Topics in Continuum Mechanics applied to Biology/2: Blood flow

Giulia Giantesio

Università Cattolica del Sacro Cuore, Brescia
giulia.giantesio@unicatt.it

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From the mathematical point of view (Eulerian)

$$\frac{d\rho}{dt} + \rho\,\mathrm{div}\,\mathbf{v} = 0 \tag{1}$$

$$\rho\frac{d\mathbf{v}}{dt} = \rho\mathbf{b} + \mathrm{div}\,\mathsf{T} \tag{2}$$

$$\mathsf{T} \quad \text{is symmetric} \tag{3}$$

Stokesian fluids

We now have to make some assumptions on the form of T.

Stokesian fluids

$\mathsf{T} = -p\mathsf{I}$

Stokesian fluids

"That the difference between the pressure on a plane in a given direction passing through any point P of a fluid in motion and the pressure which would exist in all directions about P if the fluid in its neighbourhood were in a state of relative equilibrium depends only on the relative motion of the fluid immediately about P; and that the relative motion due to any motion of rotation may be eliminated without affecting the differences of the pressures above mentioned." G.Stokes

Stokesian fluids

- V is a continuous function of D;
- V does not depend explicitely on $\mathbf{x}$;
- V is isotropic;
- $V = 0$, if $D = 0$.

Stokesian fluids

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$$\mathsf{V} = \alpha \mathsf{I} + \beta \mathsf{D} + \gamma \mathsf{D}^2$$

But . . .

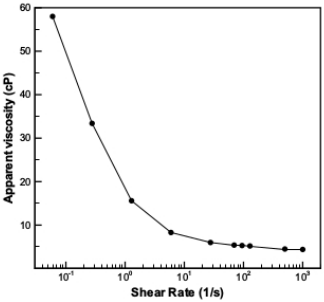

FIGURE 5. Apparent viscosity as a function of the shear rate for whole blood obtained from a 25 year old female donor with Ht = 40%, $T = 23°$C. Obtained using a Contraves LS30 (Couette) viscometer at shear rates of $\dot{\gamma} \in [0.06, 128]\ \mathrm{s}^{-1}$) and a Cannon–Manning Semi-Micro (capillary) viscometer, (Cannon Instrument Co.) at shear rates of $\dot{\gamma} \in [300, 1000]\ \mathrm{s}^{-1}$ (unpublished data from M. Kameneva, with permission).

Reiner-Rivlin models

$$\mathsf{T} = -\tilde{p}\mathsf{I} + \beta(|\mathsf{D}|, \det \mathsf{D})\mathsf{D} + \gamma(|\mathsf{D}|, \det \mathsf{D})\mathsf{D}^2$$

Reiner-Rivlin models

$$\mathsf{T} = -\tilde{p}\mathsf{I} + \beta(|\mathsf{D}|, \det \mathsf{D})\mathsf{D} + \gamma(|\mathsf{D}|, \det \mathsf{D})\mathsf{D}^2$$

Generalized Newtonian:

$$\mathsf{T} = -\tilde{p}\mathsf{I} + \beta(|\mathsf{D}|)\mathsf{D}$$

Reiner-Rivlin models

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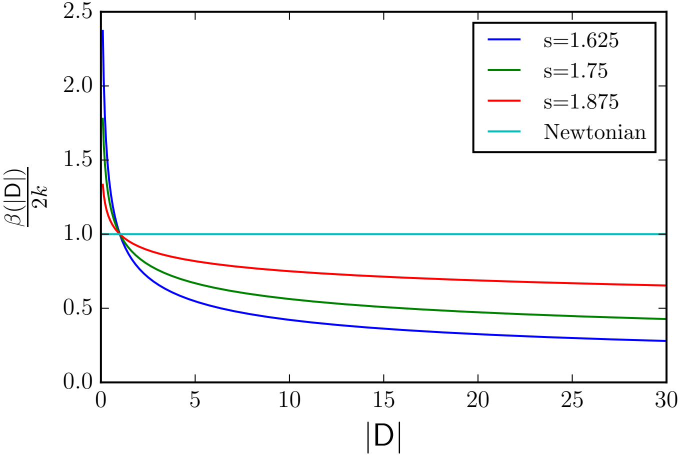
Generalized Newtonian:

$$\mathsf{T} = -\tilde{p}\mathsf{I} + \beta(|\mathsf{D}|)\mathsf{D}$$

Power-law fluids:

$$\beta(|\mathsf{D}|) = 2k|\mathsf{D}|^{s-2}$$

Power-law fluids



Non-Newtonian models

$$\beta(|\mathsf{D}|) = \mu_\infty + (\mu_0 - \mu_\infty) f(\lambda |\mathsf{D}|)$$

Non-Newtonian models

$$\beta(|D|) = \mu_\infty + (\mu_0 - \mu_\infty) f(\lambda |D|)$$

Model	Equation	Non-Newtonian Properties
Carreau-Yasuda	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{[1+(\lambda\dot{\gamma})^a]^{\frac{1-n}{a}}}$	shear thinning
Casson	$\tau^{1/2} = (k\dot{\gamma})^{1/2} + \tau_o^{1/2}$	yield stress
Power law	$\tau = k\dot{\gamma}^n$	shear thinning
Cross	$\mu = \mu_\infty + \frac{\mu_0 - \mu_\infty}{1+\lambda\dot{\gamma}^m}$	shear thinning
Herschel-Bulkley	$\tau = k\dot{\gamma}^n + \tau_o$	shear thinning, yield stress
Oldroyd-B	$\boldsymbol{\tau} + \lambda_1 \overset{\triangledown}{\boldsymbol{\tau}} = \mu_0 \left(\dot{\boldsymbol{\gamma}} + \lambda_2 \overset{\triangledown}{\dot{\boldsymbol{\gamma}}} \right)$	viscoelasticity
Quemada	$\mu = \mu_p \left(1 - \frac{k_0 + k_\infty \sqrt{\dot{\gamma}/\dot{\gamma}_c}}{2\left(1+\sqrt{\dot{\gamma}/\dot{\gamma}_c}\right)} \phi \right)^{-2}$	shear thinning
Yeleswarapu	$\mu = \mu_\infty + (\mu_0 - \mu_\infty) \frac{1+\ln(1+\lambda\dot{\gamma})}{1+\lambda\dot{\gamma}}$	shear thinning
Bingham	$\tau = k\dot{\gamma} + \tau_o$	yield stress
Eyring-Powell	$\mu = \mu_\infty + \frac{(\mu_0 - \mu_\infty)\sinh^{-1}(\lambda\dot{\gamma})}{\lambda\dot{\gamma}}$	shear thinning
Ree-Eyring	$\tau = \tau_c \sinh^{-1} \left(\frac{\mu_0 \dot{\gamma}}{\tau_c} \right)$	shear thinning

Non-Newtonian models

$$\beta(|\mathrm{D}|) = \mu_\infty + (\mu_0 - \mu_\infty) f(\lambda|\mathrm{D}|)$$

$$f(x) = \frac{\operatorname{arcsinh} x}{x}$$
$$f(x) = \frac{\log(1+x)}{x^m}$$
$$f(x) = \frac{1}{1+x^m}$$
$$f(x) = (1+x^a)^{\frac{n-1}{a}}$$

Non-Newtonian models

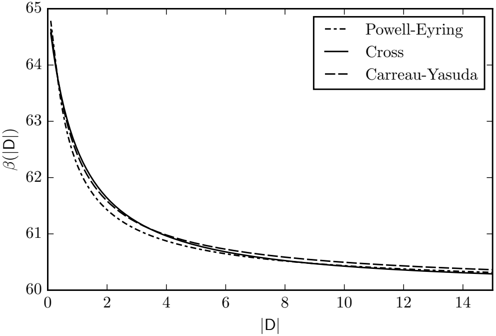
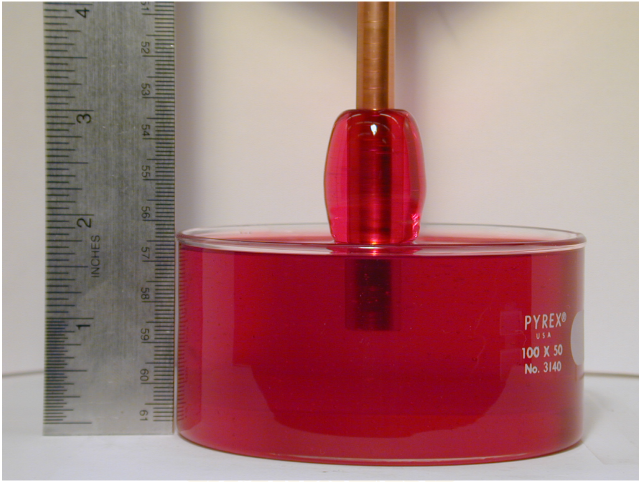


Figure: Un confronto fra i tre modelli newtoniani generalizzati principali per il sangue, con $\mu_0 = 65$ kg m^{-1} s^{-1} e $\mu_\infty = 60$ kg m^{-1} s^{-1}. Inoltre, per il modello di Powell-Eyring si è scelto $\lambda = 5.383$ s, per quello di Cross $\lambda = 1.007$ s e $m = 1.028$, per quello di Carreau-Yasuda $\lambda = 1.902$ s, $n = 0.22$ e $a = 1.25$.

Rod climbing



$$\nabla f = \frac{\partial f}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial f}{\partial \vartheta}\mathbf{e}_\vartheta + \frac{\partial f}{\partial z}\mathbf{e}_z \,,$$

$$\operatorname{div}\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\vartheta}{\partial \vartheta} + \frac{\partial u_z}{\partial z} \,,$$

$$\begin{aligned}
\nabla\mathbf{u} = {} & \frac{\partial u_r}{\partial r}\mathbf{e}_r\otimes\mathbf{e}_r + \frac{\partial u_\vartheta}{\partial r}\mathbf{e}_\vartheta\otimes\mathbf{e}_r + \frac{1}{r}\left(\frac{\partial u_r}{\partial \vartheta} - u_\vartheta\right)\mathbf{e}_r\otimes\mathbf{e}_\vartheta + \frac{1}{r}\left(\frac{\partial u_\vartheta}{\partial \vartheta} + u_r\right)\mathbf{e}_\vartheta\otimes\mathbf{e}_\vartheta \\
& + \frac{\partial u_r}{\partial z}\mathbf{e}_r\otimes\mathbf{e}_z + \frac{\partial u_\vartheta}{\partial z}\mathbf{e}_\vartheta\otimes\mathbf{e}_z + \frac{\partial u_z}{\partial z}\mathbf{e}_z\otimes\mathbf{e}_z + \frac{\partial u_z}{\partial r}\mathbf{e}_z\otimes\mathbf{e}_r + \frac{1}{r}\frac{\partial u_z}{\partial \vartheta}\mathbf{e}_z\otimes\mathbf{e}_\vartheta \,,
\end{aligned}$$

$$\Delta\mathbf{u} = \left[\Delta u_r - \frac{1}{r^2}\left(u_r + 2\frac{\partial u_\vartheta}{\partial \vartheta}\right)\right]\mathbf{e}_r + \left[\Delta u_\vartheta - \frac{1}{r^2}\left(u_\vartheta - 2\frac{\partial u_r}{\partial \vartheta}\right)\right]\mathbf{e}_\vartheta + \Delta u_z\mathbf{e}_z \,,$$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \vartheta^2} + \frac{\partial^2 f}{\partial z^2} \,,$$

$$\begin{aligned}
\operatorname{div}\mathsf{T} = {} & + \left[\frac{\partial \mathsf{T}_{rr}}{\partial r} + \frac{\mathsf{T}_{rr} - \mathsf{T}_{\vartheta\vartheta}}{r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\vartheta r}}{\partial \vartheta} + \frac{\partial \mathsf{T}_{zr}}{\partial z}\right]\mathbf{e}_r + \\
& + \left[\frac{\partial \mathsf{T}_{r\vartheta}}{\partial r} + \frac{2\mathsf{T}_{r\vartheta}}{r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\vartheta\vartheta}}{\partial \vartheta} + \frac{\partial \mathsf{T}_{z\vartheta}}{\partial z} + \frac{\mathsf{T}_{\vartheta r} - \mathsf{T}_{r\vartheta}}{r}\right]\mathbf{e}_\vartheta + \\
& + \left[\frac{\partial \mathsf{T}_{rz}}{\partial r} + \frac{\mathsf{T}_{rz}}{r} + \frac{1}{r}\frac{\partial \mathsf{T}_{\vartheta z}}{\partial \vartheta} + \frac{\partial \mathsf{T}_{zz}}{\partial z}\right]\mathbf{e}_z \,,
\end{aligned}$$