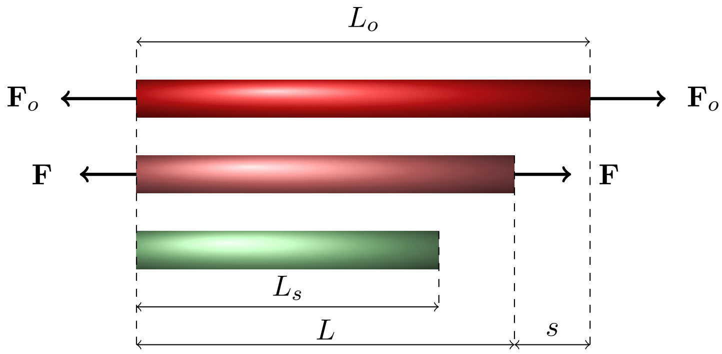

Figure 2: Schematic of Hill's experiment: a muscle is tetanized at constant length L_o (top); then, it is left free to shorten a distance s under a constant load $F < F_o$ until a given length L is reached (middle). During the shortening from L_o to L a measurable amount of heat is released. The green bar represents the slack length (bottom).

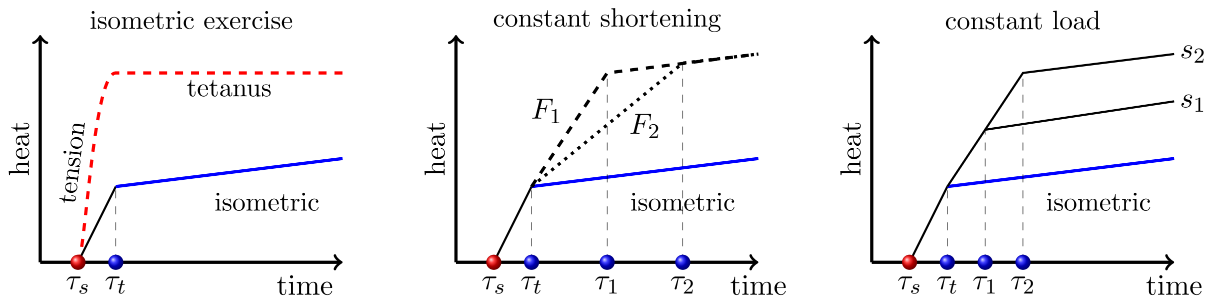

Figure 3: Stimulus starts at time τ_s and tetanus is reached at τ_t; heat rate in (τ_s, τ_t) is higher with respect the rate after tetanus (left). Shortening starts at τ_t and stops at τ_i (i=1,2). Heat rate is proportional to the load ($F_2 > F_1$), but eventually the same amount of heat is released (middle). Shortening different distances ($s_2 > s_1$) under same load yields same heat rate; total amount of heat released is proportional to distance (right).

Key findings by Hill (see Fig. 6,7 in [Hill, (1938)]) are represented in Fig.3, and listed in the following:

- Isometric exercises show two distinct energetic regimes: heat is released at high rate immediately after activation, between stimulus and tetanus; after tetanus is reached, the rate at which heat is released becomes much smaller, see Fig. 3, left. It is worth noting that isometric exercises are characterized by a same heat rate: after shortening, when muscle is back again in isometric regime, heat is released at the same rate as during the isometric benchmark.

$$\begin{aligned} &\dot{h} = A \quad \text{Activation thermal-power, almost constant in } (\tau_s, \tau_t); \\ &\dot{h} = B \quad \text{Tension-time thermal-power, with } B < A \text{ (isometric exercise).} \end{aligned} \tag{6}$$

- Isotonic exercises show a further different regime, characterized by large heat rates. Fig. 3, (center, right), shows:

$$\dot{h} > B \quad \text{heat rate during shortening is larger that isometric heat rate.} \tag{7}$$

- Shortening same distance under different loads shows that the amount of heat released does not depend on the load, while both heat rate and shortening velocity are proportional to the load, see Fig. 3, center:

$$\begin{aligned} &V_i = s/(\tau_i - \tau_t) \propto F_i \quad \text{shortening velocity is proportional to load;} \\ &\dot{h} \propto V \propto F \qquad \text{shortening heat rate is proportional to load and thus to velocity.} \end{aligned} \tag{8}$$

- Shortening different distances under same load shows that heat rate during shortening does not vary, while the amount of heat released is proportional to shortened distances.

$$h(\tau_i) \propto s_i \quad \text{heat released is proportional to distance shortened.} \tag{9}$$

We define the rate of *extra* energy liberation $\dot{E}$ during shortening as the sum of the shortening thermal-power $a\,V$, and the mechanical power W expended by the isotonic load F on the shortening velocity V: $W = F\,V$. Experimental findings show that $\dot{E}$ is a linear function of the difference between the load at tetanus F_o and the load sustained during isotonic shortening F

$$\dot{E} = W + a\,V = b(F_o - F)\,. \tag{10}$$

Relation (10) yields the Hill's equation in its original forms

$$(F + a)\,V = b\,(F_o - F) \quad \text{or} \quad (V + b)\,(F + a) = b\,(F_o + a)\,; \tag{11}$$

it is worth noting that a is a force-like parameter, related to heat production during contraction, while b is a velocity-like parameter. Let us note that, to an unloaded ($F = 0$) isotonic condition, there corresponds a maximal shortening speed $V_o = b\, F_o/a$; thus, the parameter b can be expressed in terms of the two measurable quantities V_o, F_o. Using this representation for b, equation (11) can be rewritten highlighting non dimensional ratios as follows:

$$(a+F)\,V = \frac{V_o\, a}{F_o}\,(F_o - F) \quad \Rightarrow \quad (1+\frac{F}{a})\,\frac{V}{V_o} = 1 - \frac{F}{F_o}\,. \tag{12}$$

We then define the non dimensional quantities $v = V/V_o$, $f = F/F_o$, $\alpha = a/F_o$ and write a nondimensional version of (12)

$$(1+\frac{f}{\alpha})\,v = 1 - f\,. \tag{13}$$

The change of variables $\xi = f + \alpha$, $\eta = v + \alpha$ yields the following hyperbolic function

$$\xi\,\eta = \alpha\,(1+\alpha)\,. \tag{14}$$

Hill's equation contains three parameters, namely F_o, b, and a, or, alternatively, F_o, V_o and α; these parameters are function of the initial muscle length L_o, and of other experimental conditions such as temperature and composition of physiological bath.

3.1 Isometric force-length relationship

A noteworthy dependence becomes apparent during the isometric exercise: a non-monotone relationship between the isometric length L_o and the maximal force F_o. Let us denote with L_s the slack length; a tetanized muscle cannot generate any appreciable force when $L_o \leq 0.6\,L_s$; then, maximal force increases with L_o, until a plateau is reached for $L_o \sim L_s$; then, maximal force decreases again, and becomes negligible for $L_o \geq 1.8\,L_s$. Thus, it happens that, in order to develop the maximum force, a muscle must exercise around its slack length. The isometric force-length curve was first presented in [Gordon et al., (1966)], and is showed in Fig.4.

Contrary to F_o, the other two parameters, V_o and α, do not show any appreciable dependence on length L_o.

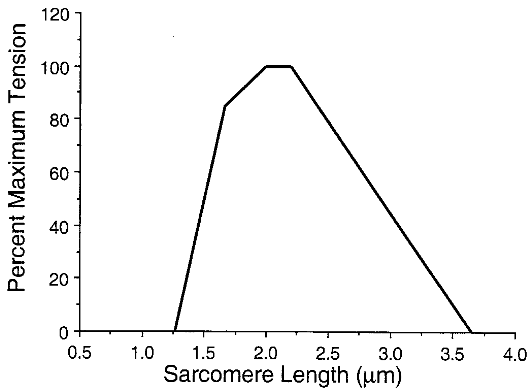

Figure 4: Sarcomere tension-length relation for isolated frog skeletal muscle.

4 Hill's three-element model

Hill's model represents a muscle as composed of two elastic elements acting in parallel; one element, dubbed *passive*, is a simple elastic spring; the other one, called *active*, is a spring whose *ground* length, that is, whose *zero elastic-energy* length, can be shortened; moreover, it is usually assumed that this active spring could sustain a force only under positive straining. The force exerted by the system is thus the sum of two contributions, which are related to the strains suffered by the passive and the active springs. The model retains its name, three-element model, from the fact that originally the active component was conceived as composed of a spring and a contractile element in series, thus totaling three components.

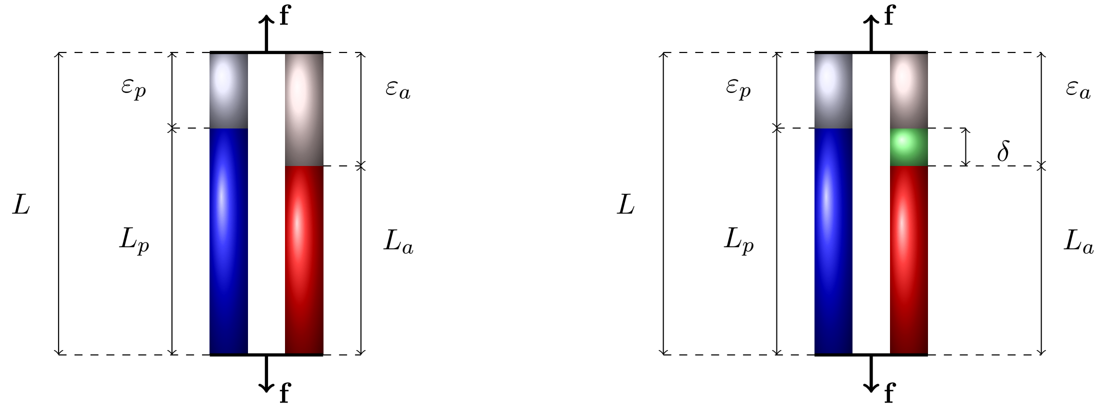

Figure 5: Hill's model.

The model is zero-dimensional, and the state of the system is described, at any instant τ, by two variables:

$$\begin{aligned} &L(\tau) \quad && \textit{visible} \text{ length of the two parallel elements,} \\ &L_a(\tau) \quad && \textit{ground} \text{ length of the active component;} \end{aligned} \tag{15}$$

omitting the time dependence in the notation, we define the strains as:

$$\Delta L_p = L - L_p\,, \quad \textit{passive} \text{ strain;} \quad \Delta L_a = L - L_a\,, \quad \textit{active} \text{ strain,} \tag{16}$$

where L_p is the ground length of the passive spring. The overall force f is the sum of a passive force f_p and an active one f_a; assuming linear elastic spring, we may write

$$f = f_p + f_a = k_p\,\Delta L_p + k_a\,\Delta L_a\,H(\Delta L_a)\,, \tag{17}$$

with k_p and k_a the stiffness of the passive and active springs, respectively, and $H(\cdot)$ the Heaviside function. We note that $L = L_p = L_a$ yields $\Delta L_p = \Delta L_a = 0$ and thus $f = 0$; it follows that L_p may be considered as the slack state for the system. We can define the actual ground length L_a in terms of the shortening δ with respect to the slack length L_p:

$$L_a = L_p - \delta\,, \quad \Delta L_a = L - L_p = \Delta L_p + \delta\,. \tag{18}$$

The parameter δ is somehow interpreted as a coarse description of the overlapping between the microscopic filaments which the sarcomere is composed of; the non-monotonicity of the force-length relationship for tetanized muscles may be described by assuming the stiffness of the active spring to be a function of δ:

$$f_a = f_a(\Delta L_a, \delta) = k_a(\delta)\, \Delta L_a\, H(\Delta L_a)\,. \tag{19}$$

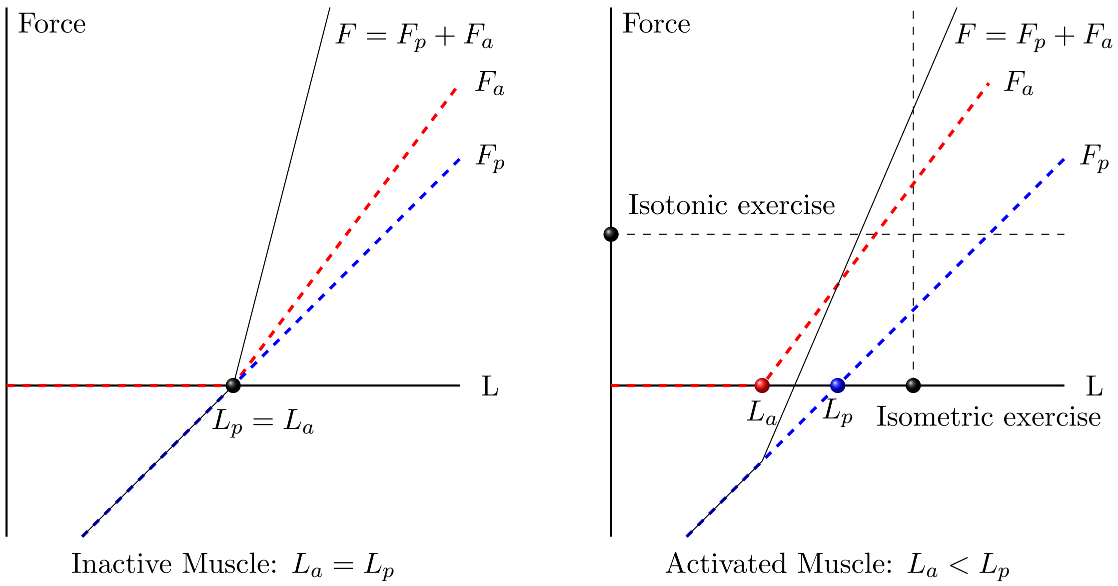

Figure 6: Force-length relationships corresponding to a non active muscle (left) and to an activated one (right); total force (solid, black) is the sum of the passive contribution (dashed, blue) and the active one (dashed, red). On the right, horizontal and vertical dashed lines represent isotonic and isometric exercises, respectively. Plots correspond to $k_a = 4/3\, k_p$; the activated response is plotted for $L_a = 2/3\, L_p$.

4.1 Gym session

Time derivative of (18) yields the following relation between strain rates, velocity and contraction velocity

$$\frac{d}{dt}\Delta L_p = \dot{L}\,, \quad \frac{d}{dt}\Delta L_a = \dot{L} + \dot{\delta}\,. \tag{20}$$

thus, from (17), (19), it follows (discarding the Dirac impulse $\dot{H}$ in $\Delta L_a = 0$)

$$\dot{f} = k_p \frac{d}{dt}\Delta L_p + k_a \frac{d}{dt}\Delta L_a + \frac{d\, k_a}{d\,\delta}\,\dot{\delta} = (k_p + k_a)\,\dot{L} + (k_a + \frac{d\, k_a}{d\,\delta})\,\dot{\delta}\,. \tag{21}$$

Isometric. From $\dot{L} = 0$, it follows

$$\dot{f} = (k_a + \frac{d\,k_a}{d\,\delta})\,\dot{\delta}\,, \tag{22}$$

that is, the time rate of force is proportional to contraction rate.
Isotonic. From $\dot{f} = 0$, it follows

$$\frac{\dot{L}}{\dot{\delta}} = -\frac{k_a + \dfrac{d\,k_a}{d\,\delta}}{k_p + k_a}\,, \tag{23}$$

that is, visible velocity $\dot{L}$ and contraction velocity $\dot{\delta}$ are related each other. For $d\,k_a/d\,\delta = 0$, it follows that the two velocities must have different sign.

5 Active tension

Hill's model dominated the field of mechanical muscle modeling since its proposal in 1938. Despite many improvements have been added, and full fledged 3D settings have been proposed, one key assumption still survive: the additive splitting of the force in a passive component and an active one.

Here, as prototypical example of enhanced muscle model developed in the cultural heritage of Hill's model, we present that proposed in [Hunter et al., (1998)]; the model is aimed at interpreting some experimental tests, and focuses on the relationships between the tension developed in a cardiac muscle and its physiological state, summarized by a coarse description of crossbridge kinetics. In particular, it focuses on the following mechanical experiments:

- tension-length relation in resting and activated muscles;
- time course of tension development under isometric conditions;
- tension recovery under length step tests (1% sudden shortening yields 100% tension drop);
- isotonic shortening at constant velocity (Hill experiments as benchmark);
- frequency response.

A key assumption is that the overall tension T sustained by a muscle be the sum of a passive component T_p, described by an appropriate elastic energy (pole-zero law) based on biaxial test results, plus an active component T_a, assumed to depend on the strain and the level of activation

$$T = T_p(\text{strain}) + T_a(\text{strain, activation})\,. \tag{24}$$

Most of the effort is done on the characterization of a response function for T_a(strain, activation), based on physiological phenomena and capable of describing the aforementioned mechanical experiments.

5.1 Physiology

Muscle physiology is described through the following phenomena: diffusion of free calcium ions through the cell (intracellular ions, C_{ai}); binding and release of calcium ions to Troponin C (bound ions, C_{ab}); thin filaments kinetics (percentage of available actin sites, z).

$$\text{free ions } C_{ai} \Rightarrow \text{bound ions } C_{ab} \Rightarrow \text{actin sites available } z \Rightarrow \text{active tension } T_a\,. \tag{25}$$

Ca^{2+} kinetics. Intracellular Calcium ions (C_{ai}) is released from the junctional sarcoplasmatic reticulum (jSR) as a consequence of a twitch; it diffuses into the cell, and bounds to Troponin C (TnC) which in turn regulates myofilaments force development. The kinetics of C_{ai} is assumed to be almost independent from the state of the muscle, and described once and for all by the following relation

$$C_{ai} = C_{ao} + (C_{am} - C_{ao})\,\frac{\tau}{\tau_c}\,\exp\left(1 - \tau/\tau_c\right). \tag{26}$$

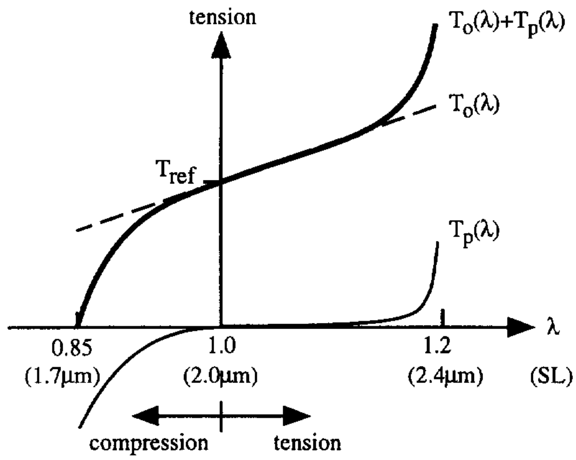

Figure 7: Schematic of tension-length relation for passive and active response of muscle. The non linear passive response T_p is added to a linear function whose intercept with vertical axis depends on the activation level. From [Hunter et al., (1998)].

Relation (30) describes a fast increase of C_{ai} from the resting level C_{ao} to a maximum value C_{am}, attained at $\tau = \tau_c$, followed by a slower decay to the resting value.

TnC-Ca^{2+} binding kinetics. Binding of C_{ai} to TnC is very fast and limited by diffusion through the cell; its release, in turns, depends on the state of the muscle. As example, isometric or isotonic twitches terminate differently, and the load sustained by a muscle may alter the release rate. It follows that the amount of bound Calcium ions C_{ab} should be sensible at least to the intracellular concentration C_{ai} and to the load T sustained; moreover, binding and release should have different time rates

Thin filaments kinetics. Binding of C_{ai} to TnC triggers myosin–actin interactions; its macroscopic effect is what is called 'force generation'. The level of activation of the muscle should be sensible to the percentage of actin sites available for cross-bridge binding

Moreover, a fading memory effect is included to capture some key features observed in experiments that involve vary fast responses, as in length step or frequency test. These effects are somehow related to crossbridge kinetics.

5.2 State variables

The model is essentially 1D, and considers a bar of length l_s, meant to represent the muscle in its slack state, that is, unloaded and non activated. The muscle state is described by four variables

$$\begin{array}{ll} l & \textit{visible} \text{ length;} \\ C_{ab} & \text{concentration of bound } Ca^{2+}; \\ z & \text{proportion of actin sites that are available for crossbridging.} \end{array} \tag{27}$$

Defined the strain $\lambda = l/l_s$, and denoted with $\psi(\lambda)$ the elastic energy density, we may write

$$T = T_p + T_a = \partial\psi(\lambda) + T_a(\lambda, z)\,. \tag{28}$$

5.3 Evolution equations for physiological varibles

The physiological variables C_{ab} and z satisfy two coupled evolution equations. Kinetics of bound ions is described by

$$\dot{C}_{ab} = \rho_o\, C_{ai}\,(C_{abm} - C_{ab}) - \rho_1\,(1 - \frac{T}{\gamma\, T_o}\,)\, C_{ab}\,. \tag{29}$$

where ρ_o, ρ_1 are binding and release rate constants, respectively, C_{abm} is the maximum value of C_{ab} attained for $T = \gamma\, T_o$ Kinetics of actin sites available is described by

$$\dot{z} = \alpha_o \left[\left(\frac{C_{ab}}{C_{50}} \right)^n (1 - z) - z \right]. \tag{30}$$

where α_o describes tropomyosin ‘mobility’, C_{50} is the value of bound ions yielding a 50% availability, and $n = n(\lambda)$ governs the steepness of the relation $z \mapsto \dot{z}$. For the active tension T_a. it is assumed a linear dependence on both the strain λ and the actin sites z

$$T_a = T_{ref}\,(1 + \beta_o\,(\lambda - 1)\,)\, z\,. \tag{31}$$

where T_{ref} is the reference tension at $\lambda = 1$, and β a non dimensional parameter reflecting the stiffening effects of activated myofilaments.

6 Mechanics in a Page!

Kinematics.

The motion of a *body manifolds* in the *ambient space* is described by *state variables*. Rigid motions have a peculiar role; thus, a definition of *strain* is needed to discriminate those motions from general ones. Definition of velocity and test velocity follows straightforward from the choice of state variables.

$$u \quad \text{state variables} \quad \Rightarrow \quad \dot{u} \quad \text{velocity}, \quad \tilde{u} \quad \text{test velocity}\,. \tag{32}$$

Working.

A *force* is primarily a continuous linear real-valued functional on the space of test velocities, whose value is the working expended by that force. The assumption of any specific functional expresses a strong intention about the outcome of the mechanical model under development.

$$< \text{force, test velocity} > = < f, \tilde{u} > = \mathcal{P}(\tilde{u}) \quad \text{power expended by } f \text{ on } \tilde{u}\,. \tag{33}$$

Balance law.

All balance laws are systematically provided by the principle of null working: the total working expended on any test velocity should be zero, i.e., the total force should be the null functional.

$$\mathcal{P}(\tilde{u}) = 0\,, \quad \forall \tilde{u}\,. \tag{34}$$

Constitutive issues.

Working is additively split into inner, outer, and inertia working:

$$\mathcal{P}(\tilde{u}) = \mathcal{P}^{in}(\tilde{u}) + \mathcal{P}^{out}(\tilde{u}) - \mathcal{P}^{ine}(\tilde{u})\,. \tag{35}$$

Two fundamental principles are enforced to deliver strict selection rules on the constitutive prescription for the inner actions. Such a priori restrictions do not apply to the outer actions, which are regarded as adjustable controls on the process.

Constitutive principle #1: material indifference to change in observer.

Inner working must be indifferent to change in observer:

$$\mathcal{P}^{in}(\tilde{u}) = \mathcal{P}^{in}(\tilde{u}^*)\,, \quad \forall\, \tilde{u}\,, \tilde{u}^* \text{ related by a change in observer}\,. \tag{36}$$

Constitutive principle #2: dissipation principle.

Along any realizable process, the time rates of kinetic energy ($\dot{k}$) plus free energy ($\dot{\psi}$) must be less or equal the outer power expended along the same process.

$$\dot{k}(u) + \dot{\psi}(u) \leq \mathcal{P}^{out}(u) = \mathcal{P}^{ine}(u) - \mathcal{P}^{in}(u)\,, \quad \forall \text{ realizable process } \tau \mapsto u(\tau)\,. \tag{37}$$

By definition, $\dot{k}(u) = \mathcal{P}^{ine}$; thus dissipation principle can be re-formulated as

$$\dot{\psi}(u) \leq -\mathcal{P}^{in}(u)\,, \quad \forall \text{ realizable process } \tau \mapsto u(\tau)\,. \tag{38}$$

or, introducing a dissipation functional $\mathscr{D}(u) = -(\dot{\psi}(u) + \mathcal{P}^{in}(u))$, by asserting that the power dissipated should be non-negative, for all body-parts, at all times, along all processes.

7 Active Contraction: 1D setting

We model a muscle as a single elastic bar, whose rest length can change. It is worth noting that the main difference with respect to the Hill's three element model will rely on dynamics, as we shall consider the rest length of the bar as an additional kinematical descriptor; in this sense, we may say that this model is endowed with a double-layer kinematics. The omission of the passive bar is just a minor issue.

7.1 State variables

We consider a bar of length l_s, meant to represent the muscle in its *slack* state, that is, unloaded and non activated, and assuming the bar to be uniformly stretched, we describe its state using two kinematical variables, that we call *extended motion*: $p = (l, l_o)$, with

$$\begin{array}{ll} l & \textit{visible} \text{ length;} \\ l_o & \textit{ground} \text{ length.} \end{array} \tag{39}$$

We can introduce three stretch measures: two compares the ground and visible lengths with respect to the slack one; the last one compares the actual and the ground length (see Fig. 8):

$$\begin{array}{ll} \lambda = \dfrac{l}{l_s} & \text{(visible) stretch;} \\ \lambda_o = \dfrac{l_o}{l_s} & \text{active stretch;} \\ \varphi = \dfrac{l}{l_o} = \dfrac{l}{l_s}\dfrac{l_s}{l_o} = \lambda\,\lambda_o^{-1} & \text{total stretch.} \end{array} \tag{40}$$

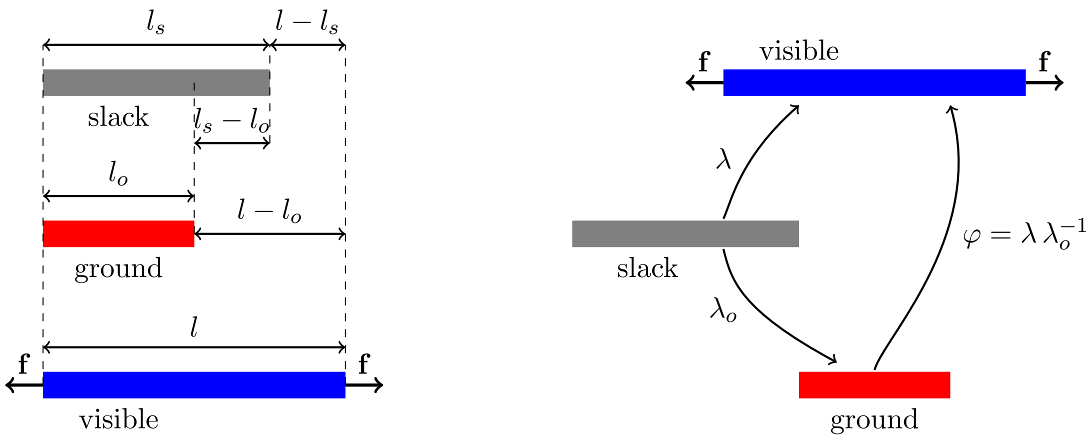

Figure 8: Schematic of the muscle model. The load f sustained during activation depends on the difference between the visible and the ground length. Cartoon at right shows the role of the three stretch measures.

We define the remodeling velocity as $V = \dot{l}_o\, l_o^{-1}$, and denote the actual and the test velocity associated to the state variables with the ordered pairs

$$v = (\dot{l}, V) \quad \text{velocity}, \qquad \widetilde{v} = (\tilde{l}, \tilde{V}) \quad \text{test velocity}, \tag{41}$$

with $\tilde{V} = \tilde{l}_o\, l_o^{-1}$ the remodeling test velocity.

7.2 Power

We postulate the following linear form on velocity as representative of the power expended

$$\mathcal{P}(\widetilde{v}) = -T\,\tilde{l} + R\,\tilde{V} + f\,\tilde{l}\,, \tag{42}$$

with

$$\begin{array}{ll} T & \text{muscle tension (positive for traction);} \\ R & \text{remodeling force;} \\ f & \text{external load.} \end{array} \tag{43}$$

The balance principle we enforce requires that the power expended be null for any test velocity:

$$\mathcal{P}(\widetilde{v}) = 0 \quad \forall \widetilde{v}\,. \tag{44}$$

This request, together with (**??**), yield the following two balance laws

$$T = f\,; \quad R = 0\,. \tag{45}$$

7.3 Constitutive recipes

Inner and Outer forces. We begin by distinguishing between *inner* and *outer* forces; the former represent the dofs described by the model, the latter represent the interaction between the dofs accounted for and the 'cosmos'. We list:

$$\begin{array}{ll} \text{inner forces} & T\,,\, R^i\,; \\ \text{outer forces} & f\,,\, R^o\,. \end{array} \tag{46}$$

As a consequence, we may split the power as the sum of an internal and an external contribution $\mathcal{P} = \mathcal{P}^i + \mathcal{P}^o$, with

$$\mathcal{P}^i(\widetilde{v}) = -T\,\tilde{l} + M^i\,\tilde{V}\,, \quad \mathcal{P}^o(\widetilde{v}) = R^o\,\tilde{V} + f\,\tilde{l}\,. \tag{47}$$

Energetics. The free energy density ψ is assumed to be entirely mechanical, and to depend on the total stretch φ

$$\psi = J_o\,\psi_o(\varphi)\,, \tag{48}$$

with ψ_o the energy density at ground state, and $J_o = l_o/l_s$ the Jacobian of the deformation from the slack to the ground length (in such a case, it is equivalent to the stretch λ_o).

Response functions. The inner actions are to be prescribed by appropriate response function

$$T = T(p)\,, \quad R^i = R^i(p)\,. \tag{49}$$

Dissipation Principle. Any constitutive prescription for $T(p)$, $R^i(p)$ must satisfy the following dissipation principle: along any realizable motion $\tau \mapsto p(\tau)$, the time rate of the free energy must be less or equal than the power expended by the external actions, which in turn, is the opposite to the internal power

$$\dot{\psi} \leq \mathcal{P}^o(v) = -\mathcal{P}^i(v)\,, \forall v \tag{50}$$

The dissipation principle stands as an explicit regulation governing the selection of appropriate constitutive prescriptions.

Reduced dissipation. We write here some useful relations, consequences of (40) and (48):

$$\dot{l} = (\varphi\, l_o)^{\cdot} = \dot{\varphi}\, l_o + \varphi\, \dot{l}_o = (\dot{\varphi} + \varphi\, V)\, l_o\,, \quad \dot{J}_o = \frac{\dot{l}_o}{l_s} = J_o\, V\,. \tag{51}$$

Thus, from (50), with (48) and (49), it follows a reduced dissipation inequality

$$[\, J_o\, \psi_o(\varphi) - \varphi\, T(p)\, l_o + R^i(p)\,]\, \dot{V} + [\, J_o\, \psi_o'(\varphi) - T(p)\, l_o\,]\, \dot{\varphi} \leq 0\,. \tag{52}$$

We now define

$$\begin{array}{ll} S_o(p) = J_o^{-1}\, T(p)\, l_o & \text{reference tension;} \\ S^d(p) = \psi_o'(\varphi) - S_o(p) & \text{dissipative tension;} \\ E(p) = \psi_o(\varphi) - \varphi\, S_o(p) & \text{Eshelby-like tension;} \\ R^d(p) = J_o\, E(p) + R^i(p) & \text{dissipative remodeling force.} \end{array} \tag{53}$$

and rewrite (52) as follows

$$R^d(p)\, \dot{V} + S^d(p)\, \dot{\varphi} \leq 0\,. \tag{54}$$

Let us note that the Eshelby-like tension constitutes a key coupling between the remodeling inner force R^i and the tension S_o. A further assumption that tension is purely energetic, $S^d(p) = 0$, and dissipative remodeling is viscous-like, yields constitutive prescriptions for $S_o(p)$, $R^i(p)$, and $E(p)$ that identically satisfy (54)

$$\begin{array}{l} S^d(p) = 0 \\ \\ R^d(p) = -\dfrac{V}{M}\,, \quad M > 0 \end{array} \quad \Rightarrow \quad \left\{ \begin{array}{l} S_o(p) = \psi_o'(\varphi) \\ \\ R^i(p) = -(J_o\, E(p) + \dfrac{V}{M}) \\ E(p) = \psi_o(\varphi) - \varphi\, \psi_o'(\varphi) \end{array} \right. \tag{55}$$

We can now write balance in equations in terms of the motion

$$\begin{array}{ll} J_o\, \psi_o'(\varphi)\, l_o^{-1} = f & \text{balance of force;} \\ V = M\, (R^o - J_o\, E) & \text{balance of remodeling force.} \end{array} \tag{56}$$

We conclude this presentation by considering a K-L like energy

$$\psi_o = \frac{1}{2}\, k\, (\varphi^2 - 1)^2\,. \tag{57}$$

It follows:

$$
\begin{aligned}
&S_o(p) = 2\,k\,\varphi\,(\phi^2 - 1) && \text{reference tension;}\\
&T(p) = J_o\,k\,\varphi\,l_o^{-1} = \dots && \text{tension;}\\
&E(p) = \dots && \text{Eshelby-like tension.}
\end{aligned}
\tag{58}
$$

7.4 Gym session

We deal with the following evolution equations

$$
\begin{aligned}
&\frac{1}{l_s}\,k\,\varphi = f && \text{balance of force;}\\
&V = M\,(R^o + \frac{l_o}{l_s}\,k\,\varphi^2) && \text{balance of remodeling force.}
\end{aligned}
\tag{59}
$$

Let us note that both visible and ground lengths are involved in tension development

$$
T = T(l, l_o)\,, \tag{60}
$$

thus, we can control both load and position independently from each other.

Isometric. From $\dot{l} = 0$, it follows $\dot{\varphi}\,l_o + \varphi\,\dot{l}_o$=0. A remodeling force such that $R^o = -l_o/l_s\,k\,\varphi^2$ yields $V = 0$ and thus, $\dot{l}_o = 0$

$$
\dot{f} = (k_a + \frac{d\,k_a}{d\,\delta})\,\dot{\delta} \tag{61}
$$

that is, the time rate of force is proportional to contraction rate.
Isotonic. From $\dot{f} = 0$, it follows $\dot{\varphi} = \dot{l}/l_o - l\,\dot{l}_o/l_o^2 = 0$; thus, isotonic exercise can be done provided

$$
\frac{\dot{l}}{l} = \frac{\dot{l}_o}{l_o}\,, \tag{62}
$$

that is, visible velocity $\dot{l}$ and contraction velocity $\dot{l}_o$ are related each other.

8 Active Contraction: 3D setting

We shall briefly present non linear elasticity with *growing* large distortions, as presented in [DiCarlo and Quiligotti,(2002)]. The evolution of distortion is used to describe muscle contraction as in [Nardinocchi and Teresi, (2007)], [DiCarlo et al., (2009)].

9 Kinematics

9.1 Motion

Let $\mathscr{B}$ be the body manifold (identified with its reference shape), $\mathscr{T}$ the time line, $\mathscr{E}$ the 3D Euclidean ambient space, with $\mathrm{V}\mathscr{E}$ the associated vector space, $\mathrm{Lin} = \mathrm{V}\mathscr{E} \otimes \mathrm{V}\mathscr{E} = \mathrm{Sym} \oplus \mathrm{Skw}$ the space of double tensors on $\mathrm{V}\mathscr{E}$ (linear maps of $\mathrm{V}\mathscr{E}$ into itself). An *extended motion* is described by the pair motion and distortion $(p, \mathbf{F}_o)$; the motion is a position-valued fieldd field

$$p : \mathscr{B} \times \mathscr{T} \quad \rightarrow \quad \mathscr{E} \tag{63}$$

$$(X, t) \quad \mapsto \quad x = p(X, t) = X + \mathbf{u}(X, t) \tag{64}$$

associating to any material point $X \in \mathscr{B}$ its position in space $x = p(X, t) \in \mathscr{E}$; the vector-valued field $\mathbf{u} = x - X$ represents the displacement of the material point X. The set $\mathscr{B}_t = p(\mathscr{B}, t)$ is the actual configuration of $\mathscr{B}$ at time t. The distortion is a tensor-valued field

$$\mathbf{F}_o : \mathscr{B} \times \mathscr{T} \quad \rightarrow \quad \mathrm{Lin} \tag{65}$$

$$(X, t) \quad \mapsto \quad \mathbf{F}_o(X, t) \tag{66}$$

endowed with a noteworthy constitutive assumption: the image of body elements under $\mathbf{F}_o$ are volume elements at ground state, that is at zero elastic energy. Let us remark the distortions are not required to be compatible, not even locally, that is, a ground configuration may not exists.

Given a motion $(p, \mathbf{F}_o)$, the associated *exended velocity* and *extended test velocity* are given by

$$(\dot{p}, \dot{\mathbf{F}}_o \mathbf{F}_o^{-1}), \quad (\tilde{p}, \tilde{\mathbf{F}}_o \mathbf{F}_o^{-1}). \tag{67}$$

Key geometrical functions are performed by the *gradient* of the motion $\mathbf{F}$, its *adjugate* $\mathbf{F}^*$, and its *Jacobian determinant* J:

$$\begin{aligned} \mathbf{F} &:= \nabla p = \mathbf{I} + \nabla \mathbf{u}\,, \\ \mathbf{F}^* &:= J\, \mathbf{F}^{-\top}\,; \\ J &:= \det(\mathbf{F})\,. \end{aligned} \tag{68}$$

Let a hierarchy of infinitesimally small one-, two-, and three-dimensional parallelepipedal cells, built out of the vectors $a, b, c \in \mathrm{V}\mathscr{E}$, be attached to a place $X \in \mathscr{B}$:

i) a *line element*, gauged by the vector a;

ii) a *facet* (a, b), gauged by its Gibbs representative $a \times b$; and

iii) a *volume element* (a,b,c), gauged by its (oriented) volume $a\times b\cdot c$.

Then, their images under the action of p, obviously attached to $x=p(X,t)$, are gauged respectively by:

i) $\mathbf{F}(X,t)\,a$;

ii) $\mathbf{F}^*(X,t)(a\times b)=(\mathbf{F}(X,t)\,a)\times(\mathbf{F}(X,t)\,b)$;

iii) $J(X,t)(a\times b\cdot c)=(\mathbf{F}(X,t)\,a)\times(\mathbf{F}(X,t)\,b)\cdot(\mathbf{F}(X,t)\,c)$.

Let us note that from (ii), it follows a rule relating the oriented normal $\mathbf{m}$ to a facet and the oriented normal $\mathbf{n}$ to the image of that facet

$$\mathbf{n}=\frac{\mathbf{F}^*\,\mathbf{m}}{|\mathbf{F}^*\,\mathbf{m}|}\,. \tag{69}$$

Finally, introducing a standard notation for line, surface and volume elements, we may write

$$dx=\mathbf{F}\,dX\,,\quad da=|\mathbf{F}^*\,\mathbf{m}|\,dA\,,\quad dv=J\,dV. \tag{70}$$

The same geometrical functions are performed by the distortion $\mathbf{F}_o$, its adjugate $\mathbf{F}_o^*$ and its Jacobian determinant J_o, with the important caveat that, in general, it does not exists any displacement $\mathbf{u}_o$ such that $\mathbf{I}+\nabla\mathbf{u}_o=\mathbf{F}_o$.

9.2 Anisotropic distortions

Let us note that it is possible to define transversely-anisotropic distortion field $\mathbf{F}_o$: denoted with $\mathbf{e}$ a unit vector field, we have

$$\mathbf{F}_o=\lambda_{\|}\,\mathbf{e}\otimes\mathbf{e}+\lambda_{\perp}\,(\mathbf{I}-\mathbf{e}\otimes\mathbf{e})\,. \tag{71}$$

Here $\lambda_{\|}$ measures the *contraction* along the fibers, and $\lambda_{\perp}$ a possible deformation orthogonal to the fibers, whose choice represents an important constitutive assumption; it is worth associating the parameter $\lambda_{\perp}$ to the variation of volume $J_o=\det\mathbf{F}_o$ induced by the distortion, and write

$$\lambda_{\perp}=\sqrt{\frac{J_o}{\lambda_{\|}}}\,; \tag{72}$$

$J_o=1$ yields a volume preserving contraction, while $J_o=\lambda_{\|}$ yields a uni-axial one; in such a case, we have

$$\mathbf{F}_o=\mathbf{I}+(\lambda_{\|}-1)\,\mathbf{e}\otimes\mathbf{e}\,. \tag{73}$$

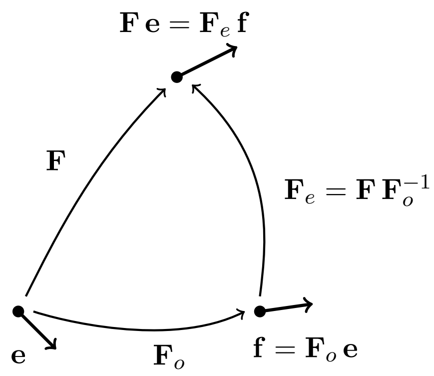

Figure 9: Deformations $\mathbf{F}$ and distortions $\mathbf{F}_o$ suffered by a material fiber $\mathbf{e}$.

9.3 Elastic Deformation

We define the *elastic deformation* $\mathbf{F}_e$ of the body elements as the difference between the distortions $\mathbf{F}_o$ and the *visible* deformations $\mathbf{F}$ in the sense of multiplicative composition:

$$\mathbf{F}_e = \mathbf{F}\,\mathbf{F}_o^{-1}\,; \tag{74}$$

it is worth noting that $\mathbf{F}_o$ and $\mathbf{F}_e$ are not, in general, gradients of any fields. We define the following three different metric tensors to be used as strain measures (left Cauchy-Green strains):

$$\mathbf{C} = \mathbf{F}^\top\mathbf{F}\,,\quad \mathbf{C}_e = \mathbf{F}_e^\top\,\mathbf{F}_e = \mathbf{F}_o^{-\top}\,\mathbf{C}\,\mathbf{F}_o^{-1}\,,\quad \mathbf{C}_o = \mathbf{F}_o^T\,\mathbf{F}\,. \tag{75}$$

Let us note that $\mathbf{C}$ and $\mathbf{C}_o$ measure the strain suffered by a reference fibre $\mathbf{e}$ once embedded in the actual state or distorted, respectively; alternatively, $\mathbf{C}_e$ measures the strain λ_e suffered by a distorted fiber $\mathbf{f}$ embedded in the actual state, see Fig. 9:

$$\mathbf{e} \mapsto \mathbf{f} = \mathbf{F}_o\,\mathbf{e}\,,\ \mathbf{F}_e = \mathbf{F}\,\mathbf{F}_o^{-1} \quad\Rightarrow\quad \lambda_e^2 = \mathbf{F}_e\,\mathbf{f}\cdot\mathbf{F}_e\,\mathbf{f} = \mathbf{C}_e\cdot\mathbf{f}\otimes\mathbf{f} = \mathbf{F}\,\mathbf{e}\cdot\mathbf{F}\,\mathbf{e} = \mathbf{C}\cdot\mathbf{e}\otimes\mathbf{e}\,. \tag{76}$$

10 Elastic Energy

Let us introduce three different stress measures

$$\begin{aligned} &\mathbf{S} && \text{reference stress, aka, first P-K}\,;\\ &\mathbf{S}_o && \text{ground stress}\,;\\ &\mathbf{T} && \text{“the stress”, aka, actual or Cauchy stress}\,. \end{aligned} \tag{77}$$

The reference stress $\mathbf{S}$ is related to the other ones by a pull-back as follows

$$\mathbf{S} = \mathbf{T}\,\mathbf{F}^* = \mathbf{S}_o\,\mathbf{F}_o^*\,. \tag{78}$$

We assume that the hyperelastic response of the body is described by a strain energy density per unit ground volume ψ_o whose value at each body point $X \in \mathscr{B}$ depends only on the present

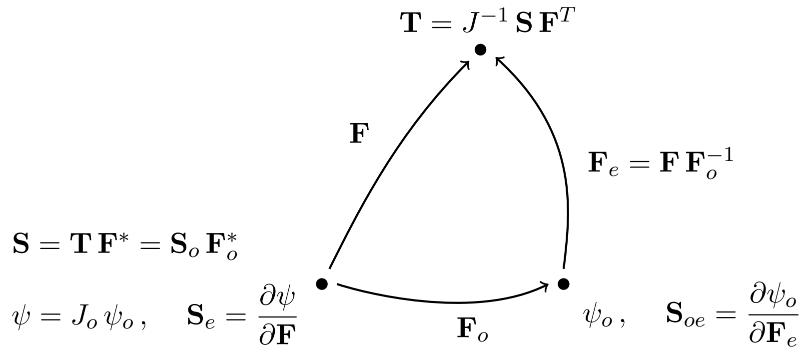

Figure 10: Stress measures and energy densities.

value of the elastic deformation $\mathbf{F}_e$ at that point; moreover, we assume the elastic deformation to be isochoric:

$$\psi_o : \mathbf{F}_e \mapsto \psi_o(\mathbf{F}_e)\,, \quad \& \quad J_e = \det(\mathbf{F}_e) = 1\,. \tag{79}$$

Given ψ_o, the strain energy density per unit reference volume is defined by $\psi = J_o\,\psi_o$, with $J_o = \det(\mathbf{F}_o)$; the energetic part of the stress is thus defined by

$$\mathbf{S}_{oe} := \frac{\partial\psi_o}{\partial\mathbf{F}_e}\,, \text{ground measure}\,; \quad \mathbf{S}_e := \frac{\partial\psi}{\partial\mathbf{F}}\,, \text{reference measure}\,. \tag{80}$$

We have, see Fig. 10:

$$\begin{aligned}
\mathbf{S}_e \quad &= \mathbf{S}_{oe}\,\mathbf{F}_o^*\,, && \text{pull back of the energetic stress}\,;\\
\mathbf{S} \quad &= \mathbf{S}_e + \mathbf{S}_v\,, \quad \text{with} \quad \mathbf{S}_v = -p\,\mathbf{F}^*\,, && \text{reference stress}\,;\\
\mathbf{T} \quad &= \mathbf{S}\,(\mathbf{F}^*)^{-1} = \mathbf{S}_e\,(\mathbf{F}^*)^{-1} - p\,\mathbf{I}\,, && \text{actual stress}\,.
\end{aligned} \tag{81}$$

Here p denotes the indeterminate pressure, reaction to the isochoric constraint. Let us note that energy is usually represented in terms of the strain measure $\mathbf{C}_e$ or $\mathbf{E}_e = 1/2(\mathbf{C}_e - \mathbf{I})$; from the representation formula for the time rate of $\mathbf{C}_e$, $\mathbf{E}_e$

$$\dot{\mathbf{C}}_e = \dot{\mathbf{F}}_e^T\,\mathbf{F}_e + \mathbf{F}_e^T\,\dot{\mathbf{F}}_e = 2\,\mathrm{sym}(\mathbf{F}_e^T\,\dot{\mathbf{F}}_e)\,, \quad \dot{\mathbf{E}}_e = \dot{\mathbf{C}}_e/2\,. \tag{82}$$

we have the following useful relation between energy derivatives:

$$\dot{\psi}_o = \frac{\partial\psi_o}{\partial\mathbf{C}_e}\cdot\dot{\mathbf{C}}_e = 2\,\mathbf{F}_e\,\frac{\partial\psi_o}{\partial\mathbf{C}_e}\cdot\dot{\mathbf{F}}_e = \frac{\partial\psi_o}{\partial\mathbf{F}_e}\cdot\dot{\mathbf{F}}_e \quad \Rightarrow \quad \mathbf{S}_{oe} = \frac{\partial\psi_o}{\partial\mathbf{F}_e} = 2\,\mathbf{F}_e\,\frac{\partial\psi_o}{\partial\mathbf{C}_e} = \mathbf{F}_e\,\frac{\partial\psi_o}{\partial\mathbf{E}_e}\,. \tag{83}$$

The derivative of the energy with respect to $\mathbf{E}_e$ is usually dubbed as the second P-K stress:

$$\frac{\partial\psi_o}{\partial\mathbf{E}_e} = \mathbf{F}_e^{-1}\,\mathbf{S}_{oe}\,. \tag{84}$$

Finally, let us note that from $\psi = J_o\,\psi_o$, it follows

$$\dot{\psi} = \underbrace{\frac{\partial \psi}{\partial \mathbf{F}}}_{\mathbf{S}_e} \cdot \dot{\mathbf{F}} = J_o \,\frac{\partial \psi_o}{\partial \mathbf{F}_e} \cdot \dot{\mathbf{F}}_e = J_o \,\frac{\partial \psi_o}{\partial \mathbf{F}_e}\,\mathbf{F}_o^{-\top} \cdot \dot{\mathbf{F}}_e = \underbrace{\frac{\partial \psi_o}{\partial \mathbf{F}_e}}_{\mathbf{S}_{oe}}\,\mathbf{F}_o^{*} \cdot \dot{\mathbf{F}}_e\,. \tag{85}$$

Remark: Here, we have assumed distortions as given once and for all; by assuming distortional processes $t \mapsto \mathbf{F}_o(t)$ you need to consider their time derivative, and this yields noteworthy consequences on balance equations.

11 Attach-detach-contract model (release 2.8)

We model a muscle as a telescoping unit, composed by two bars, overlapping each other to some extent, that we dub attach-detach-contract model, as presented in [DiCarlo, (2008)]; moreover, we assume one of the two bars to be active, that is, endowed with a variable ground length.

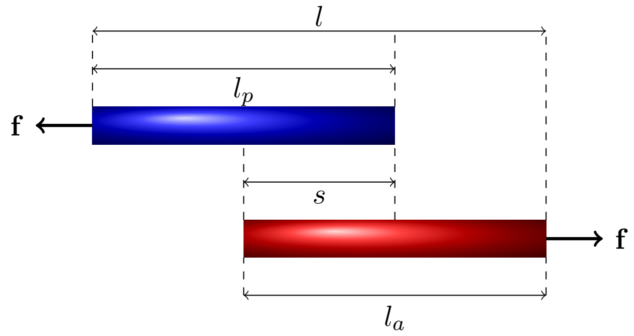

Figure 11: adc's model.

11.1 State variables

We assume the bars to be uniformly stretched, and we describe the system using four state variables, that we call *extended motion* $p = (l_p, l_a, s, l_o)$, with

$$\begin{array}{ll} l_p & \text{passive-bar (p-bar) length;} \\ l_a & \text{active-bar (a-bar) length;} \\ s & \text{overlapping between p- and a-bars;} \\ l_o & \text{ground length of the a-bar.} \end{array} \tag{86}$$

The overall length of the system, that is, its *visible* length, is given by

$$l = l_a + l_m - s\,. \tag{87}$$

The stretches in the bars are defined as the ratios between their actual and *relaxed* length

$$\lambda_a = \frac{l_a}{l_o}\,, \quad \lambda_p = \frac{l_p}{l_p^*}\,, \tag{88}$$

with l_p^* the ground length of the p-bar, given once and for all. We denote the (actual) velocity and the test velocity associated to the state variables with the ordered quadruplets

$$\begin{array}{ll} v = (\dot{l}_a, \dot{l}_m, \dot{s}, \dot{l}_o) & \text{velocity;} \\ \widetilde{v} = (\tilde{l}_a, \tilde{l}_m, \tilde{s}, \tilde{l}_o) & \text{test velocity;} \end{array} \tag{89}$$

given (87), we may represent the visible velocity $\dot{l}$ in terms of v as follows: $\dot{l} = \dot{l}_a + \dot{l}_m - \dot{s}$.

11.2 Power

Power is a linear form on velocity; prompted by the 'serial' disposition of the elements, we postulate the following representation formula for the power expended

$$\mathcal{P}(\widetilde{v}) = -T_a\,\tilde{l}_a - T_p\,\tilde{l}_p + C\,\tilde{s} + R\,\tilde{l}_o + f\,\tilde{l}\,, \tag{90}$$

with

$$\begin{array}{ll} T_a\,, T_m & \text{tension (positive for traction);} \\ C & \text{coupling force (between a- and m-elements);} \\ R & \text{remodeling force:} \\ f & \text{external load.} \end{array} \tag{91}$$

The balance principle we enforce requires that the power expended be null for any test velocity:

$$\mathcal{P}(\widetilde{v}) = 0 \quad \forall \widetilde{v}\,. \tag{92}$$

This request, together with (87), yield the following four balance laws

$$T_a = T_p = C = f\,; \quad R = 0\,. \tag{93}$$

11.3 Constitutive recipes

Inner and Outer forces. We distinguish between *inner* and *outer* forces:

$$\begin{array}{ll} \text{inner forces} & T_a\,, T_p\,, C\,,\, R^i\,; \\ \text{outer forces} & f\,,\, R^o\,. \end{array} \tag{94}$$

As a consequence, we may split the power as the sum of an internal and an external contribution $\mathcal{P} = \mathcal{P}^i + \mathcal{P}^o$, with

$$\mathcal{P}^i(\widetilde{v}) = -T_a\,\tilde{l}_a - T_p\,\tilde{l}_p + C\,\tilde{s} + R^i\,\tilde{l}_o\,, \quad \mathcal{P}^o(\widetilde{v}) = R^o\,\tilde{l}_o + f\,\tilde{l}\,. \tag{95}$$

Energetics. The free energy density, assumed to be entirely mechanical, is defined as the sum of the elastic energies of the two bar

$$\psi = \psi_a(\lambda_a) + \psi_p(\lambda_p) \tag{96}$$

Response functions. The inner actions are prescribed by appropriate response function

$$T_a = \hat{T}_a(p)\,, \quad T_p = \hat{T}_p(p))\,, \quad C = \hat{C}(p)\,, \quad R^i = \hat{R}^i(p)\,, \tag{97}$$

Dissipation Principle. We enforce that, along any realizable motion $\tau \mapsto p(\tau)$, the time rate of the free energy be less or equal than the power expended by the external actions, which in turn, is the opposite to the internal power

$$\dot{\psi} \leq \mathcal{P}^o(v) = -\mathcal{P}^i(v)\,, \forall v \tag{98}$$

Inserting relations (95), (96) and (97) in (98), it follows a reduced dissipation inequality

$$[\psi_a' \frac{1}{l_o} - \hat{T}_a(p)]\,\dot{l}_a + [\,\psi_p' \frac{1}{l_p^*} - \hat{T}_p(p)]\,\dot{l}_p + \hat{C}(p)\,\dot{s} + [\,R^i(p) - \psi_a' \frac{l_a}{l_o^2}\,]\,\dot{l}_o \,\leq\, 0\,. \tag{99}$$

A further assumption that no dissipation is related to T_a, T_p, yields

$$\hat{T}_a(p) = \psi_a'(\lambda_a)\,\frac{1}{l_o}\,, \quad \hat{T}_p(p) = \psi_p'(\lambda_p)\,\frac{1}{l_p^*}\,, \quad C(p)\,\dot{s} + R^d(p)\,\dot{l}_o \leq 0\,, \tag{100}$$

with $R^d(p)$ a dissipative term, endowed with an Eshelbian coupling between elasticity, ψ_a', and remodeling force, R^i, given by

$$R^d(p) = R^i(p) - \psi_a'(\lambda_a)\,\frac{l_a}{l_o^2} = R^i(p) - \hat{T}_a(p)\,\lambda_a\,. \tag{101}$$

Thus, it remains to prescribe a constitutive recipe for the coupling force $C(p)$ and the inner remodeling action $R^i(p)$, such that $(103)_3$ hold. A simple recipe fitting the reduced dissipation inequality is as follows

$$C(p) = -\frac{1}{M}\,\dot{s}\,, \quad R^d(p) = -D\,\dot{l}_o,, \quad \text{with} \quad M > 0\,, D > 0\,. \tag{102}$$

In such a case, reduced dissipation inequality rewrites as

$$-\frac{\dot{s}^2}{M} - D\,\dot{l}_o \leq 0\,. \tag{103}$$

11.4 Evolution equations.

$$\begin{aligned} \dot{s} &= -M\,f \\ D\,\dot{l}_o &= \lambda_a\,T_a(\lambda_a) + R^o \\ 0 &= -T_a(\lambda_a) + f \end{aligned} \tag{104}$$

References

[Hill, (1938)] Hill, A.V., 1938. The heat of shortening and the dynamic constants of muscle. Proc R Soc London B Biol Sci 126: 136-195.

[Huxley, 1957] Huxley A.F., 1957. Muscle structure and theories of contraction. Prog Biophys Biophys Chem 7: 255-318.

[Gordon et al., (1966)] Gordon, A.M., Huxley, A.F., Julian, F.J., 1966. The variation in isometric tension tension with sarcomere length in vertebrate muscle fibers. J. Physiol. 185, 170–192.

[Hunter et al., (1998)] Hunter P.J., McCulloch A.D., ter Keurs H.E.D.J., 1998. Modelling the Mechanical properties of cardiac muscle. Prog. Biophys. Molec. Bio. 69, 289–331.

[DiCarlo, (2008)] DiCarlo A., 2008. Elementary mechanics of muscular exercise. Mini-Workshop: The mathematics of growth and remodelling of soft biological tissues, Mathematisches Forschungsinstitut Oberwolfach Report 39/2008, 2232–2235

[DiCarlo and Quiligotti,(2002)] DiCarlo A., Quiligotti S., 2002. Growth & Balance. Mechanics Res. Comm. 29:449–456.

[Nardinocchi and Teresi, (2007)] Nardinocchi P., Teresi L., 2007. On the Active Response of Soft Living Tissues. J. Elasticity, 88:27–39.

[DiCarlo et al., (2009)] A. DiCarlo, P. Nardinocchi, T. Svaton, L. Teresi, 2009. Passive and active deformation process in cardiac tissue. In Proceedings of the Int. Conf. on Computational Methods for Coupled Problems in Science and Engineering (COUPLED PROBLEMS 2009), Eds. B. Schreer, E. Onate and M.Papadrakakis, CIMNE, Barcelona.

[Schneck and Bronzino, (2003)] Schneck D.J., Bronzino J.D., 2003. Biomechanics. Principles and Applications. CRC Press.