Onset of Chaos in a Model of Quantum Computation

G.P.Berman[a],F.Borgonovi[b,c], F.M.Izrailev[d], V.I.Tsifrinovich[e]

[a]Theoretical Division and CNLS, Los Alamos National Laboratory, Los Alamos, NM 87545, USA
[b]Dipartimento di Matematica e Fisica, Università Cattolica, via Musei 41, 25121 Brescia, Italy
[c] I.N.F.M., Gruppo Collegato di Brescia and I.N.F.N., Sezione di Pavia Italy
[d]Instituto de Fisica, Universidad Autonoma de Puebla, Apdo. Postal J-48, Puebla 72570, Mexico
[e] IDS Department, Polytechnic University, Six Metrotech Center, Brooklyn, NY 11201

Recently, the question of a relevance of *quantum chaos* has been discussed in applications to quantum computation [1, 2]. Indeed, according to the general approach to closed systems of finite number of interacting Fermi-particles (see, e.g. [3, 4]), as the interaction between *quibits* increases a kind of chaos is expected to emerge in the energy spectra and structure of many-body states. Specifically, the fluctuations of energy levels and components of the eigenstates turn out to be very large and they are described by the random matrix theory. Clearly, if this happens in a quantum computer, it may lead to a destruction of the coherence (due to an internal decoherence inside many-body states) required for quantum computations. It is important to stress that the quantum chaos occurs not only in the systems with *random* interactions, but also for purely *dynamical* interactions. In the latter case, the mechanism of chaos is the non-linear two-body interaction represented in the basis of non-interacting particles.

Numerical analysis [1] of the simplest model of a quantum computer (2D model of 1/2-spins with a random interqubit interaction J) shows that as the number, L, of qubits increases, the chaos threshold J_{cr} *decreases* as $J_{cr} \propto 1/L$. Consequently, it was claimed that the onset of quantum chaos is a real danger for the quantum computers with large $L \gg 1$. On the other hand, in [2] is was argued that in order to treat this problem properly, one needs to distinguish between the chaotic properties of *stationary states* and the dynamical process of quantum computation.

Below, we report our theoretical and numerical results for a realistic model of quantum computer, described in [5, 6]. We consider *both* stationary and dynamical approaches to the model in the region of a *non-selective* excitation which prepares a homogeneous superposition of $N = 2^L$ states needed to implement both the Shor and the Grover algorithms.

The model describes a 1-dimensional chain of L interacting 1/2-spins in the constant magnetic field B^z, subjected to a sum of $p = 1, ..., P$ time-dependent rectangular pulses of a circular polarized magnetic field rotating in the x, y-plane [7, 8]. Each of the pulses has the amplitude $b_\perp^p$, frequency ν_p, phase φ_p, and is non-zero during the time $T_p = t_{p+1} - t_p$. The Hamiltonian has the form,

$$\mathcal{H} = -\sum_{k=0}^{L-1}(\omega_k I_k^z + 2\sum_{n>k} J_{k,n} I_k^z I_n^z) -$$
$$\frac{1}{2}\sum_{p=1}^{P}\Theta_p(t)\Omega_p\sum_{k=0}^{L-1}\left(e^{-i\nu_p t - i\varphi_p} I_k^- + e^{i\nu_p t + i\varphi_p} I_k^+\right), \tag{1}$$

where the "pulse function" $\Theta_p(t)$ is 1 during the p-th pulse. The quantities $J_{k,n}$ are the constants of Ising interactions between two qubits, ω_k are the frequencies of the spin precession in the B^z-field, and Ω_p is the Rabi frequency corresponding to the p-th pulse. The operators $I_k^\pm = I_k^x \pm i I_k^y$ are defined by the relations $I_k^{x,y,z} = (1/2)\sigma_k^{x,y,z}$, the latter being the Pauli matrices.

The Hamiltonian for a single pulse can be written in the coordinate system, rotating around z-axes with the frequency ν_p. Thus, for one pulse the model is described by the *stationary* Hamiltonian (below, $\varphi_p = \pi/2, \Omega_p = \Omega, \nu_p = \nu$). We mainly study the nearest-neighbor interaction (*N-interaction*) between qubits for the *dynamical* case, $J_{k,n} = J\ \delta_{n,k+1}$, and when all $J_{k,k+1}$ are random. In contrast to the model with homogeneous magnetic field [1], we consider a constant gradient magnetic field with linear dependence on the position of the k-th qubit, $\delta_k = |\omega_{k+1} - \omega_k| \ll \omega_k = ak$, with $\Omega_p \ll J_{k,n} \ll \delta\omega_k \ll \omega_k$. Thus, for the dynamical $N-$interaction the Hamiltonian reads,

$$H = \sum_{k=0}^{L-1}\Big[-\delta_k I_k^z + \Omega I_k^y\Big] - 2J\sum_{k=0}^{L-2} I_k^z I_{k+1}^z; \qquad \delta_k = \omega_k - \nu. \tag{2}$$

For this Hamiltonian we have developed a theory [5, 6] which predicts two transitions which depend on the interaction J. The first transition was called in [6] the *delocalization border* which corresponds to the transition to *weak chaos* for,

$$J > J_{cr} \approx \frac{4a^2}{\Omega}. \tag{3}$$

By weak chaos we mean a kind of randomness in many-body states, together with the absence of the Wigner-Dyson (WD) distribution $P(s)$ for the spacings between energy levels of the Hamiltonian (2). The latter distribution is a strong evidence of quantum chaos in the energy spectra of chaotic quantum systems. It typically emerges above the delocalization border [4]. Instead, the form of $P(s)$ in our model is very close to Poisson, which is known to occur in integrable systems. Our analytic approach allows one to explain this unexpected result by showing that, indeed, the model (2) is close to the integrable one, even in the case of a completely random N-interaction [6].

The estimate (3) turns out to be very different from that obtained in [1] for a homogeneous magnetic field. Indeed, according to (3), the (weak) chaos border is independent of the number of qubits. Therefore, a magnetic field with a constant gradient strongly reduces the unwanted effects of quantum chaos. Numerical data show that one needs to have a relatively weak interaction, $J \ll J_{cr}$, in order to avoid large errors in the structure of many-body states, which appear as a result of weak chaos.

Another unexpected analytical prediction which is confirmed by the numerical data, is that the delocalization border (3) remains the same for the case when *all* qubits interact with each other with random interactions, $J_{k,n}$. However, in this case, the delocalization border (3) coincides with the onset of *strong chaos*. The latter is characterized by strong (almost Gaussian) fluctuations of the components of eigenstates, and by a WD-distribution for $P(s)$. Theoretical analysis shows that the mechanism of this transition to strong chaos is related to a strong overlap of energy bands in the spectra of the Hamiltonian (2).

We have also studied the errors that arise when preparing the uniform many-body state from the ground state. For this, we computed the evolution of the wave function in the model (1), during one pulse with $\varphi = \pi/2$. Without the interaction, $J = 0$, and at the absence of the magnetic field gradient, at the end of the pulse all components of the wave function are the same, $\psi_n^0 = 1/\sqrt{N}$. The interaction causes in some errors which can be characterized by the amplitude, $\eta = \langle ||\psi_n| - \psi_n^0|\rangle_n$, and the phase, $\phi = \langle \arctan(Im\psi_n/Re\psi_n)\rangle_n$, where $\langle ... \rangle_n$ means the average over different n components. Numerical data show that the errors decrease with an increase of Ω as $\eta \propto \Omega^{-2}$, and $\phi \propto \Omega^{-1}$ in agreement with simple analytical estimates. As one can see, the delocalization border does not influence the errors. This means that weak chaos is not important for this kind of evolution ("non-selective") of our system. Indeed, this evolution lasts only a short time ($\tau = \pi/\Omega$) compared with the inverse distance between nearest levels inside the energy band. Therefore, when the bands are non-overlapped, weak chaos does not influence the dynamics. On the other hand, as Ω decreases, the bands start to overlap which strongly increases errors in the wave function.

The work of GPB and VIT was supported by the Department of Energy (DOE) under the contract W-7405-ENG-36, by the National Security Agency (NSA) and by the Advanced Research and Development Activity (ARDA). FMI acknowledges the support by CONACyT (Mexico) Grant No. 34668-E.

References

[1] B. Georgeot and D.L. Shepelyansky, Phys. Rev. E., **62**, 3504 (2000); ibid, 6366.

[2] V.V. Flambaum. Aust. J.Phys. **53**, N4, (2000); P.G. Silvestrov, H. Schomeraus, and C.W.J. Beenakker, quant-ph/0012119.

[3] B.L. Altshuler, Y. Gefen, A. Kamenev and L.S. Levitov, Phys. Rev. Lett., **78**, 2803 (1997); V.V. Flambaum and F.M. Izrailev, Phys. Rev. **E 56**, 5144 (1997).

[4] F.M. Izrailev, to appear in Proceedings of the Nobel Simposia "Quantum Chaos Y2K", Physica Scripta, 2001; cond-mat/0009207.

[5] G.P. Berman, F. Borgonovi, F.M. Izrailev, and V.I. Tsifrinovich, quant-ph/0006095.

[6] G.P. Berman, F. Borgonovi, F.M. Izrailev, and V.I. Tsifrinovich, quant-ph/0012106.

[7] G.P. Berman, G.D. Doolen, R. Mainieri, and V.I. Tsifrinovich, *Introduction to Quantum Computers*, World Scientific Publishing Company, 1998.

[8] G.P. Berman, G.D. Doolen, G.V. Lopez, V.I. Tsifrinovich, Phys. Rev. B, **61**, 2305 (2000); Phys. Rev. A, **61**, 2307 (2000).