(α, β)-geometries from polar spaces

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Chapter 1

Introduction

The purpose of this course is to give an overview of results on (α, β)-geometries with an emphasis on those geometries related to polar spaces.
Before we treat these geometries, we want to give a quick overview of some general concepts we will need in the rest of this course. This will also give us the opportunity to fix the notation we are using, although sometimes the notation is not consistent over the whole course.

1.1 Generalities on geometries

An *incidence structure* (a terminology going back to Dembowski [33]) is a triple $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}(\neq \emptyset)$ the set of points, $\mathcal{L}(\neq \emptyset)$ the set of lines and a symmetric incidence relation $\mathrm{I} \subseteq (\mathcal{P} \times \mathcal{L}) \cup (\mathcal{L} \times \mathcal{P})$. The elements of I will also be called the *flags* of $\mathcal{S}$. In this course both the sets $\mathcal{P}$ and $\mathcal{L}$ will be finite. In a lot of cases, lines will be subsets of the point set $\mathcal{P}$ and the incidence I will be the symmetrized membership.
The *dual* of an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is the incidence structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D)$ with $\mathcal{P}^D = \mathcal{L}$, $\mathcal{L}^D = \mathcal{P}$, and I^D=I.
Isomorphisms (or collineations), anti-isomorphisms (or correlations), automorphisms, anti-automorphisms, involutions and polarities of incidence structures are defined in the usual way, and we will not explicitly define them here.
In more recent standard works (for instance in [8]) an incidence structure is called a *rank* 2 *geometry* $\mathcal{S}$ or a $\{0, 1\}$-geometry (note the curly brackets!), being a special case of what is commonly known as a *rank n geometry*, where n sets $\mathcal{P}_0, \mathcal{P}_1, \ldots, \mathcal{P}_{n-1}$ are involved with an incidence relation between any two of them. Elements of $\mathcal{P}_i$ are then called elements of *type i*. As we are only dealing with 2 sets $\mathcal{P}_0 = \mathcal{P}$ and $\mathcal{P}_1 = \mathcal{L}$ we will rather use the Dembowski terminology. We will sometimes call $\mathcal{S}$ for short a *geometry.*
An incidence structure $\mathcal{S}$ is called a *partial linear space*, if each point is on at least 2 lines, if all lines have at least two points and if any two distinct

points in $\mathcal{P}$ are incident with at most one line, or equivalently, if any two distinct lines are incident with at most one point. Some authors call this a *semi-linear space*. Lines incident with only 2 points, are called *thin lines*. If all lines are thin lines, then $\mathcal{S}$ is called a *thin partial linear space*. If all lines are incident with at least 3 points and if every point is incident with at least 3 lines, the partial linear space is called *thick*. Two points are said to be *collinear* if they are incident with a common line. Note that a point is collinear with itself. Dually, two lines are said to be *concurrent* if they are incident with a common point. We will denote collinear points x and y (resp. concurrent lines L and M) by $x \sim y$, (resp., $L \sim M$) while $x \not\sim y$ (resp., $L \not\sim M$) means that x and y are not collinear (resp., L and M are not concurrent). If $x \sim y$ (resp., $L \sim M$) we may also say that x (resp., L) is *orthogonal* or *perpendicular* to y (resp., M). The line (resp., point) which is incident with distinct collinear points x, y (resp., distinct concurrent lines L, M) is denoted by xy (resp., LM or $L \cap M$). We will often use the notation $x^\perp$ for the set of points collinear with a point x (hence $x \in x^\perp$). Dually, $L^\perp$ will denote the set of lines concurrent to a line L.
If any two different points are collinear, then $\mathcal{S}$ is called a *linear space*.
In this course we will mainly deal with a quite special class of partial linear spaces. They will have the next two properties:

($\mathcal{S}_1$): Each point is incident with $t+1\,(t \geq 1)$ lines.

($\mathcal{S}_2$): Each line is incident with $s+1\,(s \geq 1)$ points.

A partial linear space $\mathcal{S}$ satisfying these two properties will be called a partial linear space of *order* (s,t), if $s = t$, then the partial linear space will be said to have order s. Let (x, L) be an anti-flag of $\mathcal{S}$, i.e. x is a point and L is a line of $\mathcal{S}$, such that x is not incident with L. We denote by $\alpha(x, L)$ the number of points on L collinear with x, or equivalently the number of lines through x concurrent with L. We will sometimes call $\alpha(x, L)$ the *incidence number* of the anti-flag (x, L). Of special interest here, will be those partial linear spaces of order (s,t) in which $\alpha(x, L)$ can take only a few values.
For a lot of the examples we will encounter in this course the points and lines are points and lines of a projective or affine space. To be more precise, an incidence structure $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ will be called *projective* if $\mathcal{P}$ is a subset of the point set of some projective space $\mathsf{PG}(d,q)$, $\mathcal{L}$ is a set of lines of $\mathsf{PG}(d,q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathsf{PG}(d,q)$. We also say that $\mathcal{S}$ is *embedded* in $\mathsf{PG}(d,q)$. If $\mathsf{PG}(d',q)$ is the subspace of $\mathsf{PG}(d,q)$ generated by all points of $\mathcal{P}$, then we say that $\mathsf{PG}(d',q)$ is the *ambient space* of $\mathcal{S}$.
In the same way we define $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ to be *affine* or to be *embedded* in the finite affine space $\mathsf{AG}(d,q)$ if $\mathcal{P}$ is a subset of the point set of $\mathsf{AG}(d,q)$, $\mathcal{L}$ is a set of lines of $\mathsf{AG}(d,q)$, $\mathcal{P}$ is the union of all members of $\mathcal{L}$, and the incidence relation I is the one induced by that of $\mathsf{AG}(d,q)$. If $\mathsf{AG}(d',q)$ is the

subspace of $\mathsf{AG}(d,q)$ generated by all points of $\mathcal{P}$, then we say that $\mathsf{AG}(d',q)$ is the *ambient space* of $\mathcal{S}$.
A special type of affine embedding is the so-called *linear representation* of a geometry of order (s,t) in $\mathsf{AG}(n+1,s+1)$. It is an embedding of $\mathcal{S}=(\mathcal{P},\mathcal{L},\mathrm{I})$ in $\mathsf{AG}(n+1,s+1)$ such that the line set $\mathcal{L}$ of $\mathcal{S}$ is a union of parallel classes of lines of $\mathsf{AG}(n+1,s+1)$ and so the point set $\mathcal{P}$ of $\mathcal{S}$ is the point set of $\mathsf{AG}(n+1,s+1)$. Then the lines of $\mathcal{S}$ define a set of points $\mathcal{K}$ of size $t+1$ in the hyperplane at infinity $\Pi_\infty = \mathsf{PG}(n,s+1)$ of $\mathsf{AG}(n+1,s+1)$. If (x,L) is an anti-flag of $\mathcal{S}$, then the line $xL \cap \Pi_\infty$ intersects $\mathcal{K}$ in $\alpha(x,L)+1$ points. A linear representation of an incidence structure $\mathcal{S}$ in $\mathsf{AG}(n+1,s+1)$ will be denoted by $T_n^*(\mathcal{K})$. We shall give several examples in this course.

1.2 Graphs

1.2.1 Some general definitions from graph theory

A finite *graph* $\Gamma=(X,E)$ is a structure consisting of a set $X(\neq\emptyset)$ with v elements and a set E of unordered pairs of X. The elements of X are called the *vertices* of the graph Γ, while the elements of E are called the *edges*. If x and y are two different vertices such that $\{x,y\}\in E$, then x and y are called *adjacent* and we write $x\sim y$; if $\{x,y\}\notin E$ then we denote this by $x \not\sim y$. Remark that $x\not\sim x$. If E is the set of all unordered pairs of X, then Γ is called the *complete graph on v vertices* and is denoted by K_v. The *complement* Γ^C of a graph $\Gamma=(X,E)$ is the graph $\Gamma^C=(X^C,E^C)$ with $X^C=X$ and $E^C=X^{|2|}\setminus E$. The *line graph* $L(\Gamma)$ of a graph Γ is the graph with vertices the edges of Γ, two distinct edges being adjacent if and only if they have a common vertex.
A *path* of length m from x to y, is a sequence of vertices $x=x_0,x_1,x_2,\ldots,x_m=y$ in the graph, such that $x_i\neq x_{i+2}$, $0\le i\le m-2$ and $x_i\sim x_{i+1}$, $0\le i\le m-1$. If $x=y$ then any such path of length at least 3 will be called a *circuit*. Two distinct vertices x and y of a graph Γ are at *distance* $\delta(x,y)$, provided there exists a path of length $\delta(x,y)$ between these vertices and there exists no shorter one; by definition a vertex has distance 0 from itself. A vertex has distance 1 from all its adjacent vertices. We will denote by $\Gamma_i(x)$ the set of all vertices of Γ at distance i from x. For convenience we will use $\Gamma(x)$ for the set $\Gamma_1(x)$. If $|\Gamma(x)|=k$ for all vertices x, then Γ is called *regular* of *valency* k. A graph is *connected* if and only if for any two distinct vertices x and y, there is at least one path connecting these 2 vertices. The *diameter* d of a connected graph Γ is the maximum value of the distance function $\delta(x,y)$. The *girth* of a graph Γ having at least one circuit, is the length of its shortest circuit.
For any graph Γ of diameter d, and vertex set $\{x_1,\ldots,x_v\}$, the *distance*

matrices A_h, $h = 0, \ldots, d$, are the $v \times v$ matrices defined as follows:

$$(A_h)_{ij} = \begin{cases} 1 & \text{if } \delta(x_i, x_j) = h \\ 0 & \text{otherwise.} \end{cases}$$

Given a partial linear space $\mathcal{S}$, one may define the *point graph* or *collinearity graph* $\Gamma(\mathcal{S})$, by taking as vertices the points of $\mathcal{S}$. Two different vertices are adjacent whenever they are collinear. Remark that we are using the same symbol ($\sim$) for the collinearity relation as for the adjacency relation, although a point x is collinear with itself but not adjacent to itself. A geometry $\mathcal{S}$ is connected whenever its point graph $\Gamma(\mathcal{S})$ is. From now on, we will only deal with connected geometries.
The *incidence graph* $\mathcal{I}(\mathcal{S})$ is the graph with vertices the elements of $\mathcal{P} \cup \mathcal{L}$, and 2 vertices are adjacent if and only if the corresponding elements are incident; hence edges of $\mathcal{I}(\mathcal{S})$ are the flags of $\mathcal{S}$. Unlike the case of the collinearity graph, the geometry is completely determined (up to duality) by its incidence graph. Obviously, two vertices of the same type (i.e. either points or lines) in the incidence graph are connected by paths of even length. In particular a circuit in an incidence graph has even length and hence the girth is an even positive integer, say $2n$. By definition, n is called the *gonality* or *geometric girth* of $\mathcal{S}$. A *geodesic* (*based at* x) is a path γ in the incidence graph starting in x and such that the length of γ is equal to the distance $\delta(x, y)$, where y is the last element of γ. A *maximal geodesic* is a geodesic that is not properly contained in another one. The *local diameter* $d(x)$ is the length of the longest geodesic based at x, whether x be a point or a line. The *point-diameter* d_p (resp. *line-diameter* d_l) of $\mathcal{S}$ is the greatest value taken by $d(x)$ for x a point (resp. a line). The *diameter* d of a geometry $\mathcal{S}$ is the diameter of the incidence graph $\mathcal{I}(\mathcal{S})$, hence it is the largest of the two numbers d_p, d_l.
Finally, the *flag graph* of a geometry $\mathcal{S}$ is the graph with as vertices the flags of $\mathcal{S}$, where two flags are adjacent whenever they share exactly one element. The *flag-diameter* d^* of $\mathcal{S}$ is the diameter of the flag graph.

1.2.2 Strongly regular graphs

A regular graph Γ is called a strongly regular graph (notation $\mathsf{srg}(v, k, \lambda, \mu)$) provided:

1. any two vertices x and y, $x \sim y$, are both adjacent to a constant number λ of vertices (independent of the choice of the adjacent pair $\{x, y\}$);

2. any two distinct vertices x and y, $x \not\sim y$, are both adjacent to a constant number μ of vertices (independent of the choice of the non-adjacent pair $\{x, y\}$).

As we will exclude disconnected graphs and their complements, we may assume $0 < \mu < k < v - 1$. It is easy to check that the complement of a $\mathsf{srg}(v, k, \lambda, \mu)$ is a $\mathsf{srg}(v, v - k - 1, v - 2k + \mu - 2, v - 2k + \lambda)$.
The distance matrix $A_1 = A$ (or the $(0, 1)$ adjacency matrix) of a $\mathsf{srg}(v, k, \lambda, \mu)$ satisfies

$$AJ = kJ, \qquad A^2 + (\mu - \lambda)A + (\mu - k)I = \mu J,$$

where J is the all-one matrix. Hence A has the valency k as an eigenvalue with multiplicity 1, and two other eigenvalues r and l ($r > 0$ and $l < 0$) with $r + l = \lambda - \mu$ and $rl = \mu - k$.
There are some necessary conditions known for the existence of a $\mathsf{srg}(v, k, \lambda, \mu)$. We will summarize the most important ones in the next theorem. For the proofs and more information on strongly regular graphs we refer to [2], [4], [13], [15], [55] and [71].

Theorem 1.2.1
If Γ is a $\mathsf{srg}(v, k, \lambda, \mu)$ then:

1. $k(k - \lambda - 1) = \mu(v - k - 1)$.

2. *The multiplicities of the eigenvalues r and l of A are respectively*

$$f = \frac{-k(l + 1)(k - l)}{(k + rl)(r - l)} \quad \text{and} \quad g = \frac{k(r + 1)(k - r)}{(k + rl)(r - l)};$$

   they clearly have to be integers.

3. *The eigenvalues $r > 0$ and $l < 0$ are both integers, except for one family of graphs, the so-called conference graphs, which are $\mathsf{srg}(2k + 1, k, \frac{k}{2} - 1, \frac{k}{2})$. For a conference graph the number of vertices can be written as a sum of two squares, and the eigenvalues are $\frac{-1 \pm \sqrt{v}}{2}$.*

4. *The Krein conditions:*

$$(r + 1)(k + r + 2rl) \leq (k + r)(l + 1)^2,$$

$$(l + 1)(k + l + 2rl) \leq (k + l)(r + 1)^2.$$

5. *The absolute bound:*

$$v \leq \frac{1}{2}f(f + 3), \qquad v \leq \frac{1}{2}g(g + 3).$$

A lot of examples of strongly regular graphs are known, see for instance [44] and [4].

1.3 (α, β)-geometries

An (α, β)-geomety $\mathcal{S}$ is a connected partial linear space of order (s, t), such that for any anti-flag (x, L) the incidence number $\alpha(x, L)$ can have at most 2 values α or β.
If $\alpha = \beta$, then $\mathcal{S}$ is known under the name *partial geometry* and was introduced by Bose [2]. The numbers s, t and α are called the *parameters* of $\mathcal{S}$, and we will use the notation $\mathsf{pg}(s, t, \alpha)$.
It is an easy excercise to prove that the point graph $\Gamma(\mathcal{S})$ of a $\mathsf{pg}(s, t, \alpha)$ $\mathcal{S}$ is a

$$\mathsf{srg}\left((s+1)\frac{(st+\alpha)}{\alpha}, s(t+1), s-1+t(\alpha-1), \alpha(t+1)\right).$$

Remarks

1. If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry with parameters s, t, α, then the dual structure $\mathcal{S}^D = (\mathcal{P}^D, \mathcal{L}^D, \mathrm{I}^D) = (\mathcal{L}, \mathcal{P}, \mathrm{I})$, is a partial geometry with parameters $s^D = t$, $t^D = s$ and $\alpha^D = \alpha$.
2. $|\mathcal{P}| = v = (s+1)\frac{(st+\alpha)}{\alpha}$ and $|\mathcal{L}| = b = (t+1)\frac{(st+\alpha)}{\alpha}$.
3. The partial geometries can be divided into four (non-disjoint) classes.
   - (a) The partial geometries with $\alpha = 1$, that are the generalized quadrangles. See for instance [53] for more information on generalized quadrangles.
   - (b) The partial geometries with $\alpha = s+1$ or dually $\alpha = t+1$, i.e. the 2–$(v, s+1, 1)$ designs and their duals. We refer to any standard book on design theory for more details.
   - (c) The partial geometries with $\alpha = s$ or dually $\alpha = t$. The partial geometries with $\alpha = t$ are the *nets of order* $s+1$ *and degree* $t+1$, introduced by Bruck [6] (note that we are here in conflict with our former definition of order).
   - (d) Finally, the so–called *proper* partial geometries with $1 < \alpha < \min(s, t)$.

A second class of (α, β)-geometries $\mathcal{S}$ are the ones with $\beta = 0$. However the point graph $\Gamma(\mathcal{S})$ of a $(0, \alpha)$ geometry is not necessarily a strongly regular graph. If $\Gamma(\mathcal{S})$ is strongly regular, then the geometry is called a *semipartial geometry* and was introduced by Debroey and Thas [29]. A special case of such geometries corresponds to $\alpha = 1$; these geometries are called *partial quadrangles* and were introduced by Cameron [12] as a generalization of the generalized quadrangles (hence the name, although in our context we better should call them *semigeneralized quadrangles*). The $(0, s)$-geometries are known under the name of *copolar spaces* (see [37]). The $(1, s+1)$-geometries

are called *polar spaces*, and were introduced and classified in [10] (see next section). Under mild nonsingularity conditions, also the copolar spaces are classified, see later.
The partial and semipartial geometries can be regarded as generalizations of the generalized quadrangles. But generalized quadrangles, together with the projective planes, can also be seen as special cases of the more general *(weak) generalized polygons*. These incidence structures are incidence structures whose diameter is equal to the gonality, but we will not treat them as such.

1.4 Polar spaces: definition and properties

Polar spaces describe the geometry of vector spaces carrying a reflexive sesquilinear form or a quadratic form in much the same way as projective spaces describe the geometry of vector spaces. In the sequel we will always assume that everything is finite. There are three types of forms with an associated polar space. The space associated with the alternating bilinear form is called the *symplectic* polar space; the one with the Hermitian form is called the *unitary or Hermitian* polar space; finally with the quadratic form is associated the *orthogonal* polar space. These polar spaces are commonly called *classical polar spaces*. Although we assume here some preknowledge on the theory of polar spaces, we will give some of the basic definitions and properties. The reader is referred to the literature for more information, e.g. [8], [14], [41].
Let $V(n+1,q)$ be a vector space carrying a reflexive sesquilinear form σ of one of the three types. Recall that the sesquilinear form defines uniquely the polarity in the associated projective space $\mathsf{PG}(n,q)$, unless q is even and σ is orthogonal, in which case one has to use the *quadratic form* which we will also denote by σ. A subspace W of V is called *totally isotropic* if the sesquilinear form σ vanishes identically on W, i. e. if $W \subseteq W^\sigma$. In case of an orthogonal polarity, a subspace on which the quadratic form σ vanishes is called a *totally singular subspace*. We shall often call the maximal totally isotropic subspaces or maximal totally singular subspaces of a polar space $\mathcal{S}$, the *generators* of $\mathcal{S}$.
One can prove that the classical polar space can be regarded as a geometry whose elements are called subspaces satisfying the following properties (see for instance [14] for a proof).

[P1] Each subspace is isomorphic to a projective space of dimension at most $r-1$.

[P2] The intersection of any family of subspaces is a subspace.

[P3] If W is a subspace of dimension $r-1$, and p a point not in W, then the set of points p' such that the line pp' is totally isotropic or totally singular, is a hyperplane in W, and the union of those lines pp' is a subspace of dimension $r-1$.

[P4] There exist two disjoint subspaces of dimension $r-1$.

A geometry consisting of a set of points with a collection of distinguished subsets called subspaces, satisfying the axioms **[P1]-[P4]** is called an *abstract polar space of rank* r.
An important simplification of the axioms **[P1]-[P4]** was obtained by Buekenhout and Shult [10]. Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be an incidence structure of points and lines (or a point-line geometry). A *subspace* of $\mathcal{S}$ will be a point set which contains any line through two of its points; it is *singular* provided any two of its points are collinear.

Theorem 1.4.1 ([10])
Suppose that an incidence structure $\mathcal{S}$ has the following properties:

(i) if p is a point not on a line L, then p is collinear with one or all points of L;

(ii) any line contains at least three points;

(iii) no point is collinear with all others;

(iv) any chain (w.r.t. inclusion) of singular subspaces is finite.

Then the singular subspaces constitute an abstract polar space.

Remark

It is clear that the generalized quadrangles are the polar spaces of rank 2, and these are far of being completely classified. We might expect however a classification to be possible for sufficiently large rank. This has been proved indeed by Tits, building on work of Veldkamp [72].

Theorem 1.4.2 ([68])
All finite polar spaces of rank at least 3 are classical.

We shall use the following notation for classical polar spaces:

$\mathsf{W}_n(q)$: the polar space arising from a symplectic polarity of $\mathsf{PG}(n,q)$, n odd and $n \geq 3$: here $r = (n+1)/2$;

$\mathsf{Q}(2n,q)$: the polar space arising from a non-singular quadric in $\mathsf{PG}(2n,q)$, $n \geq 2$: here $r = n$;

$\mathsf{Q}^+(2n+1,q)$: the polar space arising from a non-singular hyperbolic quadric in $\mathsf{PG}(2n+1,q)$, $n \geq 1$: here $r = n+1$;

$\mathsf{Q}^-(2n+1,q)$: the polar space arising from a non-singular elliptic quadric in $\mathsf{PG}(2n+1,q)$, $n \geq 2$: here $r = n$;

$\mathsf{H}(n,q^2)$: the polar space arising from a non-singular Hermitian variety H in $\mathsf{PG}(n,q^2)$, $n \geq 3$: for n odd $r = (n+1)/2$, for n even $r = n/2$.

Theorem 1.4.3
For q even, the polar space $\mathsf{Q}(2n,q)$ is isomorphic to the polar space $\mathsf{W}_{2n-1}(q)$.

Proof. The theorem is easily proved by projection of $\mathsf{Q}(2n,q)$ from the nucleus onto a $\mathsf{PG}(2n-1,q) \subset \mathsf{PG}(2n,q)$ not containing the nucleus. ■

Let $|\mathcal{S}|$ denote the number of points of the polar space $\mathcal{S}$, and let $\Sigma(\mathcal{S})$ be the set of generators of $\mathcal{S}$; all elements of $\Sigma(\mathcal{S})$ have dimension $r-1$. For a proof of the following theorems we refer e.g. to [41] or [59].

Theorem 1.4.4
The numbers of points of the finite classical polar spaces are given by the following formulae:

$$\begin{aligned}
|\mathsf{W}_n(q)| &= (q^{n+1}-1)/(q-1),\\
|\mathsf{Q}(2n,q)| &= (q^{2n}-1)/(q-1),\\
|\mathsf{Q}^+(2n+1,q)| &= (q^n+1)(q^{n+1}-1)/(q-1),\\
|\mathsf{Q}^-(2n+1,q)| &= (q^n-1)(q^{n+1}+1)/(q-1),\\
|\mathsf{H}(n,q^2)| &= (q^{n+1}+(-1)^n)(q^n-(-1)^n)/(q^2-1).
\end{aligned}$$

Theorem 1.4.5
The numbers of generators of the finite classical polar spaces are given by the following formulae:

$$\begin{aligned}
|\Sigma(\mathsf{W}_n(q))| &= (q+1)(q^2+1)\dots(q^{(n+1)/2}+1),\\
|\Sigma(\mathsf{Q}(2n,q))| &= (q+1)(q^2+1)\dots(q^n+1),\\
|\Sigma(\mathsf{Q}^+(2n+1,q))| &= 2(q+1)(q^2+1)\dots(q^n+1),\\
|\Sigma(\mathsf{Q}^-(2n+1,q))| &= (q^2+1)(q^3+1)\dots(q^{n+1}+1),\\
|\Sigma(\mathsf{H}(2n,q^2))| &= (q^3+1)(q^5+1)\dots(q^{2n+1}+1),\\
|\Sigma(\mathsf{H}(2n+1,q^2))| &= (q+1)(q^3+1)\dots(q^{2n+1}+1).
\end{aligned}$$

1.5 m-systems, ovoids and spreads of polar spaces

Let $\mathcal{S}$ be a finite classical polar space of rank r, with $r \geq 2$. A *partial m-system* [56] of $\mathcal{S}$, with $0 \leq m \leq r-1$, is any set $\{\pi_1, \pi_2, \dots, \pi_k\}$ of $k(\neq 0)$ totally singular m-spaces of $\mathcal{S}$ such that no generator containing π_i has a point in common with $(\pi_1 \cup \pi_2 \cup \dots \cup \pi_k) \setminus \pi_i$, $i = 1, 2, \dots, k$. Note that a partial 0-system is a k-cap, while a partial $(r-1)$-system, a set of generators that are pairwise disjoint, is called a *partial spread*.

Theorem 1.5.1 ([56])
Let M be a partial m-system of the finite classical polar space $\mathcal{S}$, then

$$\begin{array}{lcll} \textit{for } \mathcal{S} & = & \mathsf{W}_n(q), & |M| \leq q^{(n+1)/2}+1, \\ \textit{for } \mathcal{S} & = & \mathsf{Q}(2n,q), & |M| \leq q^n+1, \\ \textit{for } \mathcal{S} & = & \mathsf{Q}^+(2n+1,q), & |M| \leq q^n+1, \\ \textit{for } \mathcal{S} & = & \mathsf{Q}^-(2n+1,q), & |M| \leq q^{n+1}+1, \\ \textit{for } \mathcal{S} & = & \mathsf{H}(2n,q^2), & |M| \leq q^{2n+1}+1, \\ \textit{for } \mathcal{S} & = & \mathsf{H}(2n+1,q^2), & |M| \leq q^{2n+1}+1. \end{array}$$

If for $|M|$ the upper bound is reached, which is by the way independent of m, then M is called an *m-system* of $\mathcal{S}$. A 0-system is called *an ovoid* of $\mathcal{S}$, while an $(r-1)$-system is called a *spread*.

Chapter 2

Examples of (α, β)-geometries from polar spaces

2.1 Partial and semipartial geometries

We have seen in section 1.3 that the point graph of a (semi)partial geometry is strongly regular. The other way round, if a strongly regular graph has the parameters of the point graph of a (semi)partial geometry, such a graph is called a *pseudo-(semi)geometric graph* then we can try to check whether it is indeed the point graph of a (semi)partial geometry or not. A pseudo-(semi)geometric graph which is indeed the point graph of at least one (semi)partial geometry is called *(semi)geometric.* We have to look for a collection of maximal cliques in a pseudo-(semi)geometric graph that could yield lines of a putative (semi)partial geometry. So the chosen maximal cliques can intersect in at most one point. In general these questions are quite difficult but for some graphs one can do a very detailed study. Especially the graphs related to classical geometrical objects such as quadrics and other polar spaces are candidates for such a study.
Note that unlike a pseudo-geometric graph, a graph can be pseudo-semigeometric for more than one set of values s, t, α, μ. And so it is more difficult to check whether an infinite class of graphs is pseudo-semigeometric or not.

Theorem 2.1.1

1. *A pseudo-geometric (s, t, α)–graph Γ is geometric if and only if there is a collection $\mathcal{C}$ of maximal cliques of Γ such that every edge is contained in a unique element of $\mathcal{C}$.*

2. *Let Γ be a pseudo-semigeometric (s, t, α, μ)-graph and let $\mathcal{C}$ be a collection of cliques of Γ of size $s + 1$ such that every edge is contained in a unique element $\mathcal{C}$. Then Γ is semigeometric if and only if for every $C \in \mathcal{C}$,*
$$|(\cup_{x\in C}\Gamma(x)) \setminus C| = \frac{s(s+1)t}{\alpha}.$$

Remark

Most attempts (with some exceptions) to construct a (semi)partial geometry from a pseudo-(semi)geometric graph were not successful. We refer for example to De Clerck, Gevaert and Thas [23], De Clerck and Tonchev [26], Spence [57].

2.2 The collinearity graph of a polar space

2.2.1 Pseudo-geometric graphs

Consider a polar space P of rank at least two. Define the graph $\Gamma(P)$ with vertex set the set of points of the polar space, two vertices being adjacent whenever they are contained in a line of P. It is well known (see for example [44]) that the graphs $\Gamma(P)$ are strongly regular.
Using the parameters of the collinearity graphs one can check the following results.

Theorem 2.2.1

1. *The graph $\Gamma(\mathsf{Q}^-(2m+1,q))$, $m \geq 2$, is a pseudo-geometric*
$$(q\frac{q^{m-1}-1}{q-1}, q^m, \frac{q^{m-1}-1}{q-1})\text{–graph.}$$

2. *The graph $\Gamma(\mathsf{Q}(2m,q))$, $m \geq 2$, and the graph $\Gamma(W_{2m-1}(q))$, $m \geq 2$, are both pseudo-geometric*
$$(q\frac{q^{m-1}-1}{q-1}, q^{m-1}, \frac{q^{m-1}-1}{q-1})\text{–graphs.}$$

3. *The graph $\Gamma(\mathsf{Q}^+(2m+1,q))$ is pseudo-geometric if and only if $m=1,2$. If $m=1$, then it is a pseudo-geometric $(q,1,1)$-graph, if $m=2$, then it is a pseudo-geometric $(q(q+1),q,q+1)$-graph.*

4. *The graph $\Gamma(\mathsf{H}(2m+1,q^2))$, $m \geq 1$, is a pseudo-geometric*
$$(q^2\frac{q^{2m}-1}{q^2-1}, q^{2m-1}, \frac{q^{2m}-1}{q^2-1})\text{–graph.}$$

5. *The graph $\Gamma(\mathsf{H}(2m,q^2))$, $m \geq 2$, is a pseudo-geometric*
$$(q^2\frac{q^{2m-2}-1}{q^2-1}, q^{2m-1}, \frac{q^{2m-2}-1}{q^2-1})\text{–graph.}$$

Note that if the graph $\Gamma(P)$, with P a polar space in $\mathsf{PG}(n,q)$, is geometric, then the lines of the geometry containing a given point define a spread of a polar space P' in $\mathsf{PG}(n-2,q)$ with P and P' of the same type.
Several authors have investigated whether those collinearity graphs of polar spaces that are pseudo-geometric are indeed geometric. We refer to [23], [32], [50], [52], [53], and [67] for more details. The known results can be summarized as follows.

Theorem 2.2.2

1. *The points and lines of the classical polar spaces* $\mathsf{Q}(4,q), \mathsf{Q}^+(3,q), \mathsf{Q}^-(5,q), W_3(q), \mathsf{H}(3,q^2)$ *and* $\mathsf{H}(4,q^2)$ *yield generalized quadrangles and so the corresponding graphs are geometric.*

2. *The points of* $\mathsf{Q}^+(5,q)$*, together with the planes of one family of generators form a dual 2-design, and so the corresponding graph is geometric.*

3. *If the graph* $\Gamma(\mathsf{Q}(2m,q))$*,* $m \geq 3$*, is geometric, then also the graph* $\Gamma(\mathsf{Q}^-(2m-1,q))$ *is geometric.*

4. *The following pseudo-geometric graphs are not geometric:*

   - $\Gamma(W_5(q))$*,* $\Gamma(W_7(q))$ *and* $\Gamma(W_9(2))$*;*
   - $\Gamma(\mathsf{Q}(4n,q))$*,* $n \geq 2$*;*
   - $\Gamma(\mathsf{Q}(6,q))$*,* q *even;* $\Gamma(\mathsf{Q}(10,2))$*;*
   - $\Gamma(\mathsf{Q}(4n+2,q))$*,* q *odd;*
   - $\Gamma(\mathsf{Q}^-(7,2))$*;* $\Gamma(\mathsf{Q}^-(9,2))$*;*
   - $\Gamma(\mathsf{H}(2m+1,q^2)), m \geq 2$*;*
   - $\Gamma(\mathsf{H}(6,4))$.

So, we may conclude that trying to construct partial and semipartial geometries from the point graph of a polar space is quite unsuccesful. Looking to the complement of the point graph of the polar space does not give more results.
So, we should try to construct geometries in another way. Before we are doing this we introduce another type of graphs, the so-called *copolar graphs.*

2.2.2 Copolar spaces and copolar graphs

A (reduced) copolar space $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a $(0,s)$-geometry, it was introduced by J. I. Hall [37].
A copolar graph Γ is a (non–empty) graph such that each pair of non–adjacent vertices is contained in at most one coclique $\langle x,y\rangle$ of Γ with the property that each vertex not contained in $\langle x,y\rangle$ is adjacent to one vertex or to all vertices of $\langle x,y\rangle$. The non–collinearity graph of a copolar space is

a copolar graph where the lines correspond to the cocliques, and conversely, given a copolar graph Γ there is associated with it a copolar space by taking as lines the cocliques of Γ.
J. I. Hall [37] classified all finite (reduced) copolar spaces of order s, $s \geq 2$.

Theorem 2.2.3 ([37])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P} \neq \emptyset$, is a finite reduced copolar space of order s, $s \geq 2$, then up to an isomorphism we have one of the following possibilities.

1. *$|\mathcal{P}| = s + 1$ and $|\mathcal{L}| = 1$;*
2. *$\mathcal{P}$ is the vertex set of a Moore graph [43] of valency $s + 1$ and each element of $\mathcal{L}$ is the set of $s + 1$ vertices that are adjacent to a vertex of the graph; note that $s \in \{1, 2, 6, 56\}$.*
3. *$\mathcal{P}$ is the set of all pairs of a finite set X, $|X| \geq 5$, and an element of $\mathcal{L}$ is the set of three pairs contained in a 3–subset of X ($s = 2$).*
4. *$\mathcal{P}$ is the set of points of $\mathsf{PG}(2m-1, 2)$ ($m > 2$) not belonging to a non–singular quadric $\mathsf{Q}^{\pm}(2m-1, 2)$ and the elements of $\mathcal{L}$ are the external lines with respect to the quadric ($s = 2$).*
5. *$\mathcal{P}$ is the point set of $\mathsf{PG}(2m-1, q)$ ($m \geq 2$) and $\mathcal{L}$ is the set of hyperbolic lines of a non–singular symplectic polarity ($s = q$).*

2.3 The partial geometries $\mathbf{PQ}^+(4n-1, q), q = 2$ or $q = 3$

2.3.1 Some properties of hyperbolic quadrics in $\mathsf{PG}(2m-1, q)$

Let $\mathsf{Q}^+ = \mathsf{Q}^+(2m-1, q)$, $m \geq 2$, be a hyperbolic quadric in $\mathsf{PG}(2m-1, q)$ (the quadric with projective index $m - 1$). The set of maximal totally isotropic or singular subspaces on a hyperbolic quadric Q^+ is divided into two disjoint families $\mathcal{D}_1$ and $\mathcal{D}_2$. Two maximal totally isotropic or singular subspaces on the quadric are in the same family if and only if the codimension of their intersection has the parity of $m - 1$ (see [42] for more details on quadrics).
Assume q is odd, and let x and y be two points of $\mathsf{PG}(2m-1, q) \setminus \mathsf{Q}^+$. Then x and y are called equivalent if and only if there exists a point $z \in \mathsf{PG}(2m-1, q) \setminus \mathsf{Q}^+$ such that the lines xz and yz are tangent lines of Q^+. This relation can also be defined as follows. Embed Q^+ in the non-singular quadric Q of $\mathsf{PG}(2m, q)$. The pole of $\mathsf{PG}(2m-1, q)$ with respect to Q is denoted by p. Then x and y are equivalent if and only if the lines xp and yp are both secants or are both exterior lines with respect to Q. The proof that this relation is indeed an equivalence relation was given by Thas [60]. There are two equivalence classes E_1 and E_2. For some i, $\mathsf{Q}^+ \cup E_i$ is the

projection of the non-singular quadric Q of $\mathsf{PG}(2m, q)$, from the point p onto $\mathsf{PG}(2m-1, q)$.

2.3.2 The partial geometry $\mathbf{PQ}^+(4n-1, 2)$

In [22] an infinite class of partial geometries was constructed as follows. Define a spread Σ of the non-singular hyperbolic quadric $\mathsf{Q}^+ = \mathrm{Q}^+(4n-1, 2)$, $n \geq 2$, in $\mathsf{PG}(4n-1, 2)$ to be a (maximal) set of $2^{2n-1}+1$ disjoint $(2n-1)$–dimensional spaces on Q^+. Let Σ be a spread of $\mathsf{Q}^+ = \mathsf{Q}^+(4n-1, 2)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1, 2) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ the set of points of $\mathsf{PG}(4n-1, 2)$ not on the quadric, $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1, 2)$ is a $\mathsf{pg}(2^{2n-1}-1, 2^{2n-1}, 2^{2n-2})$.
If $n = 2$, then $\mathrm{PQ}^+(7, 2)$ is a $\mathsf{pg}(7, 8, 4)$. Cohen [19] was the first to construct a partial geometry with these parameters using the root system E_8. In [36] a $\mathsf{pg}(8, 7, 4)$ was constructed using coding theory. Kantor [46] proved that $\mathrm{PQ}^+(7, 2)$ and the dual of the geometry of Haemers–van Lint are isomorphic. Later on Tonchev [69] showed by computer that the model of Cohen and the dual of the geometry of Haemers–van Lint are isomorphic. In [23] this isomorphism is proved without the use of a computer.
Remark that non-isomorphic spreads of the quadric $\mathrm{Q}^+(4n-1, 2)$ will produce non-isomorphic partial geometries. If $2n-1$ is composite, then $\mathrm{Q}^+(4n-1, 2)$ has non-isomorphic spreads, and probably this is true for all $n > 2$ (see [45]).

2.3.3 The partial geometry $\mathbf{PQ}^+(4n-1, 3)$

For $q = 3$ an analogous construction is given by Thas [60]. Again let Σ be a spread of $\mathsf{Q}^+ = \mathsf{Q}^+(4n-1, 3)$ and let $\mathcal{L}$ be the set of all hyperplanes of the elements of Σ. Consider the incidence structure $\mathrm{PQ}^+(4n-1, 3) = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ with $\mathcal{P}$ one of the sets E_i, and with $x \mathrm{~I~} L$, $x \in \mathcal{P}$ and $L \in \mathcal{L}$, if and only if x is contained in the polar space $L^\star$ of L with respect to Q^+. One can prove that $\mathrm{PQ}^+(4n-1, 3)$ is a partial geometry with parameters $s = 3^{2n-1}-1$, $t = 3^{2n-1}$, $\alpha = 2 \cdot 3^{2n-2}$.
Up to now it is only known that $\mathsf{Q}^+(7, 3)$ has a spread. This yields a $\mathsf{pg}(26, 27, 18)$ $\mathrm{PQ}^+(7, 3)$ $(v = 1080, b = 1120)$.

2.3.4 Partial geometries derived from $\mathbf{PQ}^+(4n-1, q)$, $q = 2$ or $q = 3$

Replaceable spreads of a partial geometry

Let Φ be a spread of a $\mathsf{pg}(s, t, \alpha)$ $\mathcal{S}$, i.e. a set of $st/\alpha + 1$ lines partitioning the point set, we will refer to Φ as a pg-spread. Assume $t > 1$ and let L

be any line of $\mathcal{S} \setminus \Phi$. Let Φ_L be the set of $s+1$ lines of Φ intersecting L. We call L *regular with respect to* Φ if there exists a set of $s+1$ lines $\mathcal{L}^* = \{L_0 = L, L_1, \ldots, L_s\}$ that partitions the point set $\mathcal{P}(\Phi_L)$ of Φ_L.

Lemma 2.3.1
If L is a regular line with respect to the pg*-spread Φ of a* $\mathsf{pg}(s,t,\alpha)$ $\mathcal{S}$, *then $\alpha = t$ (hence $\mathcal{S}$ is a net and $\mathcal{P} = \mathcal{P}(\Phi_L)$) or $t \geq s+1$. Moreover if $t = s+1$, then every line M not being an element of the spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points. Conversely, if every line M not being an element of the spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points, then $t = s+1$.*

Proof. Assume that a line M_i, not being an element of the pg-spread Φ neither of $\mathcal{L}^*$, intersects $\mathcal{P}(\Phi_L)$ in a_i points. Note that there are $d = \frac{t(st+\alpha)}{\alpha} - (s+1)$ such lines M_i. By counting the ordered pairs (p, M_i), $p \in \mathcal{P}(\Phi_L)$, $p \mathrm{\ I\ } M_i$, $i = 1, \ldots, d$, in two different ways, we get

$$\sum_{i=1}^{d} a_i = (s+1)^2(t-1).$$

Counting the ordered triples (p, p', M_i), $p,\ p' \in \mathcal{P}(\Phi_L)$, $p \mathrm{\ I\ } M_i \mathrm{\ I\ } p'$, $i = 1, \ldots, d$, we get

$$\sum_{i=1}^{d} a_i(a_i - 1) = (s+1)^2 s(\alpha - 1).$$

Using the variance trick, we find

$$s(\alpha - t)^2(t - s - 1) \geq 0$$

Hence either $\alpha = t$ and then $\mathcal{P} = \mathcal{P}(\Phi_L)$, hence $a_i = s+1$, $i = 1, \ldots, d$, or $t \geq s+1$. If $t = s+1$ then $a_i = \frac{\sum a_i}{d} = \alpha$.
Conversely, assume that every line M_i, $i = 1, \ldots, d$ not being an element of the pg-spread Φ neither of $\mathcal{L}^*$ intersects $\mathcal{P}(\Phi_L)$ in α points. We count in two different ways the number of ordered pairs (p, M_i) with p a point of $\mathcal{P}(\Phi_L)$ and $p \mathrm{\ I\ } M_i$. This yields

$$(s+1)^2(t-1) = \left(\frac{t(st+\alpha)}{\alpha} - (s+1)\right)\alpha.$$

This equation simplifies to $(s-t+1)(s(t-1)+(\alpha-1)) = 0$. Hence $t = s+1$.
■

Definition

Assume that Φ is a pg-spread of a $\mathsf{pg}(s, s+1, \alpha)$ such that every line is regular with respect to Φ. Then $\mathcal{L} \setminus \Phi$ is partitioned in $\frac{s(s+1)}{\alpha} + 1$ sets $\mathcal{L}_i$ $(i = 1, \ldots, \frac{s(s+1)}{\alpha} + 1)$ each containing $s+1$ mutually skew lines. The spread Φ will be called a *replaceable* pg*-spread* for reasons that will become clear very soon. This definition generalizes the definition given in [51]; they restrict their definition to spreads of $\mathsf{pg}(2\alpha - 1, 2\alpha, \alpha)$ and they call them *regular spreads*. We prefer to use another terminology as *regular spreads* were earlier introduced in another context.

Remark

If Φ is a replaceable pg-spread of a $\mathsf{pg}(s, s+1, \alpha)$, then the incidence structure $\mathcal{D}(\Phi)$ with as points the elements of Φ and as blocks the sets Φ_L, incidence being the natural incidence, is a symmetric $2-(\frac{s(s+1)}{\alpha} + 1, s+1, \alpha)$ design. This yields extra conditions on the parameters s and α. We will prove in the sequel that $\mathrm{PQ}^+(4n-1, q)$ $(q = 2$ or $3)$ has replaceable spreads, which yield $2-(2^{2n}-1, 2^{2n-1}, 2^{2n-2})$ designs in the case $q = 2$ and $2-(\frac{3^{2n}-1}{2}, 3^{2n-1}, 2 \cdot 3^{2n-2})$ designs in the case $q = 3$. These designs have the parameters of the complement of the designs of points and hyperplanes of a $\mathsf{PG}(2n-1, q)$, $(q = 2$ or $3)$.

The construction

Let $\mathcal{S}$ be a $\mathsf{pg}(s, s+1, \alpha)$ with a replaceable pg-spread Φ. Define the following incidence structure $\mathcal{S}_\Phi = (\mathcal{P}_\Phi, \mathcal{L}_\Phi, \mathrm{I}_\Phi)$. The elements of $\mathcal{P}_\Phi$ are on the one hand the points of $\mathcal{S}$ and on the other hand the sets $\mathcal{L}_i$, $i = 1, \ldots, s(s+1)/\alpha + 1$; the set of lines $\mathcal{L}_\Phi = \mathcal{L} \setminus \Phi$. Finally $p \; \mathrm{I}_\Phi \; L$ is defined by $p \; \mathrm{I} \; L$ if $p \in \mathcal{P}$ and by $L \in p$ if $p \in \{\mathcal{L}_i \| i = 1, \ldots, s(s+1)/\alpha + 1\}$.

Theorem 2.3.2
$\mathcal{S}_\Phi$ is a $\mathsf{pg}(s+1, s, \alpha)$

Proof. It is clear from the construction that $\mathcal{S}_\Phi$ is a partial linear space of order $(s+1, s)$. We only prove here that for each anti-flag (p, L) the incidence number equals α. Let p be a point of $\mathcal{S}$ and let L_p be the line of the pg-spread Φ through p. If L_p is not intersecting L in the partial geometry $\mathcal{S}$, then the α lines of $\mathcal{S}$ through p and intersecting L are all elements of $\mathcal{S}_\Phi$ while the point of type $\mathcal{L}_i$ defined by L is not collinear with p. However if L_p is intersecting L in the partial geometry $\mathcal{S}$, then there are $\alpha - 1$ lines of $\mathcal{S}$ (being also lines of $\mathcal{S}_\Phi$) through p and intersecting L. Let $\mathcal{L}_i$ be the unique set defined by L and let $L_i(p)$ be the line of $\mathcal{S}$ through p and contained in $\mathcal{L}_i$, then $p \; \mathrm{I}_\Phi \; L_i(p) \; \mathrm{I}_\Phi \; \mathcal{L}_i \; \mathrm{I}_\Phi \; L$. Hence also in the case the incidence number $\alpha(p, L)$ equals α. Assume $p \in \{\mathcal{L}_i \| i = 1, \ldots, s(s+1)/\alpha + 1\}$ then as each line L of $\mathcal{S}_\Phi$ not contained in p intersects the point set of p in α points, it follows that the incidence number is again α. ■

Remark

It has been checked by computer by Mathon and Street [51] (and also by V. Tonchev, private communication) that $\mathsf{PQ}^+(7,2)$ has, up to isomorphism, exactly 3 replaceable spreads, yielding (after dualizing) 3 non-isomorphic partial geometries $\mathsf{pg}(7,8,4)$. One of these $\mathsf{pg}(7,8,4)$ contains replaceable spreads too which yield again partial geometries $\mathsf{pg}(7,8,4)$, non-isomorphic to the former ones. In total Mathon and Street have found by this technique (they call this construction *switching*) 7 partial geometries $\mathsf{pg}(7,8,4)$ that are not isomorphic to $\mathsf{PQ}^+(7,2)$. We will give in the next section a geometric construction of the 3 replaceable spreads of $\mathsf{PQ}^+(7,2)$, a construction which is even valid in the general class $\mathsf{PQ}^+(4n-1,q)$, $q = 2$ or 3.

Replaceable spreads of $\mathsf{PQ}^+(4n-1,q)$ ($q = 2$ or $q = 3$)

Assume that $\mathcal{S}$ is a partial geometry of type $\mathsf{PQ}^+(4n-1,q)$ ($q = 2$ or 3). It is easy to check (see [23]) that two lines L and M are intersecting lines of $\mathcal{S}$ iff on the quadric $\mathsf{Q}^+(4n-1,q)$, $L \cap M^* = \emptyset$ iff $L^* \cap M = \emptyset$. Hence any subset of elements $\mathsf{PG}(2n-1,q)$ of Ω contained in an element ω_i of the orthogonal spread Σ yields a set of mutually disjoint lines of $\mathcal{S}$. If H is a subspace of ω_i, and if a pg-spread of $\mathcal{S}$ contains 2 elements through H then all elements through H are elements of the pg-spread. In [23], the following theorem has been proved for $q = 2$, but the proof can easily be modified for $q = 3$.

Theorem 2.3.3
Suppose that z is a point on $\mathsf{Q}^+(4n-1,q)$ ($q = 2$ or 3) . The set of lines V of $\mathsf{PQ}^+(4n-1,q)$ intersecting as $(2n-1)$-dimensional subspaces of $\mathsf{Q}^+(4n-1,q)$ in a point z of the quadric is contained in exactly 2 pg-spreads.

As before, assume that Ω is the set of all hyperplanes of the elements ω_i, $i = 0,\ldots,q^{2n-1}$ of the orthogonal spread Σ. Let $z \in \omega_0$. One of the pg-spreads, which we will denote by Φ_1, consists of all hyperplanes of ω_0. The other pg-spread, which we will denote by Φ_2, equals $V \cup W$, with $W = \{M_i = z^* \cap \omega_i \| i = 1,\ldots,q^{2n-1}\}$.
We construct a third type of pg-spread of $\mathsf{PQ}^+(7,q)$ ($q = 2$ or 3). Let ω' be an element of $\mathcal{D}_1 \setminus \Sigma$. Then $\omega' \cap \omega_i$, $i = 0,\ldots,q^3$ is either empty or is a line. Without loss of generality we may assume that $\omega' \cap \omega_i$, $i = 0,\ldots,q^2$, is a line l_i. The set $\Phi_3 = \{\pi_{i,j} \| i = 0,\ldots,q^2; j = 0,\ldots q\}$, with $\pi_{i,j}$, $j = 0,\ldots,q$ a plane of ω_i through l_i is a pg-spread of $\mathsf{PQ}^+(7,q)$ ($q = 2$ or 3) which is clearly not isomorphic to the other two.

Theorem 2.3.4 ([20])
The partial geometry $\mathsf{PQ}^+(7,q)$ ($q = 2$ or 3) has up to isomorphism exactly 3 pg-spreads, namely Φ_1, Φ_2 and Φ_3 each of them being replaceable.

Remarks

1. There are 9 pg-spreads of type Φ_1, 135 of type Φ_2 and 126 of type Φ_3 in $\mathsf{PQ}^+(7,2)$.

2. It is known that $\mathsf{Aut}(\mathsf{PQ}^+(7,2)) = \mathsf{Alt}(9)$. This can easily be proved by regarding $\mathsf{Aut}(\mathsf{PQ}^+(7,2))$ acting on the 9 elements of the orthogonal spread Σ. The group is transitive on the points as well as on the lines of $\mathsf{PQ}^+(7,2)$.

3. The spread $\Phi \cong \Phi_1$ is a replaceable pg-spread of $\mathsf{PQ}^+(4n-1,q); \forall n \geq 2$. For every line $L \in \mathcal{L}\backslash\Phi$, the $s+1$ elements of Φ_L are the hyperplanes of ω_0 not containing $z = L^* \cap \omega_0$. Hence there is a canonical bijection from the sets $\mathcal{L}_i$, $i = 1, \ldots, \frac{q^{2n}-1}{q-1}$ to the points of ω_0. From this it follows that the symmetric design $\mathcal{D}(\Phi_1)$ is indeed isomorphic to the complement of the design of points and hyperplanes of $\mathsf{PG}(2n-1,q)$. The partial geometry $\mathcal{S}_\Phi$ is the geometry with as point set the set of points of $\mathcal{S}$ union the set of points of a fixed element ω_0 of the orthogonal spread Σ. The line set is the set of hyperplanes contained in the other elements ω_i ($i = 1, \ldots q^{2n-1}$) of the orthogonal spread Σ. A point p is incident with a line L, if and only if the line L (as subspace on the quadric Q^+) is contained in the polar hyperplane p^* of p with respect to the quadric. Note that the point graph as well as the block graph of this geometry were known before, see [3].

4. Actually, in [51] Mathon and Street have constructed by computer seven new partial geometries $\mathsf{pg}(7,8,4)$ by starting from the partial geometry $\mathcal{S}_0 = \mathsf{PQ}^+(7,2)$ and by using spread derivation with respect to a suitable replaceable spread. The following scheme, taken from [51], shows how the eight partial geometries $\mathsf{pg}(7,8,4)$ are related to each other. The labeled arrow $\overset{\Phi_i}{\longleftrightarrow}$ means that the partial geometries are related under derivation with respect to the replaceable spread (of type) Φ_i and after dualizing.

$$
\begin{array}{ccccccccccccc}
\mathcal{S}_2 & \overset{\Phi_2}{\longleftrightarrow} & \mathcal{S}_0 & \overset{\Phi_1}{\longleftrightarrow} & \mathcal{S}_1 & \overset{\Phi_4}{\longleftrightarrow} & \mathcal{S}_4 & \overset{\Phi_6}{\longleftrightarrow} & \mathcal{S}_6 & \overset{\Phi_7}{\longleftrightarrow} & \mathcal{S}_7 \\
 & & {\scriptstyle\Phi_3}\updownarrow & & {\scriptstyle\Phi_5}\updownarrow & & & & & & \\
 & & \mathcal{S}_3 & & \mathcal{S}_5 & & & & & &
\end{array}
$$

Mathon and Street give in [51] information on the order of the automorphism groups of the geometries as well as information on the point and block graphs of these geometries. They remarked that the point graphs Γ_i of the geometries $\mathcal{S}_i, i = 1, 2, 3, 4$, were isomorphic graphs and their block graphs all are different. Actually the graph $\Gamma = \Gamma_i$, $i = 1, 2, 3, 4$, was not a new graph, it is the complement of the graph constructed in [3]. It is an element of the class of graphs called the *graphs on a quadric with a hole.* Such a graph has vertex set the points of a quadric $\mathsf{Q}^+(2m-1, q) \setminus G$, G a generator of the quadric and vertices x and y are defined to be adjacent whenever $\langle x, y\rangle$ is contained in $\mathsf{Q}^+(2m-1, q)\setminus G$. This graph is strongly regular for general dimensions and general q. Klin and Reichard [47, 54] found, again by computer, but independently from Mathon and Street, that the complement of the graph on the quadric $\mathsf{Q}^+(7, 2)$ with a hole, is indeed the point graph of exactly four partial geometries $\mathsf{pg}(7, 8, 4)$, namely $\mathcal{S}_1$, $\mathcal{S}_2$, $\mathcal{S}_3$ and $\mathcal{S}_4$. Mario Delanote [32] gave a geometrically construction of the replaceable spreads related to these geometries, moreover he proved that the graph on the quadric with a hole in $\mathsf{PG}(4n-1, 2)$ is always geometric, namely it is the point graph of the partial geometry $\mathcal{S}_4(n)$. He also proved that the computer construction of the partial geometry $\mathcal{S}_5$ of Mathon and Street could be generalized for all dimensions, yielding another $\mathsf{pg}(2^{2n-1} - 1, 2^{2n-1}, 2^{2n-2})$ $\mathcal{S}_5(n)$. Hence, from the eight known $\mathsf{pg}(7, 8, 4)$, four of them, namely $\mathcal{S}_i$, $i = 0, 1, 4, 5$, are the smallest member of an infinite class, namely $\mathcal{S}_i(n)$, $i = 0, 1, 4, 5$ (where we define $\mathcal{S}_0(n)$ to be the partial geometry $\mathsf{PQ}^+(4n-1, 2)$). And so it turns out that not $\mathcal{S}_0(n)$, but $\mathcal{S}_1(n)$ can be considered as the "central" partial geometry in the above scheme.

Chapter 3

Constructions of (α, β)-geometries from other ones

Several semipartial geometries can be constructed from some generalized quadrangles. We refer the reader to [27] and to [21] for more details. In this chapter we will restrict us to constructions related to special subsets in polar spaces.

3.1 (α, β)-geometries and (α, β)-reguli

3.1.1 Definitions and constructions

In [61] a construction method for semipartial geometries is introduced using so-called spg-reguli. This construction has afterwards been generalized for (α, β)-geometries by Hamilton and Mathon [40].
A strongly regular (α, β)-*regulus* is a set $\mathcal{R}$, $|\mathcal{R}| = r > 1$ of m-dimensional subspaces $\mathsf{PG}^{(1)}(m, q), \ldots, \mathsf{PG}^{(r)}(m, q)$, of $\mathsf{PG}(n, q)$, satisfying:

1. $\mathsf{PG}^{(i)}(m, q) \cap \mathsf{PG}^{(j)}(m, q) = \emptyset$, for all $i \neq j$.
2. If $\mathsf{PG}(m + 1, q)$ contains $\mathsf{PG}^{(i)}(m, q)$, then it has a point in common with either α or β spaces in $\mathcal{R} \setminus \{\mathsf{PG}^{(i)}(m, q)\}$. If this $\mathsf{PG}(m + 1, q)$ meets α elements of $\mathcal{R} \setminus \{\mathsf{PG}^{(i)}(m, q)\}$ it is said to be an α-secant to $\mathcal{R}$ at $\{\mathsf{PG}^{(i)}(m, q)\}$, similarly for β-secants.
3. If a point x of $\mathsf{PG}(n, q)$ is contained in an element $\mathsf{PG}^{(i)}(m, q)$ of $\mathcal{R}$, then it is contained in a constant number p of α-secant $(m + 1)$-spaces of $\mathcal{R} \setminus \{\mathsf{PG}^{(i)}(m, q)\}$.
4. If a point x of $\mathsf{PG}(n, q)$ is not contained in an element of $\mathcal{R}$, then it is contained in a constant number r of α-secant $(m + 1)$-spaces of $\mathcal{R}$.

If $\beta = 0$ then a strongly regular (α, β)-regulus is an spg-regulus.

Theorem 3.1.1
Let $\mathcal{R}$ be a strongly regular (α, β)-regulus in $\mathsf{PG}(n, q)$, the elements of $\mathcal{R}$ being of dimension m. Embed $\mathsf{PG}(n, q)$ as a hyperplane π_∞ of $\mathsf{PG}(n+1, q)$ and define an incidence stucture $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ as follows.

(i) The point set $\mathcal{P}$ is the set of points of $\mathsf{PG}(n+1, q) \setminus \pi_\infty$.

(ii) The line set $\mathcal{L}$ is the set of $(m+1)$-dimensional subspaces of $\mathsf{PG}(n+1, q)$ that meet π_∞ in an element of $\mathcal{R}$.

(iii) Incidence I is containment.

Then $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a strongly regular (α, β)-geometry of order $(s = q^{m+1} - 1, t = |\mathcal{R}| - 1)$.

Remark

The strongly regular graphs arising fom the strongly regular (α, β)-regulus $\mathcal{R}$ can be seen as follows. The vertices of the graph are the points of $\mathsf{PG}(n+1, q) \setminus \pi_\infty$. Two vertices are adjacent if and only if, the line of $\mathsf{PG}(n+1, q)$ joining them meets π_∞ in a point contained in some element of $\mathcal{R}$. It follows that the union of the points contained in $\mathcal{R}$ has exactly two intersection numbers with respect to hyperplanes in π_∞, and hence also yields a linear two-weight code (see [11] for more details on these connections).

3.1.2 (α, β)-reguli and polar spaces

Let $\mathcal{Q}$ be a non-degenerate hyperbolic or elliptic quadric in $\mathsf{PG}(2n+1, q)$. Then a plane of $\mathsf{PG}(2n+1, q)$ may meet $\mathcal{Q}$ in either a conic, a line pair, a single line, a single point or be intirely contained in $\mathcal{Q}$. Let L be a line disjoint from $\mathcal{Q}$, then a plane π on L must meet $\mathcal{Q}$ in either a conic or a single point, and so π meets the complement of $\mathcal{Q} \cap L$ in either $q^2 - q - 1$ or $q^2 - 1$ points.
It follows that a partition of the complement of $\mathcal{Q}$ in $\mathsf{PG}(2n+1, q)$ into lines is a strongly regular (α, β)-regulus with $\alpha = q^2 - q - 1$ and $\beta = q^2 - 1$. The first two conditions of the definition of a strongly regular (α, β)-regulus follow immediately, the third condition follows from the fact that $\mathcal{Q}$ has two intersection sizes with respect to hyperplanes.
For such a partition to exist it is necessary that $q+1$ devides $|\mathsf{PG}(2n+1, q) \setminus \mathcal{Q}|$, which is equivalent to $q+1$ divides $|\mathcal{Q}|$, which implies that n should be odd if $\mathcal{Q} = \mathsf{Q}^+(2n+1, q)$ and n should be even if $\mathcal{Q} = \mathsf{Q}^-(2n+1, q)$.
We summarise this in the following theorem.

Theorem 3.1.2 ([40])

1. *A partition of the points of* $\mathsf{PG}(4n+1,q) \setminus \mathsf{Q}^-(4n+1,q)$ *into lines is a strongly regular* (α,β)*-regulus and gives rise to a strongly regular* (α,β)*-geometry with parameters* $s = q^2-1$, $t = q^{2n}(q^{2n+1}+1)/(q+1)-1$, $\alpha = q^2-q-1$, $\beta = q^2-1$, $p = q^{2n-1}(q^2-q-1)(q^{2n}-1)/(q+1)$ *and* $r = q^{4n} - q^{4n-1}$.

2. *A partition of the points of* $\mathsf{PG}(4n+3,q) \setminus \mathsf{Q}^+(4n+3,q)$ *into lines is a strongly regular* (α,β)*-regulus and gives rise to a strongly regular* (α,β)*-geometry with parameters* $s = q^2-1$, $t = q^{2n+1}(q^{2n+2}-1)/(q+1)-1$, $\alpha = q^2-q-1$, $\beta = q^2-1$, $p = q^{2n}(q^2-q-1)(q^{2n+1}+1)/(q+1)$ *and* $r = q^{4n+2} - q^{4n+1}$.

Such partitions do exist, we refer to [40] for the construction.

Remark

There are other constructions of strongly regular (α,β)-reguli known. For instance a set of type (α,β) in $\mathsf{PG}(2,q)$, being a set of points in the plane such every line of the plane intersects in either α or β points is actually an (α,β)-regulus of points. Examples of such *two character sets* are widely available. Here is a short list of some classical examples. All the related geometries are linear representations in $\mathsf{AG}(3,q)$.

1. Maximal arcs of degree d, being sets of type $(0,d)$ with $q(d-1)+d$ points. If $d < q$, then $d|q$ and moreover q is even. The related geometry is a partial geometry.

2. Unitals, being sets of type $(1,\sqrt{q})$ with $q^{3/2}+1$ points, and exist for every q a square. The related geometry is a semipartial geometry.

3. Baer subplanes, their point sets being sets of cardinality $q+\sqrt{q}+1$ and of type $(1,\sqrt{q})$, and exist for every q a square. The related geometry is a semipartial geometry.

4. Disjoint unions of u Baer subplanes, being sets of type $(u,\sqrt{q}+u)$, yielding strongly regular $(u-1,\sqrt{q}+u-1)$-geometries.

3.1.3 spg-reguli and m-systems of polar spaces

For the special case of an spg-regulus, we can give some extra examples. In section 1.5 we have introduced the m-systems. E. Shult and J. A. Thas proved the following theorem.

Theorem 3.1.3 ([56])
Every m*-system of* $W_{2n+1}(q)$, $\mathsf{Q}^-(2n+1,q)$ *or* $H(2n,q^2)$ *yields two character set of points with respect to hyperplanes.*

Actually, working out the parameters of the related strongly regular graphs, N. Hamilton and R. Mathon [39] noticed that for these polar spaces, an m-system can only exist if $m \geq (n-1)/2$ (this condition is coming from the fact that the parameter λ of the graph has to be a positive integer). D. Luyckx [49] proved that every m-system of $W_{2n+1}(q)$, $\mathsf{Q}^-(2n+1,q)$ or $H(2n,q^2)$ is actually also an spg-regulus, hence these m-systems yield semipartial geometries. Semipartial geometries with the same parameters were known before, but in some of the cases the semipartial geometries coming from m-systems are not isomorphic with the ones known before. Specialising to spreads of polar spaces, one can summarize the results as follows.

1. A spread $\mathcal{R}$ of the non-singular elliptic quadric $\mathsf{Q}^-(2m+3,q)$ $(m \geq 0)$ contains $q^{m+2}+1$ elements (of dimension m) and is always an spg-regulus. It yields a $\mathsf{spg}(q^{m+1}-1, q^{m+2}, q^m, q^{m+1}(q^{m+1}-1))$. For $m=0$, this is the partial quadrangle $T_3^\star(\mathcal{O})$. For $m=1$, it yields a $\mathsf{spg}(q^2-1, q^3, q, q^2(q^2-1))$ with the same parameters as the semipartial geometry $T_2^\star(\mathcal{U})$. Indeed $T_2^\star(\mathcal{U})$ is isomorphic to the semipartial geometry arising from a regular spread $\mathcal{R}$ (see for instance [64]) of $\mathsf{Q}^-(5,q)$. However if the spread is non-regular, then the associated semipartial geometry is not isomorphic to $T_2^\star(\mathcal{U})$. If $m>1$, and q is even, then the quadric $\mathsf{Q}^-(2m+3,q)$ has spreads, hence this yields semipartial geometries. If q is odd, no spread of the quadric $\mathsf{Q}^-(2m+3,q)$ $(m>1)$ is known.

2. If the non-singular quadric $\mathsf{Q}(2m+2,q)$, $m \geq 0$, has a spread $\mathcal{R}$, then it is not an spg-regulus.

3. If $\mathcal{R}$ is a spread of the quadric $\mathsf{Q}^+(2m+1,q)$, $m \geq 1$, then necessarily m is odd, and moreover this spread is an spg-regulus, but the associated semipartial geometry is a net.

4. Let $\mathcal{H}(n,q^2)$ be a non-singular Hermitian variety of $\mathsf{PG}(n,q^2)$, $n \geq 2$. If n is odd, the Hermitian variety has no spread (see [7] for the case $n=3$ and [63] for $n \geq 5$). Assume that n is even. Then $\mathcal{R}$ is always an spg-regulus with $m = \frac{1}{2}n-1$ and $|\mathcal{R}| = q^{n+1}+1$. Hence there corresponds a semipartial geometry $\mathsf{spg}(q^n-1, q^{n+1}, q^{n-1}, q^n(q^n-1))$. However if $n=2$ then this semipartial geometry is $T_2^\star(\mathcal{U})$. Unfortunately for $n>2$ no spread of $\mathcal{H}(n,q^2)$, n even, is known. Brouwer (private communication, [1981]) proved that $\mathcal{H}(4,4)$ has no spread. For more details on spreads of polar spaces we refer to [64].

3.2 spg-systems and semipartial geometries

3.2.1 Definitions and constructions

Thas [65] has generalized the concept of spg-regulus of a polar space P to spg-systems of P. Without any doubt this concept will open new perspectives in the near future. We will restrict ourselves here to that part of the theory which yields semipartial geometries with new parameters. It is however important to underline that some of the examples (including the partial geometries $\mathsf{PQ}^+(4n-1,2)$ and $\mathsf{PQ}^+(4n-1,3)$) can be constructed from spg-systems.

Let $\mathsf{Q}(2n+2,q)$, $n \geq 1$ be a nonsingular quadric of $\mathsf{PG}(2n+2,q)$. An spg-system of $\mathsf{Q}(2n+2,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathsf{Q}(2n+2,q)$ such that the elements of $\mathcal{D}$ on any nonsingular elliptic quadric $\mathsf{Q}^-(2n+1,q) \subset \mathsf{Q}(2n+2,q)$ constitute a spread of the quadric $\mathsf{Q}^-(2n+1,q)$.

Let $\mathsf{Q}^+(2n+1,q)$ be a nonsingular hyperbolic quadric of $\mathsf{PG}(2n+1,q)$, $n \geq 1$. An spg-system of $\mathsf{Q}^+(2n+1,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathsf{Q}^+(2n+1,q)$ such that the elements of $\mathcal{D}$ on any nonsingular quadric $\mathsf{Q}(2n,q) \subset \mathsf{Q}^+(2n+1,q)$ constitute a spread of $\mathsf{Q}(2n,q)$.

Let $\mathsf{H}(2n+1,q)$ be a nonsingular Hermitian variety of $\mathsf{PG}(2n+1,q)$, $n \geq 1$, q a square. An spg-system of $\mathsf{H}(2n+1,q)$ is a set $\mathcal{D}$ of $(n-1)$-dimensional totally singular subspaces of $\mathsf{H}(2n+1,q)$ such that the elements of $\mathcal{D}$ on any nonsingular Hermitian variety $\mathsf{H}(2n,q) \subset \mathsf{H}(2n+1,q)$ constitute a spread of $\mathsf{Q}(2n,q)$.

One can prove that in each case the number of elements in $\mathcal{D}$ equals the number of points of the polar space. The dimension $n-1$ of the elements of an spg-system is also called the index of the spg-system.

The construction by Thas of the semipartial geometry is as follows. Let P be one of the above polar spaces, i.e. $\mathsf{Q}(2n+2,q)$, $\mathsf{Q}^+(2n+1,q)$, $\mathsf{H}(2n+1,q)$ $(n \geq 1)$. Let $\mathsf{PG}(d,q)$ be the ambient space of P. Hence in the first case $d = 2n+2$, in the other two cases $d = 2n+1$. Let $\mathcal{D}$ be an spg-system of P and let P be embedded in a nonsingular polar space $\bar{P}$ with ambient space $\mathsf{PG}(d+1,q)$ of the same type as P and with projective index n. Hence for $P = \mathsf{Q}(2n+2,q)$, we have $\bar{P} = \mathsf{Q}^-(2n+3,q)$; for $P = \mathsf{Q}^+(2n+1,q)$, we have $\bar{P} = \mathsf{Q}(2n+2,q)$ and for $P = \mathsf{H}(2n+1,q)$, we have $\bar{P} = \mathsf{H}(2n+2,q)$. If $\bar{P}$ is not symplectic and $y \in \bar{P}$, then let τ_y be the tangent hyperplane of $\bar{P}$ at y; if $\bar{P}$ is symplectic and θ is the corresponding symplectic polarity of $\mathsf{PG}(d+1,q)$, then let $\tau_y = y^\theta$ for any $y \in \mathsf{PG}(d+1,q)$.

For $y \in \bar{P} \backslash P$ let $\bar{y}$ be the set of all points z of $\bar{P} \backslash P$ for which $\tau_z \cap P = \tau_y \cap P$. Note that no two distinct points of $\bar{y}$ are collinear in $\bar{P}$. If P is orthogonal then $|\bar{y}| = 2$ except when $P = \mathsf{Q}^+(2n+1,q)$ and q even, in which case $|\bar{y}| = 1$. If P is Hermitian then $|\bar{y}| = \sqrt{q}+1$.

Let ξ be any maximal totally singular subspace of $\bar{P}$, not contained in P, such that $\xi \cap P \in \mathcal{D}$ and let $y \in \xi \setminus P$. Further let $\bar{\xi}$ be the set of all maximal totally singular subspaces η of $\bar{P}$, not contained in P, for which $\xi \cap P = \eta \cap P$ and $\eta \cap \bar{y} \neq \emptyset$.
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be the incidence structure with $\mathcal{P} = \{\bar{y} \| y \in \bar{P} \setminus P\}$; $\mathcal{L}$ contains all the sets $\bar{\xi}$ as defined above; if $\bar{y} \in \mathcal{P}$ and $\bar{\xi} \in \mathcal{L}$ then $\bar{y} \,\mathrm{I}\, \bar{\xi}$ if and only if for some $z \in \bar{y}$ and some $\eta \in \bar{\xi}$, one has that $z \in \eta$.
In [65] it is proved that this incidence structure is a $(0, \alpha)$-geometry of order (s, t) with $s + 1 = q^n$ and $t + 1$ the number of elements in a spread of P. The parameter α equals to q^{n-1} times the number of points of $\bar{P}$ in any set $\bar{y} \in \mathcal{P}$.

Theorem 3.2.1

1. *If P is the polar space $\mathsf{Q}(2n+2, q)$ then $\mathcal{S}$ is a semipartial geometry $\mathsf{spg}(q^n - 1, q^{n+1}, 2q^{n-1}, 2q^n(q^n - 1))$.*

2. *If P is the polar space $\mathsf{Q}^+(2n+1, q)$ then the point graph $\Gamma(\mathcal{S})$ is strongly regular if and only if $q = 2$ or $q = 3$. In these cases $\mathcal{S}$ is a partial geometry.*

3. *If P is the polar space $\mathsf{H}(2n+1, q)$ then $\mathcal{S}$ is a semipartial geometry $\mathsf{spg}(q^n - 1, q^n\sqrt{q}, q^{n-1}(\sqrt{q}+1), q^{n-1}(q^n - 1)\sqrt{q}(\sqrt{q}+1))$.*

3.2.2 Some facts on the existence of spg-systems

If Φ is a spread of one of the polar spaces $P \in \{\mathsf{Q}(2n+2, q), \mathsf{Q}^+(2n+1, q), \mathsf{H}(2n+1, q^2)\}$ and τ is the set of all $(n-1)$-dimensional subspaces contained in the elements of Φ then τ is an spg-system of P. Such an spg-system is called a *spread-spg-system*.
The following characterisation theorem is known.

Theorem 3.2.2 ([28])
Let τ be an spg-system of index $n - 1 \geq 2$ of a polar space P. Then τ is a spread-spg-system if and only if every two intersecting elements of τ intersect in an $(n-2)$-dimensional space.

An spg-system of index 0 of a polar space P is obviously the set of points on P. All spg-systems of index 1 are classified by Thas.

Theorem 3.2.3 ([65])
There are exactly two classes of spg-systems of index 1 on a nonsingular polar space and both are spg-systems of $\mathsf{Q}(6, q)$. One of them is a spread-spg-system, the other one consists of the lines of the classical hexagon $H(q)$ embedded in $\mathsf{Q}(6, q)$.

Note that $\mathsf{Q}(6,q)$ has at least one spread for all values of q, except possibly for the case where q is odd, with $q \equiv 1 \pmod 3$ and not prime, in which case the existence of a spread is still open, see for instance [35].
Also the spg-systems of index 2 are classified, this is done by S. De Winter.

Theorem 3.2.4 ([28])
The only spg-systems of index 2 are the spread-spg-systems on $\mathsf{Q}^+(7,q)$ and $\mathsf{Q}(8,2^h)$.

Note that $\mathsf{Q}(8,q)$ for q odd has no spreads, while $\mathsf{Q}(8,2^h)$ does indeed has spreads. As far as $\mathsf{Q}^+(7,q)$ is concerned, this quadric contains a spread if q is even, if q is odd, then spreads do exist except possibly in the case where $q \equiv 1 \pmod 3$ and not a prime, in which case the existence is still open.

3.2.3 Semipartial geometries from spg-systems

1. Let P be the polar space $\mathsf{Q}(2n+2,q)$. The geometry will be denoted by $TQ(2n+2,q)$.

   Assume $n = 1$, hence the spg-system is the complete set of points of $\mathsf{Q}(4,q)$ and the semipartial geometry was known before, it is the semipartial geometry of Metz, see [27].

   Assume $n = 2$. Then, see theorem 3.2.3 there are exactly two classes of spg-systems on $\mathsf{Q}(6,q)$. On the one hand the spread-spg-systems, i.e. the set of lines in all the planes of a spread Φ (if such a spread exists). On the other hand the line set of the classical generalized hexagon $H(q)$ embedded in $\mathsf{Q}(6,q)$, is an spg-system of $\mathsf{Q}(6,q)$.

   For any $n \geq 3$, any spread of $\mathsf{Q}(2n+2,q)$ defines an spg-system. Such a spread is known to exist if q is even.

   In [31] Delanote gives a construction of a semipartial geometry with point graph the graph on the internal points of a quadric $\mathsf{Q}(4m+2,3)$, (vertices are adjacent when non-orthogonal) under the condition of existence of an orthogonal spread. His arguments can easily be generalized for any odd q and in fact, his semipartial geometry is isomorphic to $TQ(2n+2,q)$ with $n = 2m$.

2. Let P be the polar space $\mathsf{Q}^+(2n+1,q)$; $q = 2$ or 3.

   If $n = 2m-1$ is odd and $q = 2$ then $\mathsf{Q}^+(2n+1,2)$ has a spread and the partial geometry is isomorphic to the partial geometry $\mathsf{PQ}^+(4m-1,2)$ of De Clerck, Dye and Thas [22].

   If $n = 2m-1$ is odd and $q = 3$ then the partial geometry is isomorphic to the partial geometry $\mathsf{PQ}^+(4m-1,3)$ of Thas, which only exists if $\mathsf{Q}^+(4m-1,3)$ has a spread; the existence of such a spread is open for $m \geq 3$.

3. Let P be the polar space $\mathsf{H}(2n+1,q)$. The geometry will be denoted by $TH(2n+1,q)$.

   Unfortunately, if $n \geq 2$ then no spg-system of $\mathsf{H}(2n+1,q)$ is known. If $n = 1$, then $\mathcal{D}$ is the set of points of $\mathsf{H}(3,q)$ and the semipartial geometry is the one of Thas as described in [27].

Chapter 4

Embedding of (α, β)-geometries in projective and affine spaces

In this chapter we will give an overview of the most important results on (fully) embedded (α, β)-geometries in projective and affine spaces. For the definition of embedding we refer to chapter 1.

4.1 Embeddings in projective spaces

4.1.1 On the embedding of partial and semipartial geometries

As far as projective embeddings of strongly regular (α, β) geometries is concerned only the case of partial and semipartial geometries is solved, for semipartial geometries we have to assume it is not a partial quadrangle. Although no model of a partial quadrangle embedded in a projective space is known, it is quite difficult to handle this case as one can hardly control the different types of intersections of such an embedded partial quadrangle with a general projective plane.
There exists a complete classification of partial geometries embedded in a projective space.

Theorem 4.1.1 ([24])
If $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ is a partial geometry with parameters s, t, α, which is embedded in a projective space $\mathsf{PG}(n, s)$, but not in a $\mathsf{PG}(n', s)$, with $n' < n$, then the following cases may occur:

1. *$\alpha = s + 1$, and $\mathcal{S}$ is the design of points and lines of $\mathsf{PG}(n, s)$;*
2. *$\alpha = 1$, and $\mathcal{S}$ is a classical generalized quadrangle, (see [9]);*

3. $\alpha = t+1, n = 2$ *and* $\mathcal{S}$ *is a dual design in* $\mathsf{PG}(2,s)$*;*

4. $\alpha = s$ *and* $\mathcal{S} = H_s^n (n \geq 3)$*, which is the dual net by taking as point set the set of points of* $\mathsf{PG}(n,s)$ *not contained in a fixed* $(n-2)$*-dimensional subspace* H*, and as lines those lines of* $\mathsf{PG}(n,s)$ *that are skew to* H*.*

On the embedding of semipartial geometries in projective space the following result is known.

Theorem 4.1.2
f $\mathcal{S}$ *is a proper semipartial geometry with parameters* s*,* t*,* $\alpha(>1)$*,* μ*, embedded in a* $\mathsf{PG}(n,s)$*,* $n \geq 3$ *and* $s > 2$*, but not in a* $\mathsf{PG}(n',s)$*,* $n' < n$*, then* n *is odd and* $\mathcal{S}$ *is the semipartial geometry* $\overline{W(n,s)}$*, which is the geometry of all points of* $\mathsf{PG}(n,s)$ *and those lines that are hyperbolic with respect to a fixed symplectic polarity (*$\alpha = q$ *and* $\mu = q^{n-1}(q-1)$*.*

Remark

This theorem was proved by Debroey and Thas De-Th:78b for the case $n = 3$ and by Debroey, De Clerck and Thas [66] for $n > 3$. If $\mathcal{S}$ is any semipartial geometry with $\alpha = s = 2$, then $\mathcal{S}$ is a cotriangle space and those are classified (see theorem 2.2.3). A complete classification of the embedded cotriangle spaces exists for $n = 3$ ([30]) and for $n = 4$ ([66]). The cotriangle space, often denoted by $U_{2,3}(m)$, with point set $\mathcal{P}$ the set of all pairs of a finite set X, $|X| = m \geq 5$, a line is a set of three pairs contained in a 3–subset of X has a lot of embeddings. In [48] an embedding of $U_{2,3}(n+2)$ in $\mathsf{PG}(n,2)$ is given. The lines of this geometry are hyperbolic lines, that is, lines which are not totally isotropic, of some symplectic polarity. Also an embedding of $U_{2,3}(n+3)$ in $\mathsf{PG}(n,2)$ is described. The lines of this geometry are hyperbolic for some symplectic polarity if and only if n is odd. The problem of determining all embeddings of $U_{2,3}(m)$ in $\mathsf{PG}(n,2)$ is equivalent to determining (up to equivalence) all binary codes of length m with all weights even and minimum weight greater than 4, see [38].
The dual of a proper semipartial geometry is not a semipartial geometry, those embedded in a projective space are also classified.

Theorem 4.1.3 ([25])
If $\mathcal{S}^D$ *is the dual of a semipartial geometry* $\mathcal{S}$ *with* $\alpha > 1$*, and if* $\mathcal{S}^D$ *is embedded in a projective space* $\mathsf{PG}(n,s)$*,* $n \geq 3$*, but not in a* $\mathsf{PG}(n',s)$*,* $n' < n$*, then* $n = 3$ *and* $\mathcal{S}^D$ *is the design of points and lines in* $\mathsf{PG}(3,q)$*, or* $\mathcal{S}^D = H_s^3$ *or* $\mathcal{S}^D$ *is the cotriangular space* $NQ^-(3,2)$*.*

As far as the dual of a proper partial quadrangle is concerned, there are two models known embedded in $\mathsf{PG}(3,q)$; see [34] for their construction and their characterization.

Remark

Given an embedding of an (α, β)-geometry $\mathcal{S}$ in $\mathsf{PG}(n, s)$, then intersecting $\mathcal{S}$ with a subspace $\mathsf{PG}(m, s)$ of $\mathsf{PG}(n, s)$ will not necessarily yield again an (α, β)-geometry $\mathcal{S}'$ in $\mathsf{PG}(m, s)$; it does in the case of a $(0, \alpha)$-geometry. Moreover, if $\mathcal{S}$ is a strongly regular (α, β)-geometry, then clearly the point graph of $\mathcal{S}'$ is not necessarily strongly regular. This makes a classification of strongly regular (α, β)-geometries embedded in projective spaces difficult, as we cannot use induction on the dimension of the space.
Hence in proving the classification of the semipartial geometries embedded in projective spaces, we actually had to prove the classification of the $(0, \alpha)$-geometries embedded in projective spaces. We were able to prove that as soon as the dimension of the projective space is at least 4, then the $(0, \alpha)$-geometry embedded in the projective space has to be $\overline{W(n, s)}$. As far as the embedding of $(0, \alpha)$-geometries in $\mathsf{PG}(3, s)$ is concerned, there is one example known of such a geometry which is not a semipartial geometry. This geometry $NQ^+(3, 2^h)$ has as point set the points not on a hyperbolic quadric $\mathsf{Q}^+(3, s)$, $s = 2^h$ $h \geq 2$, and as lines the set of lines of $\mathsf{PG}(3, s)$, that have no point in common with $\mathsf{Q}^+(3, s)$. One readily proves that $\alpha = 2^{h-1}$ and that $t+1 = 2^{h-1}(2^h - 1)$. A complete classification of $(0, \alpha)$-geometries embedded in $\mathsf{PG}(3, s)$ is still open, but we conjecture that $\overline{W(3, s)}$ and $NQ^+(3, 2^h)$ are the only models.

4.1.2 On the embedding of general (α, β)-geometries

Let $\mathcal{S}$ be an (α, β)-geometry fully embedded in $\mathsf{PG}(n, s)$, and $\alpha > 1$. The restriction of $\mathcal{S}$ to a plane of $\mathsf{PG}(n, s)$ is a partial linear space, but has not necessarily an order. In case it has an order, it follows that it is a partial geometry $\mathsf{pg}(s, \alpha - 1, \alpha)$ or $\mathsf{pg}(s\beta - 1, \beta)$ [24]. A plane in which the restriction of $\mathcal{S}$ is a partial geometry $\mathsf{pg}(s, \alpha - 1, \alpha)$, we call an *$\alpha$-plane*. A plane in which the restriction of $\mathcal{S}$ is a partial geometry $\mathsf{pg}(s, \beta - 1, \beta)$, we call a *$\beta$-plane*. A plane that contains an antiflag of $\mathcal{S}$ and that is not an α-plane or a β-plane, we call a *mixed* plane. In such a mixed plane, every point of $\mathcal{S}$ in the plane is incident with either α or β lines of $\mathcal{S}$ in this plane. The points and lines of a partial geometry fully embedded in a projective plane are either all points and lines of the plane, or the points not contained in a maximal arc $\mathcal{K}$ of the plane, and the lines exterior to $\mathcal{K}$. Now for q odd, there exists no non-trivial maximal arc in a Desarguesian projective plane [1]. So, if π is an α-plane or a β-plane in $\mathsf{PG}(n, s)$, then the points and lines of $\mathcal{S}$ in π are either all points and lines of π, or all points of π except one point p and all lines of π not through p. The following classification theorem is known.

Theorem 4.1.4 ([18], [17])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ be a proper (α, β)-geometry fully embedded in $\mathsf{PG}(n, s)$, s

odd and $\alpha > 1$. Assume that $\mathsf{PG}(n, s)$ *contains at least one* α*-plane or one* β*-plane. Then* $\mathcal{S}$ *is one of the following:*

1. $\mathcal{S}$ *is an* $(s, s+1)$*-geometry, with points the points of* $\mathsf{PG}(n, s) \setminus \mathsf{PG}(m, s)$*, for some* $1 \leq m < n-2$*, and lines the lines of* $\mathsf{PG}(n, s)$ *that are disjoint from* $\mathsf{PG}(m, s)$*;*

2. $\mathcal{S}$ *is an* $(s, s+1)$*-geometry, with points the points of* $\mathsf{PG}(n, s) \setminus \mathsf{PG}(m, s)$*, with* $1 \leq m \leq n-3$*. Moreover there exists a partition of the points of* $\mathcal{S}$ *in* m'*-dimensional subspaces of* $\mathsf{PG}(n, s)$ *that pairwise intersect in* $\mathsf{PG}(m, s)$*,* $m+2 \leq m' \leq n-2$*, such that the lines of* $\mathcal{S}$ *are the lines that intersect* $s+1$ *of these* m'*-dimensional spaces in a point. A necessary and sufficient condition for this partition and the geometry to exist is that* $(m'-m)|(n-m')$*;*

3. $\mathcal{S}$ *is a* $(s-1, s)$*-geometry, with points the points of* $\mathsf{PG}(n, s) \setminus \mathsf{PG}(n-2, s)$*, and lines the lines that do not contain a point of* $\mathsf{PG}(n, s) \setminus \mathcal{S}$ *and that do not belong to a partition* Σ *of the points of* $\mathsf{PG}(n, s) \setminus \mathsf{PG}(n-2, s)$ *in* r*-dimensional spaces meeting* $\mathsf{PG}(n-2, s)$ *in subspaces of dimension* $r-2$*, with* $1 \leq r \leq n-2$*. Further, such a partition exists for every* $1 \leq r \leq n-2$*, and gives a geometry;*

4. $\mathcal{S}$ *is an* $(s-1, s)$*-geometry with points the points of* $\mathsf{PG}(n, s)$ *not contained in one of two subspaces* $\mathsf{PG}(n-2, s)$ *and* $\mathsf{PG}(r, s)$ *of* $\mathsf{PG}(n, s)$*, for* $1 \leq r \leq n-2$*, for which* $\mathsf{PG}(r, s) \cap \mathsf{PG}(n-2, s)$ *is an* $(r-2)$*-dimensional space. The lines of* $\mathcal{S}$ *are either all lines of* $\mathsf{PG}(n, s)$ *that contain* $s+1$ *points of* $\mathcal{S}$*, or they are the lines not contained in a partition of the points of* $\mathcal{S}$ *in* d*-dimensional spaces pairwise intersecting in* $\mathsf{PG}(r, s)$*. A necessary and sufficient condition for such a partition to exist is that* $(d-r)|(n-r)$ *and that* $n-2 \geq d \geq r+2$*. Further, if* $(d-r)|(n-r)$ *and* $n-2 \geq d \geq r+2$*, then this partition gives a geometry;*

5. $\mathcal{S}$ *is a* $(s-\sqrt{s}, s)$*-geometry with points the points of* $\mathsf{PG}(n, s)$ *that do not belong to a Baer subspace* $B(n, s)$ *of* $\mathsf{PG}(n, s)$ *and lines the lines not intersecting* $B(n, s)$*. In this case* $n = 3$ *or* $n = 4$*.*

Remark

Note that besides the above geometries, the polar spaces (i.e $(1, s+1)$-geometries) are known, and their projective embeddings are classified. Recently S. Cauchie [16] classified the $(1, s)$-geometries ($s \geq 2$) fully embedded in $\mathsf{PG}(n, s)$. Her main theorem reads as follows.

Theorem 4.1.5 ([16])
Let $\mathcal{S} = (\mathcal{P}, \mathcal{L}, \mathrm{I})$ *be a* $(1, s)$*-geometry fully embedded in* $\mathsf{PG}(n, s)$*,* $s \geq 2$*. Then the points of* $\mathcal{S}$ *are the points of a cone* $\Pi[n-m-1]GQ$*,* $m = 3, 4, 5,$

that are not contained in the vertex $\Pi[n-m-1]$ and the lines of $\mathcal{S}$ are the lines that lie on this cone and contain $s+1$ points of $\mathcal{S}$.

4.2 Embeddings in affine spaces

On embeddings of general (α, β)-geometries in affine spaces not that much is known. There exists however a complete classification of partial geometries embedded in an affine space by Thas [58]. For the case of a generalized quadrangle we refer to [58], in this case some sporadic embeddings can occur. We will however restrict ourselves here to the case of proper partial geometries.

Theorem 4.2.1 ([58])
If $\mathcal{S}$ is a proper partial geometry embedded in an affine space $\mathsf{AG}(n, s+1)$, but not in an $\mathsf{AG}(n', s+1)$ with $n' < n$, then $n = 3$ and $\mathcal{S} = T_2^\star(\mathcal{K})$ with $\mathcal{K}$ a maximal arc in the plane at infinity.

On the embeddings of semipartial geometries in affine spaces the situation we refer for instance to [27] for a discussion on the linear representations. For the partial quadrangles the generic model is $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid in the space $\pi_\infty = \mathsf{PG}(3, s)$. There are some sporadic linear representations known for small values of s and small values of n, but if $\mathcal{S}$ is a linear representation of a partial quadrangle in $\mathsf{AG}(3, q)$, $q \geq 5$, then it was proved [70] that the partial quadrangle has to be $T_3^*(\mathcal{O})$ with $\mathcal{O}$ an ovoid.
On linear representations of semipartial geometries in affine spaces of proper semipartial geometries with $\alpha > 1$, the following models are known.

1. The set $\mathcal{K}$ is a unital $\mathcal{U}$ in the projective plane $\pi_\infty = \mathsf{PG}(2, s)$ $(s = q^2)$ at infinity, and $T_2^*(\mathcal{U})$ has parameters $s = q^2 - 1, t = q^3, \alpha = q, \mu = q^2(q^2 - 1)$.
2. The set $\mathcal{K}$ is a Baer subspace $\mathcal{B}$ of the projective space $\pi_\infty = \mathsf{PG}(n, s)$ $(s = q^2)$ at infinity, and $T_n^*(\mathcal{B})$ has parameters $s = q^2 - 1,\ t = \dfrac{q^{n+1} - 1}{q - 1} - 1,\ \alpha = q,\ \mu = q(q+1)$.

The following nice result is known.

Theorem 4.2.2 ([32])
A $(0, \alpha)$-geometry embedded in $\mathsf{AG}(n, q)$, $n > 2$, $\alpha \neq 1, 2$ is a linear representation $T_{n-1}^*(\mathcal{K})$.

Hence the affine embedding of $(0, \alpha)$-geometries is reduced to the case $\alpha = 1$ and 2. It is known that in the case $\mathsf{AG}(3, q)$ no embedding of semipartial geometry different from the known linear representations can exist. The case of embedding of $(0, 2)$-geometries in $\mathsf{AG}(3, q)$ is open, but some interesting new results have recently been proved by N. Defeyter (work in progress). It would lead us to far to give more details here. In the case $\mathsf{AG}(4,q)$ one model of semipartial geometry embedded in $\mathsf{AG}(4,q)$ is known, namely the semipartial geometry $\mathrm{TQ}(4,q)$ for q even, which is an $\mathsf{spg}(q-1,q^2,2,2q-1)$. Note that there are more geometries known with these parameters. The following theorem is proved.

Theorem 4.2.3 ([5])
Let $\mathcal{S}$ be a semipartial geometry $\mathsf{spg}(q-1,q^2,2,2q-1)$ fully embedded in $\mathsf{AG}(4,q)$. Then $q=2^h$ and $\mathcal{S}$ is isomorphic to $TQ(4,q)$.

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