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On the Strongly Damped Wave Equation

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Abstract: We prove the existence of the universal attractor for the strongly damped semilinear wave equation, in the presence of a quite general nonlinearity of critical growth. When the nonlinearity is subcritical, we prove the existence of an exponential attractor of optimal regularity, having a basin of attraction coinciding with the whole phase-space. As a byproduct, the universal attractor is regular and of finite fractal dimension. Moreover, we carry out a detailed analysis of the asymptotic behavior of the solutions in dependence of the damping coefficient.

1. Introduction

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$. Given $\omega > 0$, we consider the following initial-boundary value problem for $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$:

$$\begin{cases} u_{tt} - \omega\Delta u_t - \Delta u + \phi(u) = f, & x \in \Omega,\ t > 0, \\ u(x,0) = u_0(x), & x \in \Omega, \\ u_t(x,0) = u_1(x), & x \in \Omega, \\ u(x,t) = 0, & x \in \partial\Omega,\ t \geq 0. \end{cases} \qquad (\boldsymbol{P})$$

The semilinear wave equation with strong damping has been investigated by many authors in the last years (see, e.g., [2–4, 7, 10, 14, 16, 17]). In particular, Carvalho and Cholewa [4] have recently proved that for Problem $\boldsymbol{P}$ with the critical nonlinearity (i.e., when the growth of ϕ is of order 5), the associated semigroup possesses a universal attractor. Actually, in [4] the authors analyze a more general situation, with a term of

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the form $(-\Delta)^\theta u_t$, for $\theta \in [\frac{1}{2}, 1]$, in place of $-\Delta u_t$. This was, in our opinion, a significant progress, since the passage from the subcritical to the critical case is highly nontrivial, mainly due to the fact that in the critical situation the embeddings are no longer compact. The key ingredient of [4] is Alekseev's nonlinear variation of constants formula, which has been successfully employed also to establish an analogous result for the weakly damped semilinear wave equation (see [1]). However, the universal attractor is not shown to have the best possible regularity, even in the subcritical case. This lack of regularity prevents a more detailed asymptotic analysis.

In this paper, using a different approach, we prove the existence of a universal attractor for Problem $\boldsymbol{P}$, with a more general nonlinearity than the one used in [4]. Moreover, in the subcritical case, we demonstrate the existence of an exponential attractor of optimal regularity, and in turn the existence of a regular universal attractor of finite fractal dimension. We should mention that the basin of attraction of the exponential attractor is the *whole* phase-space, and not just a compact invariant subset. This is obtained as a consequence of a remarkable result due to Fabrie, Galusinski, Miranville and Zelik [9], who have proved that the exponential attraction enjoys a transitivity property. Indeed, after the paper [9], it is now clear that the interesting object to investigate is the exponential attractor, rather than the universal attractor, which is recovered as a byproduct. Finally, we pursue a detailed analysis of the longtime behavior of solutions in dependence of the damping coefficient $\omega > 0$.

Our technique relies on a bootstrap argument that was envisaged in [11], together with a sharp use of Gronwall-type lemmas.

2. Functional Setting

We denote the inner product and the norm on $L^2(\Omega)$ by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively, and the norm on $L^p(\Omega)$ by $\|\cdot\|_{L^p}$. Let A be the (strictly) positive operator on $L^2(\Omega)$ defined by

$$A = -\Delta \qquad \text{with domain} \qquad \mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega).$$

Identifying $L^2(\Omega)$ with its dual space $L^2(\Omega)^*$, we consider the family of Hilbert spaces $\mathcal{D}(A^{s/2})$, $s \in \mathbb{R}$, whose inner products and norms are given by

$$\langle\cdot,\cdot\rangle_{\mathcal{D}(A^{s/2})} = \langle A^{s/2}\cdot, A^{s/2}\cdot\rangle \qquad \text{and} \qquad \|\cdot\|_{\mathcal{D}(A^{s/2})} = \|A^{s/2}\cdot\|.$$

Then we have

$$\mathcal{D}(A^0) = L^2(\Omega), \qquad \mathcal{D}(A^{1/2}) = H_0^1(\Omega), \qquad \mathcal{D}(A^{-1/2}) = H^{-1}(\Omega),$$

and the compact and dense injections

$$\mathcal{D}(A^{s/2}) \hookrightarrow \mathcal{D}(A^{r/2}), \qquad \forall s > r.$$

In particular, naming α_1 the first eigenvalue of A, we get the inequalities

$$\|A^{r/2}v\| \le {\alpha_1}^{(r-s)/2}\|A^{s/2}v\|, \qquad \forall v \in \mathcal{D}(A^{s/2}). \tag{1}$$

We also recall the continuous embedding

$$\mathcal{D}(A^{s/2}) \hookrightarrow L^{6/(3-2s)}(\Omega), \qquad \forall s \in [0, \tfrac{3}{2}), \tag{2}$$

and the Ehrling lemma, that is, given $s > r > q$, for every $\nu > 0$ there exists $c_\nu > 0$ such that

$$\|A^{r/2}v\| \le \nu \|A^{s/2}v\| + c_\nu \|A^{q/2}v\|, \qquad \forall v \in \mathcal{D}(A^{s/2}). \tag{3}$$

Concerning the phase-spaces for our problem, we consider, for $s \in \mathbb{R}$, the product Hilbert spaces

$$\mathcal{H}_s = \mathcal{D}(A^{(1+s)/2}) \times \mathcal{D}(A^{s/2}),$$

endowed with the usual inner products and norms (denoted by $\|\cdot\|_s$).

Throughout the paper, we will denote by $c \ge 0$ a generic constant, that may vary even from line to line within the same equation, depending only on Ω, ϕ and the external source f. Further dependencies will be specified on occurrence. Also, we will tacitly use (1)–(3), as well as the Young and the generalized Hölder inequalities, and the usual Sobolev embeddings.

We conclude the section with two technical lemmas that will be needed in the course of the investigation.

Lemma 1. *Let X be a Banach space, and let $\mathcal{Z} \subset C(\mathbb{R}^+, X)$. Let $E : X \to \mathbb{R}$ be a function such that*

$$\sup_{t\in\mathbb{R}^+} E(z(t)) \ge -m \qquad \text{and} \qquad E(z(0)) \le M,$$

for some $m, M \ge 0$ and every $z \in \mathcal{Z}$. In addition, assume that for every $z \in \mathcal{Z}$ the function $t \mapsto E(z(t))$ is continuously differentiable, and satisfies the differential inequality

$$\frac{d}{dt}E(z(t)) + \delta\|z(t)\|_X^2 \le k,$$

for some $\delta > 0$ and $k > 0$, both independent of $z \in \mathcal{Z}$. Then

$$E(z(t)) \le \sup_{\zeta\in X}\Big\{E(\zeta) : \delta\|\zeta\|_X^2 \le 2k\Big\}, \qquad \forall t \ge \tfrac{m+M}{k}.$$

The proof can be found, for instance, in [2, Lemma 2.7].

Lemma 2. *Let Φ be an absolutely continuous positive function on $\mathbb{R}^+$, which satisfies for some $\varepsilon > 0$ the differential inequality*

$$\frac{d}{dt}\Phi(t) + 2\varepsilon\Phi(t) \le g(t)\Phi(t) + h(t),$$

for almost every $t \in \mathbb{R}^+$, where g and h are functions on $\mathbb{R}^+$ such that

$$\int_\tau^t |g(y)|dy \le m_1\big(1 + (t-\tau)^\mu\big), \qquad \forall t \ge \tau \ge 0,$$

for some $m_1 \ge 0$ and $\mu \in [0, 1)$, and

$$\sup_{t\ge 0}\int_t^{t+1} |h(y)|dy \le m_2,$$

for some $m_2 \ge 0$. Then

$$\Phi(t) \le \beta\Phi(0)e^{-\varepsilon t} + \rho, \qquad \forall t \in \mathbb{R}^+,$$

for some $\beta = \beta(m_1, \mu) \geq 1$ *and*

$$\rho = \frac{\beta m_2 e^{\varepsilon}}{1 - e^{-\varepsilon}}. \tag{4}$$

For the proof, we refer the reader to [11, Lemma 2.2].

In the applications of the above lemmas, we might not have the required regularity for E and Φ. However, this is not really a problem, since we can always suppose to work within a proper regularization scheme.

3. The Solution Semigroup

We will consider for simplicity a time-independent external source, namely

$$f \in H^{-1}(\Omega) \text{ independent of time,} \tag{5}$$

although the results could be generalized with little effort to the nonautonomous case, provided that f enjoys some translation-compactness properties. Concerning the nonlinearity, we stipulate the following set of assumptions. Let $\phi \in C(\mathbb{R})$ be such that

$$|\phi(r) - \phi(s)| \leq c|r - s|(1 + |r|^4 + |s|^4), \qquad \forall r, s \in \mathbb{R}. \tag{6}$$

Also, let ϕ admit the decomposition

$$\phi = \phi_0 + \phi_1,$$

with $\phi_0 \in C(\mathbb{R})$, $\phi_1 \in C(\mathbb{R})$, satisfying

$$|\phi_0(r)| \leq c(1 + |r|^5), \qquad \forall r \in \mathbb{R}, \tag{7}$$

$$\phi_0(r)r \geq 0, \qquad \forall r \in \mathbb{R}, \tag{8}$$

$$|\phi_1(r)| \leq c(1 + |r|^{\gamma}), \qquad \gamma < 5, \ \forall r \in \mathbb{R}, \tag{9}$$

$$\liminf_{|r|\to\infty} \frac{\phi_1(r)}{r} > -\alpha_1. \tag{10}$$

Without loss of generality, we can think γ large enough, say, $\gamma \geq 3$. Notice that, by virtue of (10), there exists $\alpha < \alpha_1$ such that

$$\phi_1(r)r \geq -\alpha r^2 - c, \qquad \forall r \in \mathbb{R}. \tag{11}$$

Remark 1. It is apparent that we can replace ϕ_0 with $\eta\phi_0$ and ϕ_1 with $\phi_1 + (1 - \eta)\phi_0$, where η is a smooth function with values in [0, 1], such that $\eta(r) = 0$ if $|r| \leq 1$, and $\eta(r) = 1$ if $|r| \geq 2$. Then ϕ_0 and ϕ_1 still fulfill (8)–(10); moreover,

$$|\phi_0(r)| \leq c(|r| + |r|^5), \qquad \forall r \in \mathbb{R}. \tag{12}$$

Thus, in the sequel, we will assume the stronger condition (12) in place of (7).

Remark 2. Analogously to what observed in [1, Lemma 1.2], a function $\phi \in C(\mathbb{R})$ such that

$$|\phi(r)| \leq c(1 + |r|^5), \qquad \forall r \in \mathbb{R},$$

$$\liminf_{|r|\to\infty} \frac{\phi(r)}{r} > -\alpha_1,$$

admits a decomposition $\phi = \phi_0 + \phi_1$ satisfying (7)–(10).

Throughout the paper, we will assume conditions (5)–(6), (8)–(10), and (12).

By [3] (see also [4, Theorem 1]), the following holds.

Theorem 1. *For every $T > 0$, and every $(u_0, u_1) \in \mathcal{H}_0$, Problem* ***P*** *admits a unique weak solution*

$$u \in C([0,T], H_0^1(\Omega)),$$

with

$$u_t \in C([0,T], L^2(\Omega)) \cap L^2([0,T], H_0^1(\Omega)),$$

which continuously depend on the initial data. In other words, Problem ***P*** *generates a strongly continuous semigroup $S(t)$ on the phase space $\mathcal{H}_0$.*

Actually, the result has been proved for $f \equiv 0$, but it holds as well in the present case.

Remark 3. As a matter of fact, it is not hard to check that, when $f \in L^2(\Omega)$, $S(t)$ is a strongly continuous semigroup also on the phase space $\mathcal{H}_1$.

For further use, let us write down explicitly the continuous dependence estimate for $S(t)$ on $\mathcal{H}_0$.

Theorem 2. *Given any $R > 0$ and any two initial data $z_0, z_1 \in \mathcal{H}_0$ such that $\|z_0\|_0 \le R$ and $\|z_1\|_0 \le R$, there holds*

$$\|S(t)z_0 - S(t)z_1\|_0 \le e^{\frac{K}{\omega}t}\|z_0 - z_1\|_0, \qquad \forall t \in \mathbb{R}^+, \tag{13}$$

for some $K = K(R)$.

Proof. Given two solutions u^1 and u^2 corresponding to different initial data, the difference $\bar{u} = u^1 - u^2$ fulfills the inequality

$$\frac{d}{dt}\big(\|A^{1/2}\bar{u}\|^2 + \|\bar{u}_t\|^2\big) + 2\omega\|A^{1/2}\bar{u}_t\|^2 = -2\langle \phi(u^1) - \phi(u^2), \bar{u}_t\rangle.$$

Taking into account the uniform energy estimates for the solutions (see the subsequent Theorem 3), from (6) we have, for every $\nu > 0$,

$$\begin{aligned}
-2\langle \phi(u^1) - \phi(u^2), \bar{u}_t\rangle &\le c\big(1 + \|u^1\|_{L^6}^4 + \|u^2\|_{L^6}^4\big)\|\bar{u}\|_{L^6}\|\bar{u}_t\|_{L^6} \\
&\le c\big(1 + \|A^{1/2}u^1\|^4 + \|A^{1/2}u^2\|^4\big)\|A^{1/2}\bar{u}\|\|A^{1/2}\bar{u}_t\| \\
&\le \frac{k}{\nu}\|A^{1/2}\bar{u}\|^2 + k\nu\|A^{1/2}\bar{u}_t\|^2,
\end{aligned}$$

for some $k = k(R)$. Therefore, setting $\nu = \frac{2\omega}{k}$, we obtain

$$\frac{d}{dt}\big(\|A^{1/2}\bar{u}\|^2 + \|\bar{u}_t\|^2\big) \le \frac{k^2}{2\omega}\big(\|A^{1/2}\bar{u}\|^2 + \|\bar{u}_t\|^2\big),$$

and the assertion follows from the Gronwall lemma. □

4. Dissipativity

We now deal with the dissipative feature of the semigroup $S(t)$. Namely, we show that the trajectories originating from any given bounded set eventually fall, uniformly in time, into a bounded absorbing set $\mathcal{B}_0 \subset \mathcal{H}_0$. In order to highlight the dependence on $\omega > 0$, we introduce the function

$$\Theta(\omega) = \begin{cases} \omega, & \omega < 1, \\ \frac{1}{\omega}, & \omega \geq 1. \end{cases} \tag{14}$$

Theorem 3. *There exists a constant $R_0 > 0$ with the following property: given any $R \geq 0$, there exist $t_0 = t_0(R, \omega)$ such that, whenever*

$$\|z_0\|_0 \leq R,$$

it follows that

$$\|S(t)z_0\|_0 \leq R_0, \qquad \forall t \geq t_0.$$

Consequently, the set

$$\mathcal{B}_0 = \left\{z_0 \in \mathcal{H}_0 : \|z_0\|_0 \leq R_0\right\}$$

is a bounded absorbing set for $S(t)$ on $\mathcal{H}_0$, that is, for any bounded set $\mathcal{B} \subset \mathcal{H}_0$, there is $t_0 = t_0(\mathcal{B}, \omega)$ such that $S(t)\mathcal{B} \subset \mathcal{B}_0$ for every $t \geq t_0$.

Remark 4. Before proceeding to the proof, let us dwell on the physical meaning of Theorem 3. Firstly, the solution corresponding to any set of initial data, after a certain time t_0 (depending only on the size of the data) is controlled in norm by the constant R_0. Notice that R_0 does not depend on the damping coefficient. What makes the difference is actually the time t_0 needed to stabilize the system. As shown in formula (21) below, t_0 is an increasing function of R, and so far this is no surprise, since the larger are the initial data, the larger is the time needed to squeeze them. Less evident, at a first glance, is the dependence on ω. Indeed, for a fixed R, $t_0 \to \infty$ both if $\omega \to 0$ and $\omega \to \infty$. This is obvious when $\omega \to 0$, for in the limiting case $\omega = 0$ the dissipation is lost. On the other hand, a very large damping has the effect of freezing the system, since the damping acts only on the velocity u_t, and this prevents the squeezing of the component u. Therefore the most dissipative situation occurs in between, that is, for a certain damping ω_*, which depends on the other coefficients of the equation (in our case, for simplicity, they are all set equal to 1).

Proof of Theorem 3. In view of a further use, throughout this proof, besides c, we will also employ the generic constant $c_0 \geq 0$, which is independent of R and vanishes if $\phi_1 \equiv 0$ and $f \equiv 0$. Denoting

$$z(t) = S(t)z_0 = (u(t), u_t(t)),$$

we consider the functional

$$\mathcal{F}(t) = \mathcal{F}(u(t)) = 2 \int_\Omega \int_0^{u(x,t)} \phi(y)dydx.$$

Set $\varpi = \min\{1, \omega\}$. Given $\varepsilon \in [0, \varepsilon_0]$, for some $\varepsilon_0 \leq 1$ to be determined later, we introduce the auxiliary variable

$$\xi(t) = u_t(t) + \frac{\varepsilon\varpi}{\omega}u(t).$$

Testing the equation with ξ yields

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}E+\frac{\varepsilon\varpi}{\omega}(1-\varepsilon\varpi)\|A^{1/2}u\|^2+\omega\|A^{1/2}\xi\|^2\\=\frac{\varepsilon\varpi}{\omega}\|\xi\|^2-\frac{\varepsilon^2\varpi^2}{\omega^2}\langle u,\xi\rangle+\frac{\varepsilon\varpi}{\omega}\langle f,u\rangle-\frac{\varepsilon\varpi}{\omega}\langle\phi(u),u\rangle,\end{aligned} \tag{15}$$

where the energy functional E is defined as

$$E(t)=E(z(t))=(1-\varepsilon\varpi)\|A^{1/2}u(t)\|^2+\|\xi(t)\|^2+\mathcal{F}(t)-2\langle f,u(t)\rangle.$$

By (5), (9) and (12), we get the bound from above

$$E(t)\le c\big(1+\|z(t)\|_0^6\big), \tag{16}$$

whereas by (5), (8) and (11), and the continuity of ϕ_1, we find the bound from below

$$E(t)\ge\lambda\|z(t)\|_0^2-c_0, \tag{17}$$

provided that ε_0 is small enough, for some (possibly very small) $\lambda>0$. We now proceed to the evaluation of the right-hand side of (15). Making use of (8) and (11),

$$\langle\phi_0(u),u\rangle\ge 0,\qquad\langle\phi_1(u),u\rangle\ge-(1-2\lambda_0)\|A^{1/2}u\|^2-c_0, \tag{18}$$

for some $\lambda_0\in(0,\frac{1}{2})$. All the constants appearing here are independent of $\varepsilon\in[0,\varepsilon_0]$. In particular, λ and λ_0 depend only on the value of the limit in (10). Using now (18) and the inequalities

$$\begin{aligned}-\frac{\varepsilon^2\varpi^2}{\omega^2}\langle u,\xi\rangle&\le\frac{\varepsilon^3\varpi^3}{4\alpha_1\omega^3}\|A^{1/2}u\|^2+\frac{\varepsilon\varpi}{\omega}\|\xi\|^2,\\\frac{\varepsilon\varpi}{\omega}\langle f,u\rangle&\le\frac{\varepsilon\varpi\lambda_0}{\omega}\|A^{1/2}u\|^2+\frac{\varepsilon\varpi}{\omega}c_0,\end{aligned}$$

we get from (15) the differential inequality

$$\frac{d}{dt}E+\frac{2\varepsilon\varpi}{\omega}\Big(\lambda_0-\varepsilon\varpi-\frac{\varepsilon^2\varpi^2}{4\alpha_1\omega^2}\Big)\|A^{1/2}u\|^2+\Big(2\alpha_1\omega-\frac{4\varepsilon\varpi}{\omega}\Big)\|\xi\|^2\le\frac{\varepsilon\varpi}{\omega}c_0,$$

which, for ε_0 small enough, becomes

$$\frac{d}{dt}E+\frac{\varepsilon\varpi\lambda_0}{\omega}\|A^{1/2}u\|^2+\Big(2\alpha_1\omega-\frac{4\varepsilon\varpi}{\omega}\Big)\|\xi\|^2\le\frac{\varepsilon\varpi}{\omega}c_0. \tag{19}$$

All the inequalities we wrote so far, hold for every $\varepsilon\in[0,\varepsilon_0]$, provided that ε_0 is small enough. However, the size of ε_0 does not depend on ω. At this point, we need to treat separately two cases. So, assume first that $\omega\ge 1$. Then it is easy to see that

$$2\alpha_1\omega-\frac{4\varepsilon\varpi}{\omega}\ge 2\alpha_1-4\varepsilon\ge\frac{\varepsilon\lambda_0}{\omega}=\frac{\varepsilon\varpi\lambda_0}{\omega},$$

up to taking ε_0 possibly smaller. On the contrary, if $\omega<1$, then

$$2\alpha_1\omega-\frac{4\varepsilon\varpi}{\omega}=2\alpha_1\omega-4\varepsilon\ge\varepsilon\lambda_0=\frac{\varepsilon\varpi\lambda_0}{\omega},$$

provided that ε_0 is of the form $\tilde{\varepsilon}\omega$, for some small $\tilde{\varepsilon}$. In either case, the term $\frac{\varepsilon\varpi\lambda_0}{\omega}$ can be given the form $2\delta\Theta(\omega)$, for some $\delta > 0$ small (independently of ω). Then, inequality (19) reads

$$\frac{d}{dt}E + 2\delta\Theta(\omega)\big(\|A^{1/2}u\|^2 + \|\xi\|^2\big) \le c_0\Theta(\omega), \tag{20}$$

which in turn (if δ is sufficiently small) entails

$$\frac{d}{dt}E + \delta\Theta(\omega)\|z\|_0^2 \le c_0\Theta(\omega).$$

Applying now Lemma 1, on account of (16)–(17), and taking c_0 strictly positive, there exists

$$t_0 = t_0(R,\omega) = \frac{c_0 + c(1+R^6)}{c_0\Theta(\omega)} \tag{21}$$

such that

$$E(z(t)) \le \sup_{\zeta\in\mathcal{H}_0}\Big\{E(\zeta) : \delta\Theta(\omega)\|\zeta\|_0^2 \le 2c_0\Theta(\omega)\Big\}, \qquad \forall t \ge t_0,$$

that is equivalent to

$$E(z(t)) \le \sup_{\zeta\in\mathcal{H}_0}\Big\{E(\zeta) : \delta\|\zeta\|_0^2 \le 2c_0\Big\}, \qquad \forall t \ge t_0.$$

The thesis then follows from (16)–(17). □

Corollary 1. *Given any $R \ge 0$, there exist $K_0 = K_0(R)$ and $\Lambda_0 = \Lambda_0(R)$ such that, whenever $\|z_0\|_0 \le R$, the corresponding solution $S(t)z_0 = (u(t), u_t(t))$ fulfills*

$$\|S(t)z_0\|_0 \le K_0, \qquad \forall t \in \mathbb{R}^+,$$

and

$$\omega\int_0^\infty \|A^{1/2}u_t(y)\|^2 dy \le \Lambda_0.$$

Proof. Set $\varepsilon = 0$ and integrate (15), on account of (16)–(17). □

Incidentally, the set $\big\{z_0 \in \mathcal{H}_0 : \|S(t)z_0\|_0 \le K_0(R_0),\ \forall t \in \mathbb{R}^+\big\}$ turns out to be a bounded absorbing set for $S(t)$ on $\mathcal{H}_0$ which is invariant under the action of the semigroup.

5. The Universal Attractor

The aim of this section is to prove the existence of a universal attractor for $S(t)$ on $\mathcal{H}_0$. Recall that the universal attractor is the (unique) compact set $\mathcal{A} \subset \mathcal{H}_0$, which is at the same time attracting, in the sense of the Hausdorff semidistance, and fully invariant for $S(t)$, that is, $S(t)\mathcal{A} = \mathcal{A}$ for all $t \in \mathbb{R}^+$ (see, e.g., [12, 15]).

Theorem 4. *For every $\omega > 0$, the semigroup $S(t)$ possesses a connected universal attractor $\mathcal{A} = \mathcal{A}(\omega) \subset \mathcal{H}_0$.*

In order to prove Theorem 4, and for further purposes, we decompose the solution u to Problem $\boldsymbol{P}$ with initial data $z_0 = (u_0, u_1) \in \mathcal{H}_0$ into the sum

$$u(t) = v(t) + w(t),$$

where v and w are the solutions to the problems

$$\begin{cases} v_{tt} + \omega A v_t + A v + \phi_0(v) = 0, \\ v(0) = u_0, \\ v_t(0) = u_1, \end{cases} \tag{22}$$

and

$$\begin{cases} w_{tt} + \omega A w_t + A w + \phi(u) - \phi_0(v) = f, \\ w(0) = 0, \\ w_t(0) = 0. \end{cases} \tag{23}$$

It is convenient to denote

$$z(t) = (u(t), u_t(t)), \qquad z_d(t) = (v(t), v_t(t)), \qquad z_c(t) = (w(t), w_t(t)).$$

As a first step, we show that z_d has an exponential decay in $\mathcal{H}_0$, which is uniform as z_0 runs into a bounded subset of $\mathcal{H}_0$.

Lemma 3. *Given any $R \geq 0$, there exist $M_0 = M_0(R) \geq 0$ and $\nu_0 = \nu_0(R) > 0$ such that, whenever $\|z_0\|_0 \leq R$, it follows that*

$$\|z_d(t)\|_0 \leq M_0 e^{-\nu_0 \Theta(\omega) t}, \qquad \forall t \in \mathbb{R}^+,$$

with $\Theta(\omega)$ given by (14)*. The constants M_0 and ν_0 depend increasingly and decreasingly, respectively, on R.*

Proof. Repeating word by word the proof of Theorem 3, that applies to the present case with $z_d(t)$ in place of $z(t)$ (with the further simplification that $c_0 = 0$, for now $\phi_1 \equiv 0$ and $f \equiv 0$), we get the differential inequality

$$\frac{d}{dt} E + \frac{\varepsilon \varpi \lambda_0}{\omega} \|A^{1/2} v\|^2 + \Big(2\alpha_1 \omega - \frac{4\varepsilon\varpi}{\omega}\Big) \|\xi\|^2 \leq 0, \tag{24}$$

for some ε_0 small enough, independent of ω. Integrating (24) for $\varepsilon = 0$ on $(0, t)$ gives

$$\sup_{\|z_0\| \leq R} \sup_{t \in \mathbb{R}^+} \|z_d(t)\|_0 < \infty.$$

Hence, we find the uniform estimate

$$\mathcal{F}(t) \leq c\big(\|v(t)\|^2 + \|v(t)\|^6_{L^6}\big) \leq k \|A^{1/2} v(t)\|^2, \qquad \forall t \geq 0,$$

for some constant $k = k(R) \geq 1$. Upon taking ε_0 small enough, we may replace the term $\|A^{1/2} v\|^2$ appearing in (24) with

$$\frac{(1 - \varepsilon\varpi)}{2k} \|A^{1/2} v\|^2 + \frac{1}{2k} \mathcal{F} = \frac{1}{2k} E - \frac{1}{2k} \|\xi\|^2,$$

so obtaining

$$\frac{d}{dt}E + \frac{\varepsilon\varpi\lambda_0}{2k\omega}E + \Big(2\alpha_1\omega - \frac{\varepsilon\varpi m}{\omega}\Big)\|\xi\|^2 \le 0,$$

where we set for simplicity $m = m(R) = 4 + \frac{\lambda_0}{2k}$. Again, arguing as in the proof of Theorem 3, we see that

$$2\alpha_1\omega - \frac{\varepsilon\varpi m}{\omega} \ge 0,$$

provided the term $\frac{\varepsilon\varpi\lambda_0}{2k\omega}$ is given the form $2\nu_0\Theta(\omega)$. The only difference here is that ν_0 is not a constant any longer, but a decreasing function of R. Thus we end up with

$$\frac{d}{dt}E + 2\nu_0\Theta(\omega)E \le 0.$$

By means of the Gronwall lemma, and using subsequently (16)–(17) (recall that $c_0 = 0$), the proof is completed. □

A straightforward consequence is

Corollary 2. *If $\phi_1 \equiv 0$ and $f \equiv 0$, then $S(t)$ decays to zero. Thus the set $\{0\} \subset \mathcal{H}_0$ is the universal attractor for $S(t)$ on $\mathcal{H}_0$.*

Next we show that, for every fixed time, the component z_c belongs to a compact subset of $\mathcal{H}_0$, uniformly as the initial data z_0 belongs to the absorbing set $\mathcal{B}_0$, given by Theorem 3.

Lemma 4. *For every time $T \in \mathbb{R}^+$ and every $\omega > 0$, there exists a compact set $\mathcal{K}_{T,\omega} \subset \mathcal{H}_0$ such that*

$$\bigcup_{z_0\in\mathcal{B}_0} z_c(t) \in \mathcal{K}_{T,\omega}, \qquad \forall t \in [0, T].$$

Proof. The constant c appearing in this proof may depend on $K_0(R_0)$ (given by Theorem 3), which is however a fixed value. Due to Corollary 1 and Lemma 3,

$$\|A^{1/2}u(t)\| + \|A^{1/2}v(t)\| \le c, \qquad \forall t \in \mathbb{R}^+.$$

Choosing

$$\sigma = \min\left\{\tfrac{1}{4}, \tfrac{5-\gamma}{2}\right\},$$

and multiplying (23) times $A^\sigma w_t$, we are led to the identity

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\|z_c\|_\sigma^2 + \omega\|A^{(1+\sigma)/2}w_t\|^2\\ &\quad= -\langle\phi(u) - \phi(v), A^\sigma w_t\rangle - \langle\phi_1(v), A^\sigma w_t\rangle + \langle f, A^\sigma w_t\rangle.\end{aligned} \tag{25}$$

By virtue of (6) we get

$$\begin{aligned}&-\langle\phi(u) - \phi(v), A^\sigma w_t\rangle\\ &\quad\le c\big(1 + \|u\|_{L^6}^4 + \|v\|_{L^6}^4\big)\|w\|_{L^{6/(1-2\sigma)}}\|A^\sigma w_t\|_{L^{6/(1+2\sigma)}}\\ &\quad\le c\big(1 + \|A^{1/2}u\|^4 + \|A^{1/2}v\|^4\big)\|A^{(1+\sigma)/2}w\|\|A^{(1+\sigma)/2}w_t\|\\ &\quad\le \frac{c}{\omega}\|z_c\|_\sigma^2 + \frac{\omega}{3}\|A^{(1+\sigma)/2}w_t\|^2.\end{aligned} \tag{26}$$

Since $\frac{\gamma}{5-2\sigma} \leq 1$, by (9) we deduce that

$$
\begin{aligned}
&-\langle \phi_1(v), A^\sigma w_t \rangle \\
&\quad \leq c\big(1 + \|v\|^\gamma_{L^{6\gamma/(5-2\sigma)}}\big)\|A^\sigma w_t\|_{L^{6/(1+2\sigma)}} \\
&\quad \leq c\big(1 + \|A^{1/2} v\|^\gamma\big)\|A^{(1+\sigma)/2} w_t\| \\
&\quad \leq \frac{c}{\omega} + \frac{\omega}{3}\|A^{(1+\sigma)/2} w_t\|^2.
\end{aligned}
\tag{27}
$$

Finally,

$$
\langle f, A^\sigma w_t \rangle \leq \|A^{-1/2} f\|\|A^{(1+\sigma)/2} w_t\| \leq \frac{c}{\omega} + \frac{\omega}{3}\|A^{(1+\sigma)/2} w_t\|^2. \tag{28}
$$

Plugging (26)–(28) into (25), we obtain

$$
\frac{d}{dt}\|z_c\|^2_\sigma \leq \frac{c}{\omega}\|z_c\|^2_\sigma + \frac{c}{\omega},
$$

and the Gronwall lemma entails

$$
\|z_c(t)\|^2_\sigma \leq e^{\frac{kt}{\omega}} - 1,
$$

which concludes the proof. □

Collecting now Theorem 3, Lemma 3 and Lemma 4, we establish that $S(t)$ is asymptotically smooth. Therefore, by means of well-known results of the theory of dynamical systems (see, e.g., [12]), Theorem 4 is proved.

Remark 5. On account of Lemma 3 and Lemma 4, for every $T \in \mathbb{R}^+$ the attractor $\mathcal{A}$ belongs to a $M_0 e^{-\nu_0 \Theta(\omega) T}$-neighborhood of $\mathcal{K}_{T,\omega}$. Note that, as $\omega \to \infty$, the set $\mathcal{K}_{T,\omega}$ shrinks, but the constant $M_0 e^{-\nu_0 \Theta(\omega) T}$ increases. This seems to suggest that the "smallest" attractor occurs for a certain ω_*, away from zero and infinity. We will come back on this in the next sections, where we discuss the dependence on ω in the subcritical case.

Remark 6. With the additional assumptions $f \in L^2(\Omega)$ and $\phi_0 \in C^1(\mathbb{R})$ with $\phi_0' \geq 0$, it is also possible to prove that the semigroup $S(t)$ possesses a universal attractor $\mathcal{A}_1$ on the phase-space $\mathcal{H}_1$. Clearly, $\mathcal{A}_1 \subset \mathcal{A}$. If we could prove that $\mathcal{A}$ is a bounded subset of $\mathcal{H}_1$, then, on account of the maximality properties of universal attractors (cf. [15]), we would have the reverse inclusion. As a consequence, $\mathcal{A}$ would not only be bounded, but also compact in $\mathcal{H}_1$. In general, one cannot have an $\mathcal{H}_1$-bound for $\mathcal{A}$ assuming only $f \in H^{-1}(\Omega)$. This follows from the fact that the stationary points (which belong to the attractor) solve the equation

$$
-\Delta \tilde{u} + \phi(\tilde{u}) = f,
$$

and are as regular as f permits. In particular, if $f \in H^{-1}(\Omega)$, but not more, then $\tilde{u} \in H^1_0(\Omega)$, but not more. In this case $\mathcal{A}$ cannot be a subset of $\mathcal{H}_\sigma$ for any $\sigma > 0$.

6. The Subcritical Case: Further Regularity

In the last two sections, we want to pursue a quite detailed asymptotic analysis when f is more regular and the nonlinearity is subcritical. More precisely, we make the extra assumptions

$$f \in L^2(\Omega) \text{ independent of time,} \tag{29}$$

$$\phi_0(r) = 0, \qquad \forall r \in \mathbb{R}, \tag{30}$$

$$\phi_1 \in C^1(\mathbb{R}) \quad \text{with} \quad |\phi_1'(r)| \leq c(1 + |r|^{\gamma-1}), \qquad \forall r \in \mathbb{R}. \tag{31}$$

Also, we focus on the case when ω is separated from zero. As we will see, this situation is much more interesting (see however Remark 9 at the end).

To this aim, we assume

$$\omega \geq \omega_0, \text{ for some } \omega_0 > 0. \tag{32}$$

All the constants and the sets appearing in the sequel are *independent* of $\omega \geq \omega_0$ (but they do depend on ω_0). Accordingly, all the estimates we will provide are understood to be *uniform* as $\omega \geq \omega_0$.

From now on, let conditions (10) and (29)–(32) hold.

Remark 7. On account of (30)–(32), Lemma 3 simplifies as follows: given any $R \geq 0$, there exist $M_0 = M_0(R) \geq 0$ and $\nu_0 > 0$ (independent of R), such that, whenever

$$\|z_0\|_0 \leq R,$$

it follows that

$$\|z_d(t)\|_0 \leq M_0 e^{-\frac{\nu_0}{\omega} t}, \qquad \forall t \in \mathbb{R}^+.$$

To be more precise, $M_0(R) = cR$, for some $c > 1$.

The goal of this section is to prove the existence of a bounded set $\mathcal{B}_1 \subset \mathcal{H}_1$ which is an attracting set in $\mathcal{H}_0$, with an exponential rate of attraction. Clearly, it is enough to prove the attraction property on the absorbing set $\mathcal{B}_0$.

Let us state the result.

Theorem 5. *There exist $M \geq 0$, $\nu > 0$, and a set $\mathcal{B}_1$, closed and bounded in $\mathcal{H}_1$, such that*

$$\text{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0, \mathcal{B}_1) \leq M e^{-\frac{\nu}{\omega} t}, \qquad \forall t \in \mathbb{R}^+,$$

where $\text{dist}_{\mathcal{H}_0}$ *denotes the usual Hausdorff semidistance in $\mathcal{H}_0$.*

In light of Remark 6, a straightforward consequence is

Corollary 3. *The universal attractor $\mathcal{A}$ of $S(t)$ on $\mathcal{H}_0$ is a compact subset of $\mathcal{H}_1$. Also, its $\mathcal{H}_1$-bound is uniform as $\omega \geq \omega_0$.*

The proof of Theorem 5 will be carried out by means of several lemmas. The main ingredient is a bootstrap procedure, along the lines of [11]. We will keep the same notation of Sect. 5 (with $\phi_0 \equiv 0$); in particular, we will use again the decomposition $z = z_d + z_c$.

Lemma 5. *Let* $\sigma \in [0,1]$ *be given. Assume that* $\|z_0\|_\sigma \le R_\sigma$*, for some* $R_\sigma \ge 0$*. Then there exist constants*

$$K_\sigma = K_\sigma(R_\sigma) \ge 0, \qquad \Lambda_\sigma = \Lambda_\sigma(R_\sigma) \ge 0, \qquad \mu_\sigma = \mu_\sigma(R_\sigma) \in [0,1)$$

such that

$$\|z(t)\|_\sigma \le K_\sigma, \qquad \forall t \in \mathbb{R}^+, \tag{33}$$

and

$$\omega \int_\tau^t \|A^{(1+\sigma)/2} u_t(y)\|^2 dy \le \Lambda_\sigma \big(1 + (t-\tau)^{\mu_\sigma}\big), \qquad \forall t \ge \tau,\ \tau \in \mathbb{R}^+. \tag{34}$$

Proof. The result for $\sigma = 0$ is already demonstrated, an account of Corollary 1. We will reach the desired conclusion by means of a bootstrap argument. Namely, assuming the result true for a certain $\sigma \in [0,1)$, we show that the thesis holds for $\sigma + s$, for all

$$s \le \min\left\{\tfrac{1}{4}, \tfrac{5-\gamma}{2}, 1-\sigma\right\}.$$

It is thus apparent that, after a finite number of steps, we get the assertion for all $\sigma \in [0,1]$. Let then $\sigma \in [0,1)$ be fixed. By the bootstrap hypothesis, (33)–(34) hold for such σ. Along the proof, the generic constant $c \ge 0$ will depend on R_σ. It is convenient to consider separately two cases.

Case 1. $\sigma < \frac{1}{2}$. Given $\varepsilon \in [0, 2\varepsilon_0]$, with $\varepsilon_0 > 0$ to be determined later, set $\xi = u_t + \frac{\varepsilon}{\omega} u$ and define

$$\Phi(t) = (1-\varepsilon)\|A^{(1+\sigma+s)/2}u(t)\|^2 + \|A^{(\sigma+s)/2}\xi(t)\|^2 + \mathcal{G}(t) + k_0,$$

for some $k_0 = k_0(R_\sigma) \ge 0$, where $\mathcal{G}$ is the functional

$$\mathcal{G}(t) = 2\langle \phi_1(u(t)), A^{\sigma+s}u(t)\rangle - 2\langle f, A^{\sigma+s}u(t)\rangle.$$

Choosing k_0 large enough and ε_0 small enough, we have

$$\frac{1}{2}\|z(t)\|^2_{\sigma+s} \le \Phi(t) \le 2\|z(t)\|^2_{\sigma+s} + c, \tag{35}$$

for all $\varepsilon \in [0, 2\varepsilon_0]$. Indeed,

$$\begin{aligned}
&2\big|\langle \phi_1(u), A^{\sigma+s}u\rangle\big| \\
&\le c\big(1 + \|u\|_{L^{6/(6+2\gamma\sigma-\gamma-4\sigma-2s)}} \|u\|^{\gamma-1}_{L^{6/(1-2\sigma)}}\big)\|A^{\sigma+s}u\|_{L^{6/(1+2\sigma+2s)}} \\
&\le c\big(1 + \|A^{q/2}u\|\|A^{(1+\sigma)/2}u\|^{\gamma-1}\big)\|A^{(1+\sigma+s)/2}u\| \\
&\le c\big(1 + \|A^{q/2}u\|\big)\|A^{(1+\sigma+s)/2}u\|,
\end{aligned}$$

where

$$q = \max\left\{\tfrac{\gamma+4\sigma+2s-3-2\gamma\sigma}{2}, 0\right\}.$$

Since $q < 1 + \sigma + s$, using (33) we get

$$\|A^{q/2}u\| \le \nu\|A^{(1+\sigma+s)/2}u\| + c_\nu,$$

for an arbitrarily small constant $\nu > 0$ and some $c_\nu = c_\nu(R_\sigma) > 0$. This gives at once the inequality

$$2\left|\langle \phi_1(u), A^{\sigma+s}u\rangle\right| \le \frac{1}{4}\|z(t)\|^2_{\sigma+s} + c.$$

Finally, it is straightforward to see that

$$2\left|\langle f, A^{\sigma+s}u\rangle\right| \le \frac{1}{4}\|z(t)\|^2_{\sigma+s} + c.$$

Multiplying the equation times $A^{\sigma+s}\xi$, we are led to the identity

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\Phi + \frac{\varepsilon}{\omega}(1-\varepsilon)\|A^{(1+\sigma+s)/2}u\|^2 + \omega\|A^{(1+\sigma+s)/2}\xi\|^2 + \frac{\varepsilon}{2\omega}\mathcal{G}\\ = \frac{\varepsilon}{\omega}\|A^{(\sigma+s)/2}\xi\|^2 - \frac{\varepsilon^2}{\omega^2}\langle A^{(\sigma+s)/2}u, A^{(\sigma+s)/2}\xi\rangle + \langle \phi_1'(u)u_t, A^{\sigma+s}u\rangle. \quad (36)\end{aligned}$$

There holds

$$-\frac{\varepsilon^2}{\omega^2}\langle A^{(\sigma+s)/2}u, A^{(\sigma+s)/2}\xi\rangle \le \frac{\varepsilon^3}{4\alpha_1\omega^3}\|A^{(1+\sigma+s)/2}u\|^2 + \frac{\varepsilon}{\omega}\|A^{(\sigma+s)/2}\xi\|^2.$$

Moreover, since $\frac{3(\gamma-1)}{2-s} \le 6$, we deduce from (9) and (35) that

$$\begin{aligned}&\langle \phi_1'(u)u_t, A^{\sigma+s}u\rangle\\ &\le c\big(1 + \|u\|^{\gamma-1}_{L^{3(\gamma-1)/(2-s)}}\big)\|u_t\|_{L^{6/(1-2\sigma)}}\|A^{\sigma+s}u\|_{L^{6/(1+2\sigma+2s)}}\\ &\le c\big(1 + \|A^{1/2}u\|^{\gamma-1}\big)\|A^{(1+\sigma)/2}u_t\|\|A^{(1+\sigma+s)/2}u\|\\ &\le c\|A^{(1+\sigma)/2}u_t\|\|A^{(1+\sigma+s)/2}u\|\\ &\le c\|A^{(1+\sigma)/2}u_t\| + c\|A^{(1+\sigma)/2}u_t\|\Phi.\end{aligned}$$

By virtue of the above inequalities, the right-hand side of (36) is less than or equal to

$$\frac{\varepsilon^3}{4\alpha_1\omega^3}\|A^{(1+\sigma+s)/2}u\|^2 + \frac{2\varepsilon}{\omega}\|A^{(\sigma+s)/2}\xi\|^2 + h\Phi + h,$$

having set

$$h(t) = c\|A^{(1+\sigma)/2}u_t(t)\|.$$

It is then clear that, fixing ε_0 small enough, we find the differential inequality

$$\frac{d}{dt}\Phi + \frac{\varepsilon}{\omega}\Phi + \omega\|A^{(1+\sigma+s)/2}\xi\|^2 \le h\Phi + h + \frac{k_0\varepsilon}{\omega}, \quad (37)$$

that holds for all $\varepsilon \in [0, 2\varepsilon_0]$. From (34) and the Hölder inequality,

$$\omega\int_\tau^t h(y)dy \le c\big(1 + (t-\tau)^\mu\big), \qquad \forall t \ge \tau,\ \tau \in \mathbb{R}^+, \quad (38)$$

with $\mu = \frac{\mu_\sigma+1}{2} < 1$. So we are in the hypotheses of Lemma 2. Setting $\varepsilon = 2\varepsilon_0$, and using (35), we obtain

$$\|z(t)\|^2_{\sigma+s} \le c\big(1 + \|z_0\|^2_{\sigma+s}\big)e^{-\frac{\varepsilon_0}{\omega}} + \rho, \qquad \forall t \in \mathbb{R}^+, \quad (39)$$

where, recalling (4), ρ is given by

$$\rho = \frac{ce^{\frac{\varepsilon_0}{\omega}}}{\omega(1-e^{-\frac{\varepsilon_0}{\omega}})} \le c \quad \text{as } \omega \ge \omega_0.$$

Hence (33) holds for $\sigma + s$. Actually, (39) says a little bit more, since the desired result is $\|z(t)\|_{\sigma+s} \le K_{\sigma+s}(R_{\sigma+s})$, whereas the constant ρ depends only on R_σ. This allows us, for instance, to prove the existence of bounded absorbing sets for $S(t)$ on the phase-space $\mathcal{H}_\sigma$, for all $\sigma \in [0, 1]$. Finally, setting $\varepsilon = 0$ in (37), and using the bound on $\|z(t)\|_{\sigma+s}$, which in turn furnishes a bound on Φ, we get

$$\frac{d}{dt}\Phi + \omega\|A^{(1+\sigma+s)/2}u_t\|^2 \le \tilde{c}h,$$

for some $\tilde{c} = \tilde{c}(R_{\sigma+s})$. Integration on (τ, t), on account of (38), entails (34) for $\sigma + s$.

Case 2. $\sigma \ge \frac{1}{2}$. Exploiting Case 1, we readily learn that the theorem holds for all $\sigma \in [\frac{1}{2}, \tilde{\sigma}]$, for some $\tilde{\sigma} > \frac{1}{2}$. Hence, if $\sigma \ge \tilde{\sigma}$, in particular we get that $\|A^{(1+\tilde{\sigma})/2}u(t)\| \le c$, and the continuous embedding $\mathcal{D}(A^{(1+\tilde{\sigma})/2}) \hookrightarrow L^\infty(\Omega)$ bears the uniform bound

$$\sup_{t\in\mathbb{R}^+} \|u(t)\|_{L^\infty} \le c. \tag{40}$$

The proof then goes exactly as in the previous case, with the difference that now the estimates are almost immediate, due to the control (40). The details are left to the reader. □

Lemma 6. *Let* $\sigma \in [0, 1)$ *be given, and set*

$$s = s(\sigma) = \min\left\{\tfrac{1}{4}, \tfrac{5-\gamma}{2}, 1-\sigma\right\}. \tag{41}$$

Given any $R_\sigma \ge 0$*, there exists* $R_{\sigma+s} = R_{\sigma+s}(R_\sigma)$ *such that, if* $\|z_0\|_\sigma \le R_\sigma$*, it follows that*

$$\|z_c(t)\|_{\sigma+s} \le R_{\sigma+s}, \qquad \forall t \in \mathbb{R}^+.$$

Proof. The argument is very similar to the one used in the previous proof. Therefore we will just detail those passages in which significant differences occur. As before, let the generic constant $c \ge 0$ depend on R_σ. Also, by virtue of Lemma 5, we have the uniform bounds (33)–(34). The energy functional considered here is

$$\Phi_c(t) = (1-\varepsilon)\|A^{(1+\sigma+s)/2}w(t)\|^2 + \|A^{(\sigma+s)/2}\xi_c(t)\|^2 + \mathcal{G}_c(t) + k_0,$$

for some $\varepsilon > 0$ and $k_0 = k_0(R_\sigma) \ge 0$, with $\xi_c = w_t + \frac{\varepsilon}{\omega}w$, and

$$\mathcal{G}_c(t) = 2\langle\phi_1(u(t)), A^{\sigma+s}w(t)\rangle - 2\langle f, A^{\sigma+s}w(t)\rangle.$$

Again, for k_0 large enough and ε small enough, we have

$$\frac{1}{2}\|z_c(t)\|^2_{\sigma+s} \le \Phi_c(t) \le 2\|z_c(t)\|^2_{\sigma+s} + c.$$

Indeed,

$$2\big|\langle\phi_1(u), A^{\sigma+s}w\rangle\big| \le c\big(1 + \|u\|^\gamma_{L^{6\gamma/(5-2\sigma-2s)}}\big)\|A^{(1+\sigma+s)/2}w\|.$$

If $\sigma < \frac{1}{2}$, on account of the inequality

$$\frac{6\gamma}{5-2\sigma-2s} \le \frac{6}{1-2\sigma},$$

we get

$$\|u\|^{\gamma}_{L^{6\gamma/(5-2\sigma-2s)}} \le \|A^{(1+\sigma)/2}u\|^{\gamma} \le c.$$

If $\sigma \ge \frac{1}{2}$, we still get the inequality

$$\|u\|^{\gamma}_{L^{6\gamma/(5-2\sigma-2s)}} \le c,$$

by means of the continuous embedding

$$\mathcal{D}(A^{(1+\sigma)/2}) \hookrightarrow L^p(\Omega), \qquad \forall p \ge 1.$$

In either case, we can conclude that

$$2\big|\langle \phi_1(u), A^{\sigma+s}w\rangle\big| \le \frac{1}{4}\|z_c(t)\|^2_{\sigma+s} + c.$$

Multiplying (23) times $A^{\sigma+s}\xi_c$, and repeating the former passages, we obtain the differential inequality

$$\frac{d}{dt}\Phi_c + \frac{\varepsilon}{\omega}\Phi_c \le h\Phi_c + h + \frac{k_0\varepsilon}{\omega},$$

for some $\varepsilon > 0$ small enough, where h fulfills (38). An application of Lemma 2 leads to the desired conclusion, since in this case (cf. (39)), $\Phi_c(0) \le c$. □

We will complete our task exploiting the transitivity property of exponential attraction [9, Theorem 5.1], that we recall below for the reader's convenience.

Lemma 7. *Let $\mathcal{K}_1, \mathcal{K}_2, \mathcal{K}_3$ be subsets of $\mathcal{H}_0$ such that*

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{K}_1, \mathcal{K}_2) \le L_1 e^{-\vartheta_1 t}, \qquad \mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{K}_2, \mathcal{K}_3) \le L_2 e^{-\vartheta_2 t},$$

for some $\vartheta_1, \vartheta_2 > 0$ and $L_1, L_2 \ge 0$. Assume also that for all $z_1, z_2 \in \bigcup_{t\ge 0} S(t)\mathcal{K}_j$ $(j = 1, 2, 3)$ there holds

$$\|S(t)z_1 - S(t)z_2\|_0 \le L_0 e^{\vartheta_0 t}\|z_1 - z_2\|_0,$$

for some $\vartheta_0 \ge 0$ and some $L_0 \ge 0$. Then it follows that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{K}_1, \mathcal{K}_3) \le L e^{-\vartheta t},$$

where $\vartheta = \frac{\vartheta_1\vartheta_2}{\vartheta_0+\vartheta_1+\vartheta_2}$ and $L = L_0 L_1 + L_2$.

We have now all the tools to proceed to the proof of the theorem.

Proof of Theorem 5. With reference to (41), notice that, starting with $\sigma = 0$, we find a strictly increasing finite sequence of numbers $\{\sigma_j\}_{j=0}^{n}$, with $n = n(\gamma)$, such that

$$\sigma_0 = 0, \qquad \sigma_{j+1} = \sigma_j + s(\sigma_j), \qquad \sigma_n = 1.$$

Choosing R_0 as in Theorem 3, let us define for $j = 0, \dots, n$

$$\mathcal{B}_{\sigma_j} = \big\{z_0 \in \mathcal{H}_{\sigma_j} : \|z_0\|_{\sigma_j} \le R_{\sigma_j}\big\},$$

where $R_{\sigma_j} = R_{\sigma_j}(R_{\sigma_{j-1}})$ are given by Lemma 6. After Remark 7 and Lemma 6, we learn at once that

$$\operatorname{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_{\sigma_{j-1}}, \mathcal{B}_{\sigma_j}) \le M_j e^{-\frac{\nu_0}{\omega}t}, \qquad \forall j = 1, \dots, n,$$

where

$$M_j = M_0\big(\alpha_1^{\sigma_{j-1}/2} R_{\sigma_{j-1}}\big).$$

Taking then into account Corollary 1 and (13), by successive applications of Lemma 7, we obtain the estimate

$$\operatorname{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0, \mathcal{B}_1) \le M e^{-\frac{\nu}{\omega}t},$$

for some $M \ge 0$ and $\nu > 0$. □

Solutions departing from $\mathcal{B}_1$ satisfy an extra regularity, which shall be needed in the sequel.

Lemma 8. *There exists $C \ge 0$ such that*

$$\sup_{z_0 \in \mathcal{B}_1} \|z_t(t)\|_0 \le C, \qquad \forall t \ge 1.$$

Proof. Let $z_0 = (u_0, u_1) \in \mathcal{B}_1$ and consider the linear nonhomogeneous problem

$$\begin{cases} \psi_{tt} + \omega A\psi_t + A\psi = -\phi_1'(u)u_t, \\ \psi(0) = u_1, \\ \psi_t(0) = -\omega A u_1 - A u_0 - \phi_1(u_0) + f, \end{cases}$$

obtained by differentiation of Problem $\boldsymbol{P}$ with respect to time. By Lemma 5 (for $\sigma = 1$) we have

$$\sup_{z_0 \in \mathcal{B}_1} \sup_{t \in \mathbb{R}^+} \|z(t)\|_1 < \infty. \tag{42}$$

Consequently, the continuous embedding $H^2(\Omega) \hookrightarrow C(\overline{\Omega})$ provides the uniform bound

$$\sup_{z_0 \in \mathcal{B}_1} \sup_{t \in \mathbb{R}^+} \|\phi'(u(t))u_t\| < \infty.$$

Thus, by standard arguments, for every $T > 0$ the above problem admits a unique solution

$$\psi \in C([0, T], L^2(\Omega)),$$

with

$$\psi_t \in C([0, T], H^{-1}(\Omega)) \cap L^2([0, T], L^2(\Omega)).$$

By comparison, $\psi(t) = u_t(t)$ for every $t \ge 0$, so, in particular, $\psi \in C(\mathbb{R}^+, H_0^1(\Omega))$. Taking the product with ψ_t, we get

$$\frac{d}{dt}\Big[\|\psi_t\|^2 + \|A^{1/2}\psi\|^2\Big] \le c.$$

Integrating the above inequality on $(r, t+1)$, for some fixed $r \in [t, t+1]$, and integrating the resulting inequality with respect to r on $(t, t+1)$, the proof follows. Notice that this procedure is a simplified version of the uniform Gronwall lemma [15, Lemma III.1.1]. □

Remark 8. Of course it is a natural question to ask why this approach fails to handle the critical case. In fact, the bootstrap procedure works as well for the critical case (clearly, it is a little bit more complicated, and an additional control on the second derivative of ϕ_0 is required), provided that we start from $\sigma > 0$. The missing passage is exactly from $\sigma = 0$ to $\sigma = s$. This means that, if we were able to prove that the attractor is bounded in some $\mathcal{H}_\sigma$, for $\sigma > 0$ no matter how small, we would obtain all the results of this paper for the critical case as well. Unfortunately, it seems a really hard task to exhibit such a regularity for the attractor when ϕ is critical. In fact, it is quite possible that there is not such a regularity.

7. Exponential Attractors for the Subcritical Case

As remarked by many authors, the universal attractor may not be for practical purposes (e.g., to get numerical results) a satisfactory object to describe the longterm dynamics. Indeed, in spite of its nice features, it is not possible in general to exhibit an actual control of the convergence rate of the trajectories to the attractor. In order to overcome the problem of quantitative control of the time needed to stabilize the system, Eden, Foias, Nicolaenko and Temam (cf. [5, 6]) introduced the notion of exponential attractor. This is a compact invariant (but not fully invariant) subset of the phase-space of finite fractal dimension that attracts a bounded ball of initial data exponentially fast. However, before the results of [9], it was not clear if, for hyperbolic systems, the exponential attractor had a basin of attraction coinciding with the whole phase-space. Clearly, this was quite a significant limitation. Nonetheless, after [9], we now know that it is possible to remove this obstacle, and this justifies the following generalization of the definition given in [5, 6].

Definition 1. *A compact set $\mathcal{E} \subset \mathcal{H}_0$ is called an exponential attractor or inertial set for the semigroup $S(t)$ if the following conditions hold:*

(i) $\mathcal{E}$ is invariant of $S(t)$, that is, $S(t)\mathcal{E} \subset \mathcal{E}$ for every $t \geq 0$;
(ii) $\dim_F \mathcal{E} < \infty$, that is, $\mathcal{E}$ has finite fractal dimension;
(iii) there exist an increasing function $J : \mathbb{R}^+ \to \mathbb{R}^+$ and $\kappa > 0$ such that, for any set $\mathcal{B} \subset \mathcal{H}_0$ with $\sup_{z_0 \in \mathcal{B}} \|z_0\|_0 \leq R$ there holds

$$\operatorname{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{E}) \leq J(R)e^{-\kappa t}.$$

We remark that, contrary to the universal attractor, the exponential attractor is not unique. As a matter of fact, if there is one, then there are infinitely many of them.

It is apparent that if there is an exponential attractor $\mathcal{E}$, then in particular the semigroup possesses a compact attracting set, and thus it has a universal attractor $\mathcal{A} \subset \mathcal{E}$ of finite fractal dimension, being $\dim_F \mathcal{A} \leq \dim_F \mathcal{E}$.

Our main result is

Theorem 6. *The semigroup $S(t)$ acting on $\mathcal{H}_0$ possesses an exponential attractor $\mathcal{E} = \mathcal{E}(\omega)$. Moreover,*

(i) $\mathcal{E}$ is a bounded subset of $\mathcal{H}_1$, and the bound is independent of $\omega \geq \omega_0$;
(ii) the rate of exponential attraction κ is proportional to $\frac{1}{\omega}$;
(iii) $J(R)$ is independent of $\omega \geq \omega_0$;
(iv) $\sup_{\omega \geq \omega_0} \left[\dim_F \mathcal{E}(\omega)\right] < \infty$.

Corollary 4. *The universal attractor $\mathcal{A}$ of the semigroup $S(t)$ has finite fractal dimension, and*

$$\sup_{\omega \geq \omega_0} \left[\dim_{\mathrm{F}} \mathcal{A}(\omega) \right] < \infty.$$

In order to prove Theorem 6, we shall use the following sufficient condition (cf. [8, Prop. 1] and [6, p.33]):

Lemma 9. *Let $\mathcal{X} \subset \mathcal{H}_0$ be a compact invariant subset. Assume that there exists a time $t_* > 0$ such that the following hold:*

(i) the map

$$(t, z_0) \mapsto S(t)z_0 : [0, t_*] \times \mathcal{X} \to \mathcal{X}$$

is Lipschitz continuous (with the metric inherited from $\mathcal{H}_0$);

(ii) the map $S(t_*) : \mathcal{X} \to \mathcal{X}$ admits a decomposition of the form

$$S(t_*) = S_0 + S_1, \qquad S_0 : \mathcal{X} \to \mathcal{H}_0, \qquad S_1 : \mathcal{X} \to \mathcal{H}_1,$$

where S_0 and S_1 satisfy the conditions

$$\|S_0(z_1) - S_0(z_2)\|_0 \leq \frac{1}{8}\|z_1 - z_2\|_0, \qquad \forall z_1, z_2 \in \mathcal{X},$$

and

$$\|S_1(z_1) - S_1(z_2)\|_1 \leq C_*\|z_1 - z_2\|_0, \qquad \forall z_1, z_2 \in \mathcal{X},$$

for some $C_* > 0$.

Then there exist an invariant compact set $\mathcal{E} \subset \mathcal{X}$ such that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{X}, \mathcal{E}) \leq J_0 e^{-\frac{\log 2}{t_*} t}, \tag{43}$$

where

$$J_0 = 2L_* \sup_{z_0 \in \mathcal{X}} \|z_0\|_0 \, e^{\frac{\log 2}{t_*}}, \tag{44}$$

and L_* is the Lipschitz constant of the map $S(t_*) : \mathcal{X} \to \mathcal{X}$. Moreover,

$$\dim_{\mathrm{F}} \mathcal{E} \leq 1 + \frac{\log N_*}{\log 2}, \tag{45}$$

where N_* is the minimum number of $\frac{1}{8C_*}$-balls of $\mathcal{H}_0$ necessary to cover the unit ball of $\mathcal{H}_1$.

In fact, [8, Prop. 1] allows to build an exponential attractor $\mathcal{E}$ that attracts $\mathcal{X}$ with an arbitrarily large attraction rate, paying the price of increasing $\dim_{\mathrm{F}} \mathcal{E}$. However, we will be interested to attract arbitrary bounded subsets of $\mathcal{H}_0$. This translates into an upper bound on the attraction rate, that depends on the velocity at which $\mathcal{X}$ attracts the absorbing set $\mathcal{B}_0$.

We remark that the original technique to find exponential attractors (cf. [5, 6]) is quite different. Indeed, it relies on the proof that the semigroup $S(t)$ satisfies the so-called *squeezing property* on $\mathcal{X}$. Besides, it works in Hilbert spaces only, since it makes use of orthogonal projections. On the contrary, this alternative approach is applicable in Banach spaces as well. In a Hilbert space setting, like in our case, the choice of which procedure to follow is just a matter of taste. Note that, to get precise numerical calculations, one has to know the number N_*, that, in general, is quite difficult to compute. Similarly, the other method requires the explicit knowledge of the eigenvalues $\{\alpha_n\}$ of A. In fact, this is actually the same problem.

We define

$$\mathcal{X} = \overline{\bigcup_{\tau \geq 1} S(\tau)\mathcal{B}_1}^{\mathcal{H}_0}.$$

Let us establish some properties of this set.

- $\mathcal{X}$ is a compact set in $\mathcal{H}_0$, bounded in $\mathcal{H}_1$, due to Lemma 5.
- $\mathcal{X}$ is invariant, for, from the continuity of $S(t)$, we have

$$S(t)\mathcal{X} \subset \overline{\bigcup_{\tau \geq 1} S(t+\tau)\mathcal{B}_1}^{\mathcal{H}_0} \subset \mathcal{X}.$$

- There holds

$$\operatorname{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_0, \mathcal{X}) \leq M e^{-\frac{\nu}{\omega} t}, \qquad \forall t \in \mathbb{R}^+, \tag{46}$$

  for some $M \geq 0$ and some $\nu > 0$. Indeed, it is apparent that

$$\operatorname{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}_1, \mathcal{X}) = 0, \qquad \forall t \geq 1.$$

  Hence (46) follows from Lemma 7, in view of Theorem 5, Lemma 5, and (13).
- There is $C \geq 0$ such that

$$\sup_{z_0 \in \mathcal{X}} \|z_t(t)\|_0^2 \leq C, \qquad \forall t \geq 0.$$

  This is a direct consequence of Lemma 8.

Therefore such a set $\mathcal{X}$ is a promising candidate for our purposes. Indeed, we have the following two lemmas.

Lemma 10. *For every $T > 0$, the mapping $(t, z_0) \mapsto S(t)z_0$ is Lipschitz continuous on $[0, T] \times \mathcal{X}$.*

Proof. For $z_1, z_2 \in \mathcal{X}$ and $t_1, t_2 \in [0, T]$ we have

$$\|S(t_1)z_1 - S(t_2)z_2\|_0 \leq \|S(t_1)z_1 - S(t_1)z_2\|_0 + \|S(t_1)z_2 - S(t_2)z_2\|_0.$$

The first term of the above inequality is handled by estimate (13). Concerning the second one,

$$\|z(t_1) - z(t_2)\|_0 \leq \left| \int_{t_1}^{t_2} \|z_t(y)\|_0 dy \right| \leq C|t_1 - t_2|.$$

Hence

$$\|S(t_1)z_1 - S(t_2)z_2\|_0 \leq L\big[|t_1 - t_2| + \|z_1 - z_2\|_0\big],$$

for some $L = L(T) \geq 0$. □

Lemma 11. *Assumption (ii) of Lemma 9 holds true.*

Proof. The constant $c \geq 0$ of this proof will depend on $\mathcal{X}$ (which, however, is a fixed set). For $z_0 \in \mathcal{X}$, let us denote by $S_0(t)z_0$ the solution at time t of the linear homogeneous problem associated to Problem $\boldsymbol{P}$, and let $S_1(t)z_0 = S(t)z_0 - S_0(t)z_0$. Given two solutions

$$z^1 = (u^1, u_t^1) \qquad \text{and} \qquad z^2 = (u^2, u_t^2),$$

originating from $z_1, z_2 \in \mathcal{X}$, respectively, set $\bar{z} = z^1 - z^2 = (\bar{u}, \bar{u}_t)$. Let us decompose $\bar{z}$ into the sum

$$\bar{z} = \bar{z}_d + \bar{z}_c = (\bar{v}, \bar{v}_t) + (\bar{w}, \bar{w}_t),$$

where

$$\begin{cases} \bar{v}_{tt} + \omega A\bar{v}_t + A\bar{v} = 0, \\ \bar{z}_d(0) = z_1 - z_2, \end{cases} \tag{47}$$

and

$$\begin{cases} \bar{w}_{tt} + \omega A\bar{w}_t + A\bar{w} = -\phi_1(u^1) + \phi_1(u^2), \\ \bar{z}_c(0) = 0. \end{cases} \tag{48}$$

It is apparent that $\bar{z}_d(t) = S_0(t)z_1 - S_0(t)z_2$ and $\bar{z}_c(t) = S_1(t)z_1 - S_1(t)z_2$. By (47) we get (cf. Remark 7),

$$\|\bar{z}_d(t)\|_0 \leq c\|z_1 - z_2\|_0 e^{-\frac{\nu_0}{\omega}t},$$

for some $c > 1$. Hence, setting

$$t_* = \frac{\omega}{\nu_0}\log 8c, \tag{49}$$

we have

$$\|\bar{z}_d(t_*)\|_0 \leq \frac{1}{8}\|z_1 - z_2\|_0. \tag{50}$$

For all trajectories departing from $\mathcal{X}$, the first component is (uniformly) bounded almost everywhere. Therefore the product of (48) and $A\bar{w}_t$ bears

$$\frac{d}{dt}\|\bar{z}_c\|_1^2 + 2\omega\|A\bar{w}_t\|^2 \leq 2\|\phi_1(u^1) - \phi_1(u^2)\|\|A\bar{w}_t\| \leq \frac{c}{\omega}\|\bar{u}\|^2 + 2\omega\|A\bar{w}_t\|^2.$$

From (13),

$$\|\bar{u}(t)\| \leq \|\bar{z}(t)\|_0 \leq e^{\frac{K}{\omega}t}\|z_1 - z_2\|_0, \qquad \forall t \in \mathbb{R}^+,$$

thus we obtain the inequality

$$\frac{d}{dt}\|\bar{z}_c(t)\|_1^2 \leq \frac{c}{\omega}e^{\frac{2K}{\omega}t}\|z_1 - z_2\|_0^2,$$

and an integration on $(0, t_*)$ yields

$$\|\bar{z}_c(t_*)\|_1^2 \leq C_*\|z_1 - z_2\|_0^2, \tag{51}$$

with

$$C_* = \frac{c}{2K}\,e^{\frac{2K}{\omega}t_*}.$$

Notice that, in light of (49), C_* is independent of $\omega \geq \omega_0$. Collecting (50)-(51), and setting $S_0 = S_0(t_*)$ and $S_1 = S_1(t_*)$, we meet the thesis. □

Proof of Theorem 6. Thanks to Lemma 10 and Lemma 11, we can apply Lemma 9, so getting a compact invariant set $\mathcal{E} \subset \mathcal{X}$ satisfying (43)–(45). In particular, due to (13) and (49), we may rewrite (43) as

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{X}, \mathcal{E}) \leq J_0 e^{-\frac{\kappa_0}{\omega} t}, \qquad \forall t \in \mathbb{R}^+, \tag{52}$$

for some $J_0 \geq 0$ and $\kappa_0 > 0$, both independent of $\omega \geq \omega_0$. In addition, N_* is independent of $\omega \geq \omega_0$, for so is C_*. This implies assertion (iv) of the theorem. In order to complete the proof, we are left to show that $\mathcal{E}$ attracts (exponentially fast) all finite subsets of the whole phase-space $\mathcal{H}_0$. Thus, let $\mathcal{B} \subset \mathcal{H}_0$ be a bounded set, and call $R = \sup_{z_0 \in \mathcal{B}} \|z_0\|_0$. By Theorem 3 (cf. (21)),

$$S(t)\mathcal{B} \subset \mathcal{B}_0, \qquad \forall t \geq \omega t_0,$$

for some t_0 depending (increasingly) only on R. Hence, by (46),

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{X}) \leq M e^{\nu t_0} e^{-\frac{\nu}{\omega} t}, \qquad \forall t \geq \omega t_0.$$

On the other hand, by Corollary 1, we easily get that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{X}) \leq k, \qquad \forall t \in \mathbb{R}^+,$$

for some $k \geq 0$ depending (increasingly) only on R. Collecting the two above inequalities, we have

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{X}) \leq \big(k + M e^{\nu t_0}\big) e^{-\frac{\nu}{\omega} t}, \qquad \forall t \in \mathbb{R}^+. \tag{53}$$

Applying once more Lemma 7, from (13), (52)–(53) and Lemma 5, we conclude that

$$\mathrm{dist}_{\mathcal{H}_0}(S(t)\mathcal{B}, \mathcal{E}) \leq J e^{-\frac{\kappa}{\omega} t}, \qquad \forall t \in \mathbb{R}^+,$$

where $J = J(R)$ is an increasing function of R, and $\kappa > 0$. Observe that both J and κ are independent of $\omega \geq \omega_0$. $\square$

Remark 9. We want to spend a few words to say what happens when $\omega \to 0$. All the results clearly hold (ω_0 can be chosen arbitrarily small), but there will be dependencies on ω. For instance, the set $\mathcal{B}_1$ (and, consequently, $\mathcal{A}$ and $\mathcal{E}$) is bounded in $\mathcal{H}_1$ with a bound that blows up as $\omega \to 0$. Precisely, the bound is proportional to ω^{-2n}, where $n = n(\gamma)$ is the number of steps required in the proof of Theorem 5. Analogously, the exponential convergence rate tends to infinity as $\omega \to 0$, as well as the upper bound for $\dim_{\mathrm{F}} \mathcal{E}$.

Remark 10. Let us conclude the paper with a consideration. We have seen that the fractal dimension of the exponential attractor (and thus of the attractor) remains bounded as $\omega \to \infty$. Clearly, our estimates provide just upper bounds. Nonetheless it seems reasonable that $\dim_{\mathrm{F}} \mathcal{E}$ tends to infinity as $\omega \to 0$. Then, as ω gets bigger, the fractal dimension decreases (at least, its upper bound) until it stabilizes. Still, the exponential convergence rate gives some information, namely, things start to get worse as soon as ω is too large. So our analysis seems to suggest, contrary to what is maintained in [17], that $\dim_{\mathrm{F}} \mathcal{E}$ is not a decreasing function of ω, but attains a minimum at some ω_*.

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