

DYNAMICAL SYSTEMS IN FINANCE

Lecture notes by Alessandro Musesti

Academic year 2026-2027

October 14, 2026

Contents

1	Discrete Dynamical Systems	2
2	Equilibrium Points, Cycles, Stability	4
3	Linear First-order One-dimensional Difference Equations	5
4	One-dimensional DDEs	7
5	The Cobweb Theory of Economics	9
6	Linear Discrete Dynamical Systems	12
7	Higher-order Discrete Dynamical Systems	13
8	Goodwin's Second-order Cobweb Model	16
9	Gambler's Ruin	17
10	Samuelson's Multiplier-accelerator Model	18
11	A Trade Model between Two Countries	20
12	Linearization of Discrete Dynamical Systems	21
13	The Discrete Logistic Map	23
14	Systems of First-order Ordinary Differential Equations	23
15	Equilibrium and Stability	25
16	Higher-order Linear One-dimensional ODEs	26

17	Two-dimensional Linear Systems	28
18	Stability of Linear Systems	30
19	Phillips' Stabilization Model for a Closed Economy	32
20	The Linearization Method for ODEs	36
21	Isocline Method	37
22	A Lotka-Volterra Model for the Price/Supply Interaction	38
23	A Model for Asset Prices	40
24	Black-Scholes Equation: Option Pricing	44
25	The Mathematical Deduction of the Black-Scholes Formulas	47
26	The Greeks	50
27	Perpetual American Put	50

1 Discrete Dynamical Systems

1.1 Definition (Discrete dynamical system). Given a set $U \subseteq \mathbb{R}^n$ and a function $F : U \rightarrow U$, a *discrete dynamical system* (or *difference equation*, or *iterated map*) is of the form

$$x_{h+1} = F(x_h). \quad (1)$$

Given an initial condition $\bar{x}_0 \in U$, the solution of (1) is a sequence (x_h) which is defined by the recursive process

$$x_0, \quad x_1 = F(x_0), \quad x_2 = F(x_1) = (F \circ F)(x_0), \quad \dots, \quad x_h = F(x_{h-1}) = (F \circ \dots \circ F)(x_0).$$

We will make use of the notation

$$F^h(x) = \underbrace{(F \circ \dots \circ F)}_{h \text{ times}}(x),$$

so that the solution of (1) starting from x_0 is given by

$$x_h = F^h(x_0). \quad \star$$

1.2 Remark. One may consider a more general class of discrete dynamical systems of the form

$$x_{h+1} = F_h(x_h) = F(x_h; h) \quad (2)$$

where the right-hand side can depend also on the time-step h . In this case the solution is recursively given by

$$x_0, \quad x_1 = F_0(x_0), \quad x_2 = F_1(x_1) = (F_1 \circ F_0)(x_0), \quad \dots, \quad x_h = F_{h-1}(x_{h-1}) = (F_{h-1} \circ \dots \circ F_0)(x_0).$$

Systems of the form (2) will be named *non autonomous*, while systems (1) will be named *autonomous*. We will typically consider only autonomous systems. ★

1.3 Example. In the 1D case ($n = 1$) let us consider the DDS

$$x_{h+1} = x_h^2,$$

that is, $F(x) = x^2$. Then for some initial conditions one has:

- $x_0 = \frac{1}{2} \Rightarrow \frac{1}{2}, \frac{1}{4}, \frac{1}{16}, \frac{1}{256}, \dots \Rightarrow x_h \rightarrow 0$
- $x_0 = 2 \Rightarrow 2, 4, 16, 256, \dots \Rightarrow x_h \rightarrow +\infty$
- $x_0 = -2 \Rightarrow -2, 4, 16, 256, \dots \Rightarrow x_h \rightarrow +\infty$
- $x_0 = 0 \Rightarrow 0, 0, 0, \dots \Rightarrow x_h \equiv 0$ (constant solution, equilibrium)
- $x_0 = 1 \Rightarrow 1, 1, 1, \dots \Rightarrow x_h \equiv 1$ (another equilibrium)
- $x_0 = -1 \Rightarrow -1, 1, 1, \dots \Rightarrow x_h \equiv 1$ (a pre-equilibrium)

In general one can prove that for $x_0 \in (-1; 1)$ one has $x_h \rightarrow 0$, for $x_0 < -1$ or $x_0 > 1$ one has $x_h \rightarrow +\infty$, for $x_0 = -1$ one has $x_h = 1$ for all $h \geq 1$, for $x_0 = 1$ one has an equilibrium. ★

1.4 Example. A more intriguing example in 1D is

$$x_{h+1} = x_h^2 - 1,$$

that is, $F(x) = x^2 - 1$.

Trying with some initial conditions we have:

- $x_0 = 0 \Rightarrow 0, -1, 0, -1, \dots$ a periodic solution of period 2, which is named *2-cycle*
- $x_0 = 2 \Rightarrow 2, 3, 8, 63, \dots \Rightarrow x_h \rightarrow +\infty$
- $x_0 = -2 \Rightarrow -2, 3, 8, 63, \dots \Rightarrow x_h \rightarrow +\infty$
- $x_0 = 1 \Rightarrow 1, 0, -1, 0, \dots \Rightarrow (x_h)$ a pre-2-cycle
- $x_0 = \sqrt{2} \Rightarrow \sqrt{2}, 1, 0, -1, 0, \dots \Rightarrow (x_h)$ another pre-2-cycle

It can be proved that $(1 \pm \sqrt{5})/2$ are two equilibrium points. ★

1.5 Example. Let us consider the DDS in $\mathbb{R}^2$

$$\begin{cases} x_{h+1} = y_h \\ y_{h+1} = x_h + y_h. \end{cases} \quad (3)$$

With the initial condition $(x_0, y_0) = (0, 1)$ one has

$$(0, 1), (1, 1), (1, 2), (2, 3), (3, 5), (5, 8), (8, 13), (13, 21), (21, 34), \dots$$

which is related to the Fibonacci sequence.

By eliminating x_{h+1} one gets

$$y_{h+2} = x_{h+1} + y_{h+1} = y_h + y_{h+1}$$

that is, a one-dimensional discrete dynamical system of order 2. Starting from $y_0 = 0, y_1 = 1$ one gets the Fibonacci numbers. ★

2 Equilibrium Points, Cycles, Stability

2.1 Definition (Equilibrium point). An *equilibrium point* of a DDS is a constant solution, that is an element $\bar{x} \in U$ such that

$$x_h = \bar{x} \quad \forall h \in \mathbb{N}. \quad \star$$

2.2 Proposition. An element $\bar{x} \in U$ is an *equilibrium point* for $x_{h+1} = F(x_h)$ if and only if it is a *fixed point* for F , that is

$$F(\bar{x}) = \bar{x}.$$

Proof. If $\bar{x} \in D$ is an equilibrium point, then

$$\bar{x} = x_1 = F(x_0) = F(\bar{x}).$$

The other implication can be proved by induction. □

2.3 Example. The one-dimensional DDS

$$x_{h+1} = x_h^3$$

has exactly three equilibria: $-1, 0, 1$. ★

2.4 Definition (Cycle). A p -cycle is a periodic solution of period p , that is

$$\{x_0, F(x_0), F^2(x_0), \dots, F^{p-1}(x_0)\}$$

such that $F^j(x_0) \neq x_0$ for any $1 \leq j \leq p-1$ and $F^p(x_0) = x_0$. ★

Notice that an equilibrium point can be seen as a 1-cycle.

It is easy to prove that every point of a p -cycle is a fixed point of the map F^p . On the other hand, the fixed points of F^p can be part of a (smaller) q -cycle if q divides p .

2.5 Definition (Stability). An equilibrium point $\bar{x}$ of the DDS

$$x_{h+1} = F(x_h)$$

is *stable* if for every neighborhood V of $\bar{x}$ there exists a neighborhood U of $\bar{x}$ such that for every $x_0 \in U$ one has

$$F^h(x_0) \in V \quad \text{for every } h \in \mathbb{N}.$$

The point $\bar{x}_0$ is *unstable* if it is not stable. ★

2.6 Definition (Asymptotic stability). An equilibrium point $\bar{x}$ of the DDS

$$x_{h+1} = F(x_h)$$

is *asymptotically stable* (a.s.) if it is stable and moreover for every $x_0 \in U$ one has

$$\lim_{h \rightarrow +\infty} F^h(x_0) = \bar{x}.$$

If the above condition holds for every $x_0 \in \mathbb{R}^n$, then the equilibrium is *globally asymptotically stable* (g.a.s.).

On the contrary, if $\bar{x}$ is stable but *not* asymptotically stable, it will be said *simply stable*. ★

3 Linear First-order One-dimensional Difference Equations

The easiest example of DDS is the one dimensional linear autonomous case

$$x_{h+1} = ax_h + b, \quad a, b \in \mathbb{R}. \quad (4)$$

In this case there is an explicit formula for the solution.

Let us first consider the *homogeneous* case $b = 0$:

$$x_{h+1} = ax_h.$$

Then, starting from x_0 , one has

$$x_0, ax_0, a^2x_0, a^3x_0, \dots \Rightarrow x_h = a^h x_0.$$

It follows that:

- $x_0 = 0$ is an equilibrium
- for $x_0 > 0$:
 - $a > 1 \Rightarrow x_h \rightarrow +\infty$
 - $a = 1 \Rightarrow x_h \equiv x_0$ constant
 - $-1 < a < 1 \Rightarrow x_h \rightarrow 0$
 - $a = -1 \Rightarrow \{x_0, -x_0\}$ 2-cycle
 - $a < -1 \Rightarrow |x_h| \rightarrow +\infty$ and the sign oscillates between plus and minus.

For $x_0 < 0$ the situation is similar.

In the non-homogeneous case

$$x_{h+1} = ax_h + b,$$

starting from x_0 one gets

$$x_0, ax_0 + b, a^2x_0 + ab + b, a^3x_0 + a^2b + ab + b, \dots$$

Then the explicit solution can be written as

$$x_h = a^h x_0 + b(a^{h-1} + \dots + a + 1) = a^h x_0 + b \sum_{j=0}^{h-1} a^j.$$

By using the formula

$$\sum_{j=0}^{h-1} a^j = \frac{a^h - 1}{a - 1}, \quad a \neq 1$$

one finds the explicit solution

$$\begin{cases} x_h = a^h x_0 + b \frac{a^h - 1}{a - 1} & a \neq 1 \\ x_h = x_0 + bh & a = 1. \end{cases} \quad (5)$$

In the case $a \neq 1$, by introducing the equilibrium point

$$\bar{x} = \frac{b}{1 - a},$$

the first formula simplifies to

$$x_h = (x_0 - \bar{x})a^h + \bar{x}. \quad (6)$$

This comes from a general formula of the kind

$$x_h = ca^h + \bar{x}$$

where c is a constant to be determined. By imposing the initial value x_0 one gets

$$x_0 = c + \bar{x} \implies c = x_0 - \bar{x}.$$

3.1 Stability of Equilibrium for Linear 1D Equations

By the formula (6) it is easy to deduce that:

- if $|a| < 1$, then $\bar{x}$ is g.a.s.
- if $|a| > 1$, then $\bar{x}$ is unstable.

If $a = 1$, then the equilibrium equation becomes $\bar{x} = \bar{x} + b$, which has no solution if $b \neq 0$. In the case $b = 0$ every position is an equilibrium: the system degenerates and any equilibrium is simply stable.

If $a = -1$, the equilibrium is $\bar{x} = \frac{b}{2}$ and every initial condition x_0 produces a 2-cycle $\{x_0, b - x_0\}$. Again, the equilibrium is simply stable.

3.2 Amortization

A loan is repaid by a sequence of periodic payments (part of interest and part to reduce the outstanding principal).

If r_h is the rate of compound interest per payment period and T_h is the payment, denoting with P_h the principal after the h -th payment, one has

$$P_{h+1} = P_h + rP_h - T_h = (1 + r_h)P_h - T_h,$$

which is a non-autonomous DDS.

If we assume that both the interest rate and the payment are constant, say r and T , then the equation simplifies to

$$P_{h+1} = (1 + r)P_h - T,$$

which is a first-order linear autonomous DDE with $a = 1 + r$ and $b = -T$. Since $a > 1$, by using the first of (5) one has

$$P_h = (1 + r)^h P_0 - T \frac{(1 + r)^h - 1}{r}.$$

Then we wonder: what would be the periodic payment T if we want to pay off the loan after N payments? To answer this question it is enough to write down the equation

$$P_N = (1 + r)^N P_0 - T \frac{(1 + r)^N - 1}{r} = 0$$

and solve for T . We get the formula

$$T = P_0 \frac{r(1 + r)^N}{(1 + r)^N - 1} = P_0 \frac{r}{1 - (1 + r)^{-N}}. \quad (7)$$

The next exercises are taken from [Elaydi, p. 8].

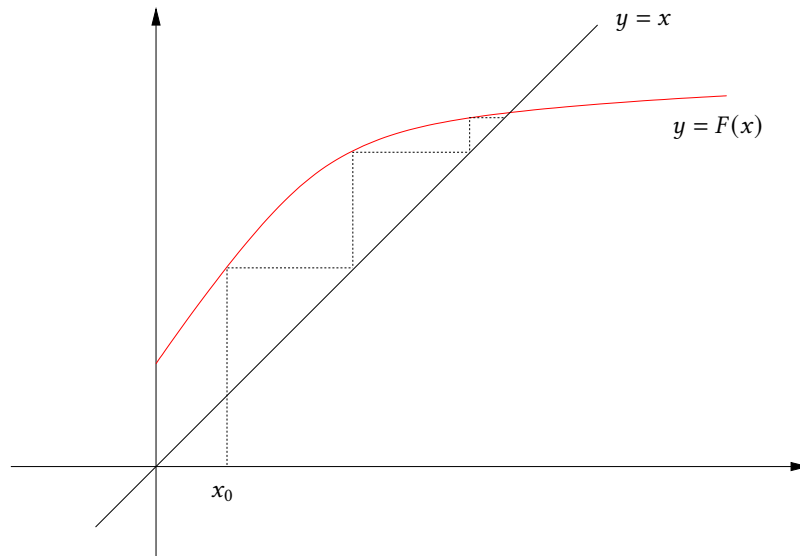


Figure 1: Starting from x_0 the solution will go towards an equilibrium point.

- 3.1 Exercise.** A loan of 12000€ is repaid by equal monthly payments of 380€ at an interest annual rate of 12% compounded monthly. How many months it will take to repay the initial debt?
[Answer: 38 plus something]
- 3.2 Exercise.** A loan of 80000€ is repaid by equal monthly payments at an interest annual rate of 10% compounded monthly. If the loan has to be repaid in 30 years, find the monthly payment
[Answer: 702€]
- 3.3 Exercise.** Suppose that you can get a 30-year mortgage at 8% interest (compounded monthly). How much can you borrow if you can afford to make a monthly payment of 1000€?
[Answer: 136270€]

4 One-dimensional DDEs

4.1 The Cobweb Method

Let $x_{h+1} = F(x_h)$ a 1D DDE. Then

- plot the function $y = F(x)$;
- plot the line $y = x$ (the bisector of the first and third quadrants);
- starting from the initial condition x_0 on the x -axis, move *vertically* toward the graph $y = F(x)$ and *horizontally* toward the bisector;
- repeat the previous step several times, trying to understand what happens to the trajectory.

Whenever one crosses the graph $y = F(x)$, the next value of the solution is obtained, hence one can get an idea on the behavior of the solution starting from x_0 when h increases. See Figures 1 and 2.

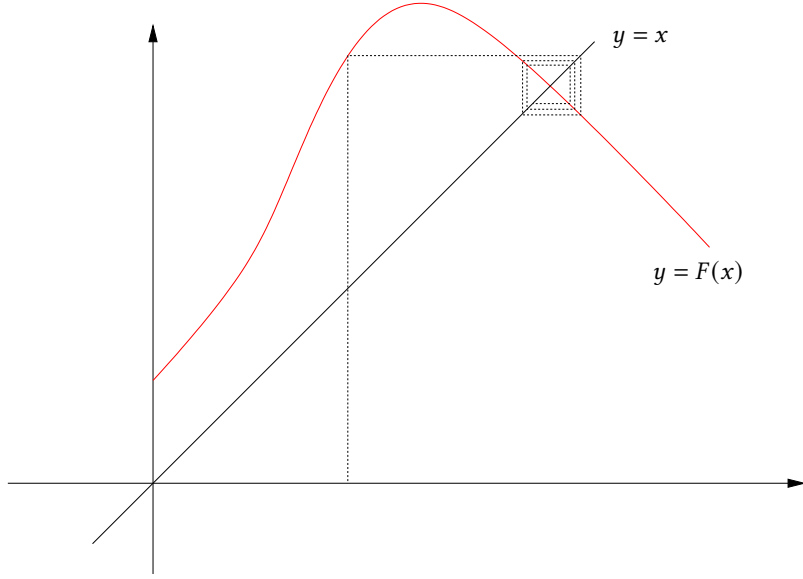


Figure 2: In this case it is not so clear where the solution is heading. It gets somewhat close to an equilibrium, but it may tend towards a “square” (that is, a 2-cycle).

4.2 Stability of Equilibria

Let now $x_{h+1} = F(x_h)$ be a one-dimensional discrete dynamical system with F of class C^1 and $\bar{x} \in \mathbb{R}$ an equilibrium point. Using the cobweb method, it can be shown that

$$\begin{aligned} |F'(\bar{x})| < 1 &\Rightarrow \bar{x} \text{ is asymptotically stable} \\ |F'(\bar{x})| > 1 &\Rightarrow \bar{x} \text{ is unstable.} \end{aligned}$$

If the condition $|F'(\bar{x})| \neq 1$ holds, then the following cases arise:

(1) if $F'(\bar{x}) = 1$ and F is of class C^3 :

- if $F''(\bar{x}) \neq 0$, then $\bar{x}$ is unstable (a saddle);
- if $F''(\bar{x}) = 0$, then we have

$$\begin{aligned} F'''(\bar{x}) < 0 &\Rightarrow \bar{x} \text{ is asymptotically stable} \\ F'''(\bar{x}) > 0 &\Rightarrow \bar{x} \text{ is unstable (a source)} \end{aligned}$$

(2) if $F'(\bar{x}) = -1$ and F is of class C^3 , then

$$\begin{aligned} 2F'''(\bar{x}) + 3F''(\bar{x})^2 > 0 &\Rightarrow \bar{x} \text{ is asymptotically stable} \\ 2F'''(\bar{x}) + 3F''(\bar{x})^2 < 0 &\Rightarrow \bar{x} \text{ is unstable.} \end{aligned}$$

Cases (1) and (2) are proven using the cobweb method, while showing (3) is somewhat more difficult.

In the case $F'(\bar{x}) = 1$, a sufficient condition for asymptotic stability that does not require further regularity is the following:

$$(F(x) - x)(x - \bar{x}) < 0 \quad \text{for } x \neq \bar{x} \text{ in a neighborhood of } \bar{x}. \quad (8)$$

See Figure 3.

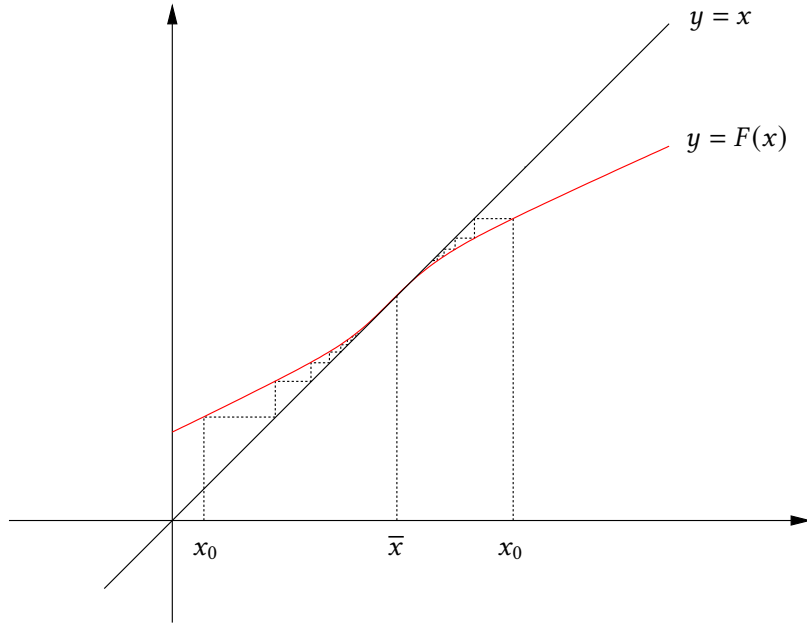


Figure 3: The case $F(\bar{x}) = 1$ with $(F(x) - x)(x - \bar{x}) < 0$.

4.1 Exercise. Please verify that the origin is:

- unstable for the map $F(x) = x + x^3$;
- unstable for the map $F(x) = x + x^2$;
- asymptotically stable for the map $F(x) = x - x^3$.

5 The Cobweb Theory of Economics

We want to model the evolution in time of the price $P_h \geq 0$ per unit of a commodity. Let us denote with S_h the number of supplied units (in period h) and with D_h the number of demanded units (in period h).

We make the following assumptions:

- D_h depends linearly on P_h :

$$D_h = -m_d P_h + b_d, \quad m_d, b_d > 0.$$

The parameter m_d is the *sensitivity* of consumers to price, and the curve $x = -m_d y + b_d$ is the so-called *demand curve* and it is always assumed to be decreasing (in this case it is a line).⁽¹⁾

- S_h depends linearly on P_{h-1} :

$$S_h = m_s P_{h-1} - b_s, \quad m_s, b_s > 0.$$

The parameter m_s is the *sensitivity* of suppliers to price. and the curve $x = m_s y - b_s$ is the so-called *supply curve* and it is always assumed to be increasing (in this case it is a line).

⁽¹⁾For historical reasons, the curve is typically represented with the price on the vertical axis.

- There is equilibrium between supply and demand, that is

$$D_h = S_h \quad \forall h \in \mathbb{N}.$$

Then we get the equation

$$-m_d P_h + b_d = m_s P_{h-1} - b_s$$

and, incrementing h by 1, we get the one-dimensional DDE

$$P_{h+1} = aP_h + b, \quad a = -\frac{m_s}{m_d}, \quad b = \frac{b_d + b_s}{m_d}. \quad (9)$$

The equation is easily studied with the methods of Sections 3 and 4. The equilibrium solution is given by

$$\bar{P} = a\bar{P} + b \quad \Rightarrow \quad \bar{P} = \frac{b}{1-a} = \frac{b_d + b_s}{m_d + m_s},$$

which is well-defined since the parameters are all positive. The stability of the position is driven by the parameter a ; since it is negative, there are three cases:

- (a) $-1 < a < 0 \quad \Rightarrow \quad m_s < m_d$ the equilibrium is g.a.s.
- (b) $a = -1 \quad \Rightarrow \quad m_s = m_d$ the equilibrium is simply stable
- (c) $a < -1 \quad \Rightarrow \quad m_s > m_d$ the equilibrium is unstable.

The **Cobweb theorem of Economics** says that: *if the suppliers are less sensitive to price than the consumers ($m_s < m_d$), the market is stable. Vice versa, the market is unstable.*

Since $a < 0$, by a cobweb diagram one can also see that in all the three cases there are oscillations in the price: in the first case the oscillations become smaller and smaller, in the second they have a constant amplitude (2-cycles), in the third they increase more and more.

Using (6), one can also express the explicit solution starting from an initial price P_0 as

$$P_h = (P_0 - \bar{P})a^h + \bar{P} = (P_0 - \bar{P})(-1)^h \left(\frac{m_s}{m_d}\right)^h + \bar{P},$$

which shows the oscillatory nature of the solution, due to the term $(-1)^h$.

One can represent the model on a diagram, by plotting the demand and supply curves introduced above (see Figure 4

5.1 Exercise. Let $D_h = 100 - 2P_h$, $S_h = -20 + 3P_{h-1}$. Find the equilibrium point and check if it is stable.

If $P_0 = 25$, compute P_1, P_2, P_3, P_4 .

[Answer: $\bar{P} = 24$, unstable, 22.5, 26.25, 20.625, 29.0625]

5.2 Exercise. Study the nonlinear model $D_h = -2P_h + 3$, $S_h = P_{h-1}^2 + 1$, finding the equation, the positive equilibria, the stability.

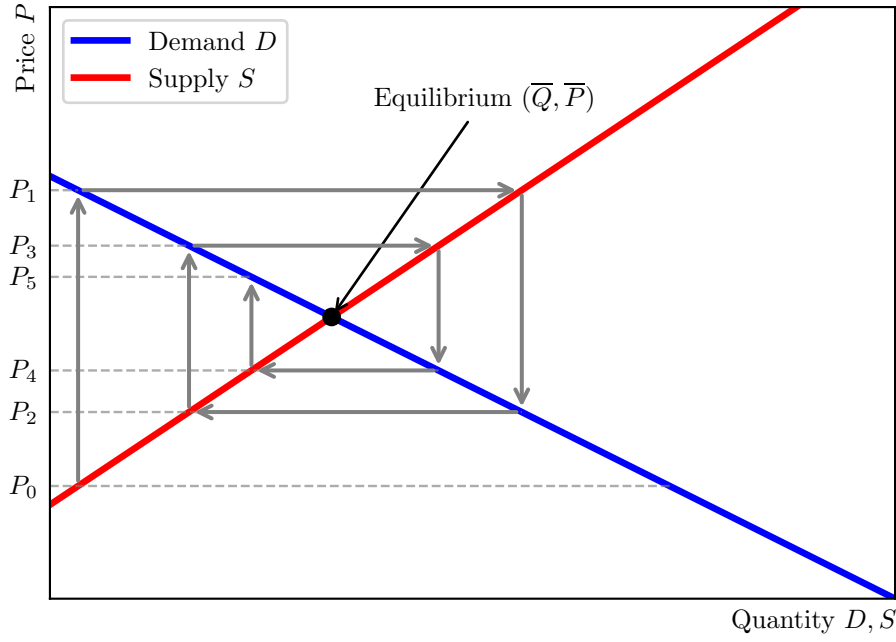


Figure 4: The cobweb diagram of Economics: one has to start and go *horizontally* toward the supply curve, then *vertically* toward the demand curve, iterating the procedure.

5.1 Cobweb Model with Expected Price

An improvement of the cobweb model is the following:

$$\begin{cases} D_h = -m_d P_h + b_d \\ S_h = m_s \hat{P}_h - b_s \\ D_h = S_h \end{cases}$$

where $\hat{P}_h$ denotes the *expected price* of the producers.

If $\hat{P}_h = P_{h-1}$ we recover the cobweb model. A more realistic assumption is to choose

$$\hat{P}_h = P_{h-1} + c(P_N - P_{h-1}), \quad 0 < c < 1$$

where P_N is the “normal” price, that is the correct price according to the producers, and c is related to the “velocity” in reaching the normal price (for $c = 1$ the price P_N is reached immediately).

Then the equation becomes

$$-m_d P_{h+1} + b_d = m_s (P_h + c(P_N - P_h)) - b_s \quad \Rightarrow \quad P_{h+1} = -\frac{m_s}{m_d} (1 - c) P_h + \frac{b_d + b_s - m_s c P_N}{m_d}.$$

Setting

$$\bar{a} = -\frac{m_s}{m_d} (1 - c), \quad \bar{b} = \frac{b_d + b_s - m_s c P_N}{m_d},$$

one has that the equilibrium position

$$\bar{P} = \frac{\bar{b}}{1 - \bar{a}} = \frac{b_d + b_s - m_s c P_N}{m_d + m_s (1 - c)}$$

is asymptotically stable if $\bar{a} > -1$, that is $m_s(1 - c) < m_d$. Since $1 - c < 1$, the domain of stability increases.

For instance we can assume that $P_N = \bar{P}$: in this case it can be verified that $\bar{\bar{P}} = \bar{P}$, that is, the equilibrium position does not change.

6 Linear Discrete Dynamical Systems

6.1 Definition. A non-homogeneous linear autonomous DDS has the form

$$\mathbf{x}_{h+1} = A\mathbf{x}_h + \mathbf{b}, \quad \mathbf{x}_h \in \mathbb{R}^n, \quad (10)$$

where A is an $(n \times n)$ -matrix and $\mathbf{b} \in \mathbb{R}^n$ is a constant vector. ★

The equilibrium points of the system solve the fixed-point equation

$$A\mathbf{x} + \mathbf{b} = \mathbf{x} \quad \Rightarrow \quad (A - I)\mathbf{x} = -\mathbf{b}.$$

Now, if $\det(A - I) \neq 0$, that is, if $\lambda = 1$ is not an eigenvalue of A , then the matrix $A - I$ is invertible and there is the unique equilibrium

$$\bar{\mathbf{x}} = (I - A)^{-1}\mathbf{b}.$$

On the other hand, if $\det(A - I) = 0$, then it can happen that there are infinitely many equilibria or that there is no equilibrium (the distinction being based on the Rouché-Capelli theorem).

Moreover, with the same technique of Section 3 it can be proved that, in the case $\det(A - I) \neq 0$, the general solution of (10) with initial condition $\mathbf{x}_0$ is given by

$$\mathbf{x}_h = A^h\mathbf{x}_0 + (A^h - I)(A - I)^{-1}\mathbf{b} = A^h(\mathbf{x}_0 - \bar{\mathbf{x}}) + \bar{\mathbf{x}}. \quad (11)$$

6.2 Definition (Spectral radius). Given a square matrix A with (complex) eigenvalues $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, the *spectral radius* of A is the real number

$$\rho(A) = \max\{|\lambda_1|, \dots, |\lambda_k|\}. \quad \star$$

It is possible to prove the following important theorem for linear discrete dynamical systems.

6.3 Theorem. Given a square matrix A , consider the linear discrete dynamical system

$$\mathbf{x}_{h+1} = A\mathbf{x}_h + \mathbf{b}.$$

Then:

- If $\rho(A) < 1$, then $(I - A)^{-1}\mathbf{b}$ is the only equilibrium point and it is globally asymptotically stable.
- If $\rho(A) > 1$, then there may be multiple equilibrium points, but they are all unstable.
- If $\rho(A) = 1$ and every eigenvalue with $|\lambda_i| = 1$ is semisimple (that is, geometric and algebraic multiplicities coincide), then all the equilibria (if any) are simply stable.

It becomes important to find a criterion to determine if $\rho(A) < 1$, that is, whether all the complex eigenvalues of a matrix lie inside the open unit disk or not. Without solving the characteristic polynomial, one can apply the *Jury criterion* (also called the *Schur-Cohn criterion*). We will mention it in the cases $n = 2, 3$.

- In the case $n = 2$, writing the characteristic polynomial as $\lambda^2 + a_1\lambda + a_2$, the condition $\rho(A) < 1$ is equivalent to

$$|a_1| < 1 + a_2 < 2,$$

which in relation with the matrix A can be written as

$$|\operatorname{tr} A| < 1 + \det A < 2$$

(try to prove it using the quadratic formula and distinguishing between real or complex conjugate solutions).

- In the case $n = 3$, writing the characteristic polynomial as $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3$, the condition $\rho(A) < 1$ is equivalent to the inequalities

$$|a_1 + a_3| < 1 + a_2, \quad |a_2 - a_1a_3| < 1 - a_3^2.$$

6.4 Exercise. For which values of the parameter $a \in \mathbb{R}$ is the system

$$\mathbf{x}_{h+1} = A\mathbf{x}_h$$

globally asymptotically stable?

$$A = \begin{bmatrix} 0 & a \\ a & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a & 1 \\ 1 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} a & a \\ a & a \end{bmatrix}.$$

[Answer: $|a| < 1$, $\nexists a$, $|a| < 1/2$.]

7 Higher-order Discrete Dynamical Systems

7.1 Definition. An autonomous DDS of order k in $\mathbb{R}^n$ is of the form

$$\mathbf{x}_{h+1} = \mathbf{F}(\mathbf{x}_h, \mathbf{x}_{h-1}, \dots, \mathbf{x}_{h+1-k}), \quad (12)$$

where $\mathbf{F} : \mathbb{R}^{nk} \rightarrow \mathbb{R}^n$. ★

It is always possible to transform a higher order DDS in a first-order system by increasing its dimensionality, in the following way: set

$$\mathbf{x}_{h-1} = \mathbf{x}_h^{(1)}, \quad \mathbf{x}_{h-2} = \mathbf{x}_h^{(2)}, \quad \dots, \quad \mathbf{x}_{h+2-k} = \mathbf{x}_h^{(k-2)}, \quad \mathbf{x}_{h+1-k} = \mathbf{x}_h^{(k-1)}.$$

Then the system

$$\begin{cases} \mathbf{x}_{h+1} = \mathbf{F}(\mathbf{x}_h, \mathbf{x}_h^{(1)}, \dots, \mathbf{x}_h^{(k-1)}) \\ \mathbf{x}_{h+1}^{(1)} = \mathbf{x}_h \\ \mathbf{x}_{h+1}^{(2)} = \mathbf{x}_h^{(1)} \\ \dots \\ \mathbf{x}_{h+1}^{(k-1)} = \mathbf{x}_h^{(k-2)} \end{cases}$$

is a $(k \cdot n)$ -dimensional DDS equivalent to (12). Hence we can speak of equilibrium points and stability even in the case of higher order systems.

In particular, an equilibrium point for (12) is found by solving

$$\bar{x} = F(\bar{x}, \bar{x}, \dots, \bar{x}).$$

In the case of *higher order linear 1D homogeneous equations*, one has that the equation

$$x_{h+1} = a_0 x_h + a_1 x_{h-1} + \dots + a_{k-1} x_{h+1-k} \quad (13)$$

is equivalent to the system

$$x_{h+1} = A x_h$$

where

$$A = \begin{bmatrix} a_0 & a_1 & a_2 & \dots & a_{k-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix}. \quad (14)$$

A higher order linear 1D equation of the form (13) has also a nice explicit formula for its solutions. First of all, rewrite the equation by increasing of $k - 1$ units all the times:

$$x_{h+k} = a_0 x_{h+k-1} + a_1 x_{h+k-2} + \dots + a_{k-1} x_h;$$

then replace every x_{h+j} with the polynomial term λ^j , obtaining the *characteristic polynomial* associated with the equation:

$$p(\lambda) = \lambda^k - a_0 \lambda^{k-1} - a_1 \lambda^{k-2} - \dots - a_{k-1}.$$

It can be proved that this polynomial corresponds to the characteristic polynomial of the matrix A above.

Now we have to compute the roots of $p(\lambda)$, paying attention to the fact that the roots can be complex-valued and can have multiplicity. If we have real solutions, we distinguish two cases:

- if all the roots $\lambda_1, \dots, \lambda_k$ are distinct real numbers (that is, the multiplicity of every root is 1), then the general solution of (13) is given by

$$x_h = c_1 \lambda_1^h + \dots + c_k \lambda_k^h = \sum_{j=1}^k c_j \lambda_j^h \quad (15)$$

where the constants c_j have to be determined by the initial conditions $x_0, \dots, x_{k-1}$

- if some root λ_j has multiplicity $m \geq 2$, we have to replace the term $c_j \lambda_j^h$ with the expression

$$\lambda_j^h (c_{j,0} + c_{j,1} h + c_{j,2} h^2 + \dots + c_{j,m-1} h^{m-1}). \quad (16)$$

In any case, the solution is given by a linear combination of exactly k terms involving the powers of the roots λ_j of the characteristic polynomial associated with the equation.

If some root λ_j is complex-valued, that is $\lambda_j = \alpha + i\beta$ for $\beta \neq 0$, then also the conjugate $\bar{\lambda}_j = \alpha - i\beta$ is a root of the characteristic polynomial, and it is possible to combine the two roots and the constants $c_j, \widehat{c}_j$ to get real quantities. For instance, by using polar coordinates one can write

$$\begin{aligned} & c_j(r \cos \vartheta + ir \sin \vartheta)^h + \widehat{c}_j(r \cos \vartheta - ir \sin \vartheta)^h \\ &= r^h [(c_j + \widehat{c}_j) \cos(h\vartheta) + i(c_j - \widehat{c}_j) \sin(h\vartheta)] \\ &= r^h [c_{j,0} \cos(h\vartheta) + c_{j,1} \sin(h\vartheta)]. \end{aligned} \tag{17}$$

7.2 Example. The discrete equation that gives the Fibonacci numbers is

$$x_{h+1} = x_h + x_{h-1}, \quad x_0 = 0, \quad x_1 = 1.$$

The associated characteristic polynomial is

$$\lambda^2 = \lambda + 1 \quad \Rightarrow \quad \lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}.$$

Then the explicit solution has the form

$$x_h = c_1 \left(\frac{1 - \sqrt{5}}{2} \right)^h + c_2 \left(\frac{1 + \sqrt{5}}{2} \right)^h$$

where the constants c_1, c_2 have to satisfy the initial conditions

$$x_0 = c_1 + c_2 = 0, \quad x_1 = c_1 \frac{1 - \sqrt{5}}{2} + c_2 \frac{1 + \sqrt{5}}{2}.$$

Hence we solve the linear system

$$\begin{cases} c_1 + c_2 = 0 \\ c_1 \frac{1 - \sqrt{5}}{2} + c_2 \frac{1 + \sqrt{5}}{2} = 1 \end{cases} \quad \Rightarrow \quad c_1 = -\frac{1}{\sqrt{5}}, \quad c_2 = \frac{1}{\sqrt{5}}.$$

Then the Fibonacci numbers are generated by the explicit formula

$$x_h = \frac{1}{\sqrt{5}} \left[\left(\frac{1 + \sqrt{5}}{2} \right)^h - \left(\frac{1 - \sqrt{5}}{2} \right)^h \right].$$

★

7.3 Exercise. Find the explicit formulae for the following second-order DDEs:

- $x_{h+1} = 4x_h + 5x_{h-1}$
- $x_{h+1} = 2x_h - x_{h-1}$
- $x_{h+1} = -x_{h-1}$

The formulae (15)–(17) have an important consequence about stability:

7.4 Theorem. A homogeneous higher order linear 1D equation of the form (13) admits 0 as an equilibrium point. Considering the roots $\lambda_1, \dots, \lambda_k$ of its characteristic polynomial, we have that:

- $\forall \lambda_j : |\lambda_j| < 1 \quad \Rightarrow \quad 0 \text{ is g.a.s.}$

- $\exists \lambda_i : |\lambda_i| > 1 \Rightarrow 0$ is unstable
- if all the roots are such that $|\lambda_j| \leq 1$ and every root with $|\lambda_i| = 1$ has multiplicity 1, then 0 is simply stable.

where $|\cdot|$ denotes the complex modulus.

Since the characteristic polynomial of (13) corresponds to the characteristic polynomial of the matrix A (14), the previous theorem corresponds to Theorem 6.3.

One can also consider the non-homogeneous case, that is

$$x_{h+1} = a_0 x_h + a_1 x_{h-1} + \dots + a_{k-1} x_{h+1-k} + c, \quad (18)$$

where we added a constant $c \neq 0$. In this case the equilibrium solution solves

$$\bar{x} = a_0 \bar{x} + a_1 \bar{x} + \dots + a_{k-1} \bar{x} + c \Rightarrow (1 - a_0 - a_1 - \dots - a_{k-1}) \bar{x} = c.$$

If $(1 - a_0 - a_1 - \dots - a_{k-1}) \neq 0$, then there is the unique equilibrium

$$\bar{x} = \frac{c}{1 - a_0 - a_1 - \dots - a_{k-1}}$$

and, by the change of variable $y_h = x_h - \bar{x}$, we obtain the equation

$$y_{h+1} = a_0 y_h + a_1 y_{h-1} + \dots + a_{k-1} y_{h+1-k}$$

which is homogeneous. In particular, the characteristic polynomial is

$$p(\lambda) = \lambda^k - a_0 \lambda^{k-1} - a_1 \lambda^{k-2} - \dots - a_{k-1}$$

that is, the polynomial associated with the original equation (18) without the constant term c , and the explicit solution in the case of simple real roots is given by

$$y_h = c_1 \lambda_1^h + \dots + c_k \lambda_k^h \Rightarrow x_h = \bar{x} + c_1 \lambda_1^h + \dots + c_k \lambda_k^h,$$

or its variants for the case with multiple or complex roots.

8 Goodwin's Second-order Cobweb Model

Another variant of the cobweb model of Section 5.1, due to Goodwin, is the following: assume that the expected price of the producers has the following form:

$$\hat{p}_h = P_{h-1} + \rho(P_{h-1} - P_{h-2}), \quad \rho > 0. \quad (19)$$

With this model the producers have an *extrapolative expectation*: if the price is increasing, it is expected to increase again.

Then the equation becomes

$$-m_d P_{h+1} + b_d = m_s P_h - b_s + m_s \rho (P_h - P_{h-1}),$$

which is a second order linear 1D discrete DDE. Since it is not homogeneous, the equilibrium is given by solving

$$-m_d \bar{p} + b_d = m_s \bar{p} - b_s + m_s \rho (\bar{p} - \bar{p}) \quad \Rightarrow \quad \bar{p} = \frac{b_d + b_s}{m_d + m_s},$$

which is the usual equilibrium of the original cobweb model.

The characteristic polynomial is found by dropping the non-homogeneous part:

$$-m_d \lambda^2 = m_s \lambda + m_s \rho (\lambda - 1) \quad \Rightarrow \quad \lambda^2 + \frac{m_s}{m_d} (1 + \rho) \lambda - \frac{m_s}{m_d} \rho.$$

By the Jury criterion, since $a_1 = \frac{m_s}{m_d} (1 + \rho) > 0$ and $a_2 = -\frac{m_s}{m_d} \rho < 0$ we have that the condition $1 + a_2 < 2$ is always satisfied and the equilibrium is asymptotically stable if

$$\frac{m_s}{m_d} (1 + \rho) < 1 - \frac{m_s}{m_d} \rho \quad \Rightarrow \quad \frac{m_s}{m_d} < \frac{1}{1 + 2\rho}$$

and it is unstable for the opposite (strict) inequality. In this case, the condition for stability is more restrictive than the original one.

With a *regressive expectation*, that is if $-1/2 < \rho < 0$, the stability condition becomes less restrictive and the market will result more stable.

9 Gambler's Ruin

Let us investigate the following situation: a gambler starts with a certain amount of euros and wins 1€ with probability q and loses 1€ with probability $1 - q$. He/she quits the game if all the money is lost (ruin) or if the goal of acquiring N euros is reached. What is the probability that the gambler gets ruined?

We can solve the problem by employing a DDS: let p_h be the probability that the gambler will be ruined if he/she possesses h euros. Then, by the Law of total probability one has that

$$p_h = qp_{h+1} + (1 - q)p_{h-1},$$

which is a second-order linear DDE. The characteristic polynomial is given by

$$\lambda = q\lambda^2 + 1 - q \quad \Rightarrow \quad q\lambda^2 - \lambda + 1 - q = 0$$

which has the two solutions

$$\lambda_1 = 1, \quad \lambda_2 = \frac{1 - q}{q}.$$

The multiplicity is 1 if $\lambda_1 \neq \lambda_2$, that is, $q \neq 1/2$. In that case, by setting $x = (1 - q)/q$, the explicit solution writes

$$p_h = c_1 + c_2 x^h,$$

which has to be complemented by the *boundary conditions* $p_0 = 1$ (if the gambler starts with no money, he is ruined with probability 1) and $p_N = 0$ (if the gambler reaches the target, he will not be ruined for sure). Then

$$c_1 + c_2 = 1, \quad c_1 + c_2 x^N = 0 \quad \Rightarrow \quad c_1 = -\frac{x^N}{1 - x^N}, \quad c_2 = \frac{1}{1 - x^N}$$

and the explicit solution becomes

$$p_h = \frac{x^h - x^N}{1 - x^N}.$$

In the case $\lambda_1 = \lambda_2$, we have the solution $\lambda = 1$ with multiplicity 2, hence

$$p_h = c_1 + c_2 h,$$

and by imposing the boundary conditions we get

$$c_1 = 1, \quad c_2 = -\frac{1}{N} \quad \Rightarrow \quad p_h = 1 - \frac{h}{N}.$$

In summary:

$$\begin{cases} p_h = \frac{x^h - x^N}{1 - x^N} & \text{for } x = \frac{1-q}{q}, \quad q \neq \frac{1}{2} \\ p_h = 1 - \frac{h}{N} & \text{for } q = \frac{1}{2}. \end{cases} \quad (20)$$

In particular, for $q < 1/2$ one has $x > 1$, hence $p_h \rightarrow 1$ as $N \rightarrow \infty$, that is, the ruin tends to be certain. The same happens for $q = 1/2$. On the other hand, for $q > 1/2$ one has $x < 1$ and $p_h \rightarrow x^h$ as $N \rightarrow \infty$.

9.1 Exercise.

- Prove that, for $N = 10$ and $q = 0.3$, one has $p_4 = 0.994$. What happens if q raises to 0.7?
- Prove that, for $N = 100$ and $q = 0.5$, one has $p_{20} = 0.8$.

10 Samuelson's Multiplier-accelerator Model

This is a macroeconomic model about national income (GDP) and the business cycle, see [Gandolfo, Ch. 5]. Let us consider:

- C_h the *consumption* (consumers' expenditure) at time h ;
- I_h the *private investment* at time h ;
- G_h the *public expenditure* at time h ;
- Y_h the national income at time h .

The period h is usually measured in years.

Then we have the relation

$$Y_h = C_h + I_h + G_h.$$

Now we make three assumptions:

- (1) $C_h = \alpha Y_{h-1}$: there is one-period lag between consumption and income, and $0 < \alpha < 1$ is the (*marginal*) *propensity to consume*;
- (2) $I_h = \beta(C_h - C_{h-1})$: the investments depend on the variation of the consumption, and $\beta > 0$ is the *acceleration coefficient*;

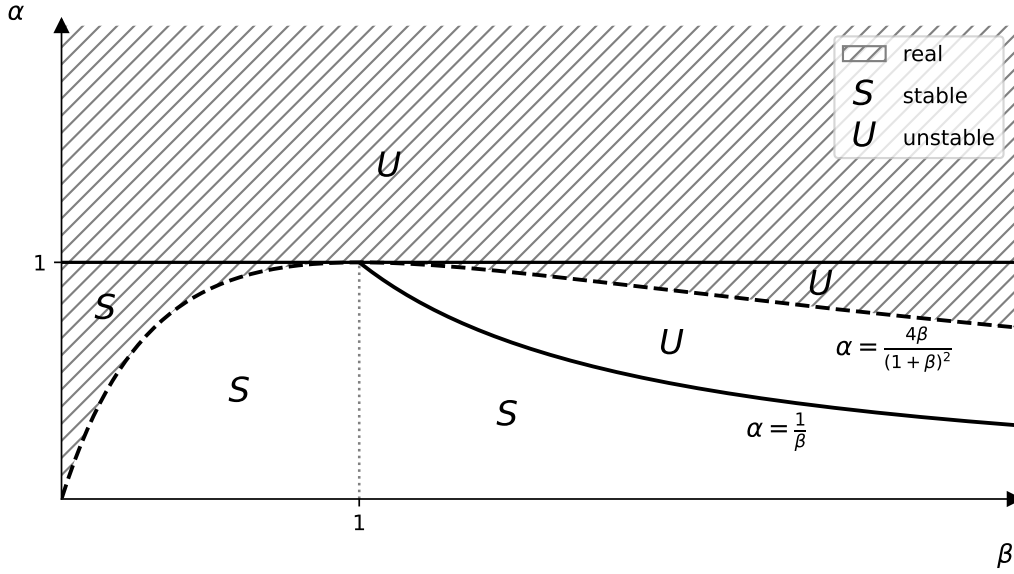


Figure 5: The stability diagram for the Samuelson's multiplier-accelerator model

(3) $G_h \equiv 1$: the public expenditure is constant and normalized to 1.

We obtain the equation for the GDP

$$Y_{h+1} - \alpha(1 + \beta)Y_h + \alpha\beta Y_{h-1} = 1 \quad (21)$$

which is a non-homogeneous second-order one-dimensional DDE.

The equilibrium position is

$$\bar{Y} = \frac{1}{1 - \alpha}$$

and the characteristic polynomial is

$$p(\lambda) = \lambda^2 - \alpha(1 + \beta)\lambda + \alpha\beta,$$

so that $\bar{Y}$ is g.a.s. if and only if

$$\alpha(1 + \beta) < 1 + \alpha\beta < 2 \quad \Rightarrow \quad \alpha\beta < 1 \quad \Rightarrow \quad \beta < \frac{1}{\alpha}$$

by the Jury criterion.

We are interested in capturing oscillations, if any. By the explicit formula for higher-order DDEs, oscillations can happen if there are negative or complex roots for $p(\lambda)$. In this case, there are no negative solutions (Descartes' rule), hence we have to look only for complex ones:

$$\Delta = \alpha^2(1 + \beta)^2 - 4\alpha\beta < 0 \quad \Rightarrow \quad \alpha < \frac{4\beta}{(1 + \beta)^2}.$$

The situation is resumed in Figure 5.

10.1 Example. Consider the previous model with $\alpha = 1/2$ and $\beta = 1$.

Then $\bar{Y} = 2$ is g.a.s. and there are (damped) oscillations. The explicit solution is given by

$$Y_h = \left(\frac{\sqrt{2}}{2}\right)^h \left(c_1 \cos\left(\frac{\pi}{4}h\right) + c_2 \sin\left(\frac{\pi}{4}h\right)\right) + 2.$$

★

11 A Trade Model between Two Countries

We want to model a trade between two (and only two) countries (see [Elaydi, 3.5.4]). Denoting with the indices $j = 1, 2$ the quantities related to the two countries, we consider:

- $Y_j(h)$ the national income in period h
- $C_j(h)$ the total consumption in period h
- I_j the total investment, assumed to be constant
- $X_j(h)$ the exports in period h
- $M_j(h)$ the imports in period h
- $D_j(h) = C_j(h) - M_j(h)$ the consumption of domestic products in period h

Then for every country we have the balance equations

$$Y_j(h) = C_j(h) + I_j + X_j(h) - M_j(h) = D_j(h) + I_j + X_j(h), \quad j = 1, 2. \quad (22)$$

Now, since there are only two countries, we assume that

$$M_1(h) = X_2(h), \quad M_2(h) = X_1(h)$$

Finally, we assume that the domestic consumption and the import in period $h + 1$ are a fixed fraction of the national income in period h :

$$\begin{aligned} D_1(h+1) &= a_{11}Y_1(h), & D_2(h+1) &= a_{22}Y_2(h) \\ M_1(h+1) &= a_{21}Y_1(h), & M_2(h+1) &= a_{12}Y_2(h) \end{aligned}$$

where the marginal propensities $a_{ij} > 0$ has to satisfy the inequalities

$$a_{11} + a_{21} < 1, \quad a_{12} + a_{22} < 1.$$

Then we can write system (22) as

$$\begin{cases} Y_1(h+1) = a_{11}Y_1(h) + a_{12}Y_2(h) + I_1 \\ Y_2(h+1) = a_{21}Y_1(h) + a_{22}Y_2(h) + I_2. \end{cases}$$

The system can be written in matrix form:

$$\mathbf{Y}_{h+1} = \mathbf{A}\mathbf{Y}_h + \mathbf{I} \quad (23)$$

where

$$\mathbf{Y}_h = \begin{bmatrix} Y_1(h) \\ Y_2(h) \end{bmatrix}, \quad \mathbf{I} = \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}.$$

We obtained a first-order two-dimensional non-homogeneous linear DDS. The matrix $\mathbf{A}$ has positive coefficients and the sum of the elements of every column is less than 1. Then:

11.1 Proposition. *Let $\mathbf{A} = [a_{ij}]$ be an $(n \times n)$ -matrix such that*

$$a_{ij} > 0, \quad \sum_{i=1}^n a_{ij} < 1 \quad \text{for every } j = 1, \dots, n.$$

Then $\rho(\mathbf{A}) < 1$.

Proof. Let λ be an eigenvalue of A and $\mathbf{x} \neq \mathbf{0}$ a corresponding eigenvector. Then one has that

$$\sum_{j=1}^n a_{ij}x_j = \lambda x_i \quad \text{for every } i = 1, \dots, n,$$

hence

$$\left| \sum_{j=1}^n a_{ij}x_j \right| = |\lambda| |x_i| \quad \text{for every } i = 1, \dots, n,$$

Summing on all the rows, one has

$$\sum_{i=1}^n \left| \sum_{j=1}^n a_{ij}x_j \right| = |\lambda| \sum_{i=1}^n |x_i|$$

and, using the triangular inequality,

$$|\lambda| \sum_{i=1}^n |x_i| \leq \sum_{i,j=1}^n |a_{ij}| |x_j| = \sum_{j=1}^n \left(\sum_{i=1}^n |a_{ij}| \right) |x_j| < \sum_{j=1}^n |x_j|.$$

Dividing both sides by $\sum_{j=1}^n |x_j|$ we get $|\lambda| < 1$, which concludes the proof. $\square$

Now we can study system (23). The equilibrium equation is

$$\bar{\mathbf{Y}} = A\bar{\mathbf{Y}} + \mathbf{I} \quad \Rightarrow \quad (I - A)\bar{\mathbf{Y}} = \mathbf{I}$$

and, by the previous proposition, $\rho(A) < 1$, hence the matrix $I - A$ is non singular and the unique equilibrium

$$\bar{\mathbf{Y}} = (I - A)^{-1} \mathbf{I}$$

is g.a.s. Hence the national incomes approach the equilibrium value whatever the initial values.

11.2 Exercise (Elaydi 3.5, Problem 11). In the trade model let $a_{11} = 0.4$, $a_{21} = 0.5$, $a_{12} = 0.3$, $a_{22} = 0.6$, $I_1 = 25$ billion dollars, and $I_2 = 20$ billion dollars. If $Y_1(h)$ and $Y_2(h)$ denote the national incomes of countries 1 and 2 in year h , respectively, and $Y_1(0) = 500$ billion dollars and $Y_2(0) = 650$ billion dollars, find $Y_1(3)$ and $Y_2(3)$. What are the equilibrium national incomes for nations 1 and 2?

12 Linearization of Discrete Dynamical Systems

In the case of a general nonlinear autonomous DDS, one can prove that in many cases the behavior of the system near an equilibrium point can be characterized by studying a particular *linear* system.

12.1 Definition. Consider the DDS

$$\mathbf{x}_{h+1} = \mathbf{F}(\mathbf{x}_h), \quad \mathbf{F} \in C^1$$

and $\bar{\mathbf{x}}$ a fixed point of $\mathbf{F}$. The *linearization* of the DDS near the equilibrium $\bar{\mathbf{x}}$ is given by

$$\mathbf{y}_{h+1} = A\mathbf{y}_h$$

where the *linearization matrix* A is given by the Jacobian of $\mathbf{F}$ in the equilibrium:

$$A = \nabla \mathbf{F}(\bar{\mathbf{x}}) = \frac{\partial \mathbf{F}}{\partial \mathbf{x}}(\bar{\mathbf{x}}) = \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(\bar{\mathbf{x}}) & \dots & \frac{\partial F_1}{\partial x_n}(\bar{\mathbf{x}}) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_n}{\partial x_1}(\bar{\mathbf{x}}) & \dots & \frac{\partial F_n}{\partial x_n}(\bar{\mathbf{x}}) \end{bmatrix}.$$

★

The importance of the linearization matrix is that it can be used to classify the stability of the equilibrium, at least when its spectral radius $\rho(A) \neq 1$.

12.2 Theorem (Linearization theorem). *Let $\bar{u}$ be an equilibrium point for the discrete dynamical system $u_{h+1} = F(u_h)$. If $\rho(\nabla F(\bar{u})) < 1$, then $\bar{u}$ is asymptotically stable.*

On the other hand, if $\rho(\nabla F(\bar{u})) > 1$, then $\bar{u}$ is unstable.

The idea of the proof is the following: apply Taylor expansion to the iterated function F^h around $\bar{x}$:

$$F^h(x) = F^h(\bar{x}) + \nabla(F^h)(\bar{x})(x - \bar{x}) + R(x - \bar{x}).$$

Since $\bar{x}$ is a fixed point for F , then

$$\nabla(F^h)(\bar{x}) = \nabla F(\bar{x})^h = A^h$$

and the expansion writes

$$F^h(x) = \bar{x} + A^h(x - \bar{x}) + R(x - \bar{x}).$$

Since $A^h \rightarrow 0 \Leftrightarrow \rho(A) < 1$, one completes the proof.

12.3 Example (A nonlinear variation on Samuelson's model). A nonlinear modification of the Samuelson's model of Sect. 10, due to T. Puu, gives rise to the following DDE:

$$Y_{h+2} = Y_{h+1} + a(Y_{h+1} - Y_h) - (a+1)(Y_{h+1} - Y_h)^3 - bY_h$$

where $a > 0$ and $0 < b < 1$.

This is a second-order nonlinear DDE, which can be transformed in a first-order system, for instance by setting

$$Z_h = Y_{h+1} - Y_h.$$

Hence we obtain

$$\begin{cases} Y_{h+1} = Y_h + Z_h \\ Z_{h+1} = aZ_h - (a+1)Z_h^3 - bY_h. \end{cases}$$

The equilibria are given by

$$\begin{cases} Y = Y + Z \\ Z = aZ - (a+1)Z^3 - bY \end{cases} \Rightarrow \begin{cases} Y = 0 \\ Z = 0. \end{cases}$$

We compute the Jacobian of the system:

$$\nabla F(Y, Z) = \begin{bmatrix} 1 & 1 \\ -b & a - 3(a+1)Z^2 \end{bmatrix} \Rightarrow A = \nabla F(0, 0) = \begin{bmatrix} 1 & 1 \\ -b & a \end{bmatrix}.$$

Employing the Jury criterion for $\rho(A) < 1$, we get

$$|\operatorname{tr} A| < 1 + \det A < 2 \Rightarrow 1 + a < 1 + a + b < 2 \Rightarrow a + b < 1.$$

Hence, for $a + b < 1$ the equilibrium $(0, 0)$ is a.s., while for $a + b > 1$ it is unstable.

The critical case $a + b = 1$ cannot be studied by linearization. It can be proved that in this case attracting cycles can appear. For instance, for $a = 0, b = 1$ there is an attracting 6-cycle, while for $a = \sqrt{2} - 1, b = 2 - \sqrt{2}$ an attracting 8-cycle appears. ★

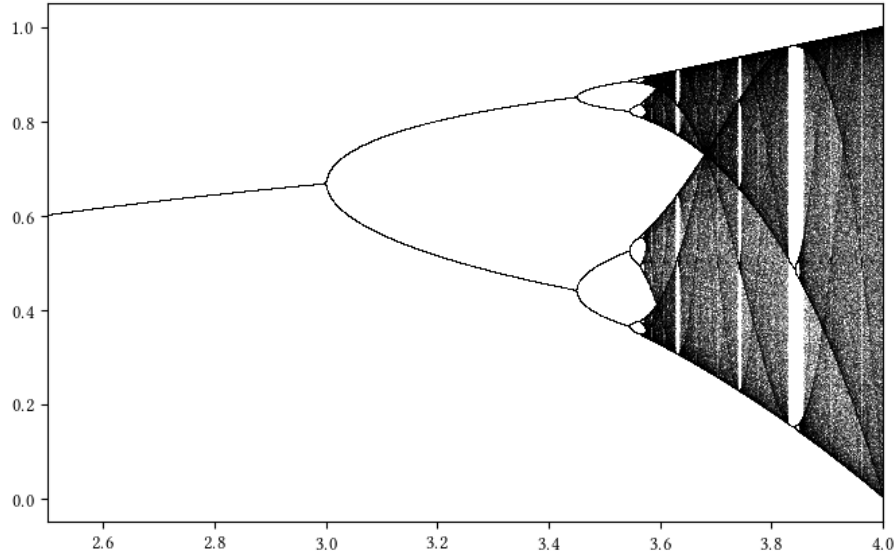


Figure 6: The bifurcation diagram of the discrete logistic map.

13 The Discrete Logistic Map

We call *discrete logistic map* the dynamical system given by $F : [0, 1] \rightarrow [0, 1]$:

$$F(x) = ax(1 - x) \quad \text{with } 0 < a \leq 4.$$

The model could represent, for instance, the dynamics of a price of an asset expressed in percentage (between 0 and 1). It is immediately verified that for $0 < a \leq 1$ there is only the equilibrium point $x_1 = 0$ which is a.s. (the case $a = 1$ cannot be treated by linearization). For $a > 1$ there are two equilibrium points: $x_1 = 0$, which is unstable, and $x_2 = 1 - \frac{1}{a}$, which is stable for $1 < a < 3$.

For $a = 3$, the equilibrium $x_2 = 2/3$ remains stable (this can be checked using higher derivatives), but for $a > 3$ it becomes unstable. In this case, a 2-cycle appears, and it can be proved that the 2-cycle is a.s. for $3 < a < 1 + \sqrt{6} \approx 3.45$.

Continuing in this way, one finds an increasing sequence (a_k) , whose first values are $a_1 = 1$, $a_2 = 3$, $a_3 = 1 + \sqrt{6}$, such that

for $a_k < a < a_{k+1}$ there exists an asymptotically stable 2^{k-1} -cycle.

This phenomenon is known as *period doubling*. The sequence is bounded and converges:

$$\lim_k a_k = a^* \approx 3.56994.$$

Beyond this value, the logistic map can exhibit chaotic behavior. In particular, for $a = 4$ the map is chaotic.

14 Systems of First-order Ordinary Differential Equations

14.1 Definition. A system of first-order ordinary differential equations (in normal form) is given by

$$\dot{u}(t) = F(u(t), t)$$

where $\mathbf{u} : I \rightarrow \mathbb{R}^n$, I is an open interval in $\mathbb{R}$ and $\mathbf{F} : \mathbb{R}^n \times I \rightarrow \mathbb{R}^n$. If $\mathbf{F}$ does not explicitly depend on t , the system is called *autonomous*.

An *initial-value problem* (or *Cauchy problem*) is defined by

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathbf{F}(\mathbf{u}(t), t) \\ \mathbf{u}(t_0) = \mathbf{u}_0 \end{cases}$$

where $\mathbf{u}_0 \in \mathbb{R}^n$ and $t_0 \in I$. ★

Typically, $\mathbf{F}$ is assumed to be regular, for example continuously differentiable, so that we have (local) existence and uniqueness of the solution to the initial-value problem.

14.2 Example. Solutions may not be globally defined: consider the one-dimensional problem

$$\begin{cases} \dot{x} = x^2 \\ x(0) = x_0 > 0. \end{cases}$$

We have

$$\dot{x}x^{-2} = 1 \quad \Rightarrow \quad -\frac{1}{x} = t + c \quad \Rightarrow \quad x(t) = \frac{x_0}{1 - x_0 t},$$

which is defined only for $t < 1/x_0$ and cannot be further extended.

However, if the right-hand side is *sublinear* in $\mathbf{u}$, that is, if

$$|\mathbf{F}(\mathbf{u}, t)| \leq a + b|\mathbf{u}|$$

for some constants $a, b > 0$, then the solution can be globally defined on $I = \mathbb{R}$. ★

14.3 Remark. Any differential system of order k can be reduced to a first-order system. Indeed, consider the system

$$\frac{d^k \mathbf{u}}{dt^k}(t) = \mathbf{F}\left(\mathbf{u}(t), \frac{d\mathbf{u}}{dt}(t), \frac{d^2 \mathbf{u}}{dt^2}(t), \dots, \frac{d^{k-1} \mathbf{u}}{dt^{k-1}}(t), t\right)$$

where $\mathbf{F} : \mathbb{R}^{nk} \times I \rightarrow \mathbb{R}^n$. By introducing the new variable

$$\mathbb{R}^{nk} \ni \mathbf{v} = (\mathbf{u}, \mathbf{v}_2, \dots, \mathbf{v}_k)$$

and the first-order system

$$\dot{\mathbf{v}}(t) = (\mathbf{v}_2(t), \dots, \mathbf{v}_k(t), \mathbf{F}(\mathbf{v}(t), t))$$

we obtain

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathbf{v}_2(t) \\ \dot{\mathbf{v}}_2(t) = \mathbf{v}_3(t) \\ \dots \\ \dot{\mathbf{v}}_{k-1}(t) = \mathbf{v}_k(t) \\ \dot{\mathbf{v}}_k(t) = \mathbf{F}(\mathbf{v}(t), t). \end{cases}$$

The last equation corresponds to the original system. ★

The important case of the initial-value problem for a second-order system

$$\begin{cases} \ddot{\mathbf{u}}(t) = \mathbf{F}(\mathbf{u}(t), \dot{\mathbf{u}}(t), t) \\ \mathbf{u}(t_0) = \mathbf{u}_0 \\ \dot{\mathbf{u}}(t_0) = \mathbf{u}_1 \end{cases}$$

thus corresponds to the first-order problem

$$\begin{cases} \dot{\mathbf{u}}(t) = \mathbf{v}_2(t) \\ \dot{\mathbf{v}}_2(t) = \mathbf{F}(\mathbf{u}(t), \mathbf{v}_2(t), t) \\ \mathbf{u}(t_0) = \mathbf{u}_0 \\ \mathbf{v}_2(t_0) = \mathbf{u}_1. \end{cases}$$

15 Equilibrium and Stability

15.1 Definition. An *equilibrium solution* is a constant solution of the differential system $\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}, t)$, that is,

$$\mathbf{u}(t) = \bar{\mathbf{u}} \quad \text{for every } t \in I. \quad \star$$

15.2 Proposition. Given a differential system $\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}, t)$, a constant function $\mathbf{u}(t) = \bar{\mathbf{u}}$ is an *equilibrium solution* if and only if

$$\mathbf{F}(\bar{\mathbf{u}}, t) = 0 \quad \text{for every } t \in I.$$

Proof. Immediate. □

In particular, if the system is autonomous, the equilibrium solutions are the zeroes of $\mathbf{F}$.

15.3 Example. The constant functions $x = k\pi$, $k \in \mathbb{Z}$, are equilibrium solutions for the autonomous one-dimensional equation

$$\dot{x} = \sin x. \quad \star$$

15.4 Example. The two-dimensional system

$$\begin{cases} \dot{x} = xy - y^2 \\ \dot{y} = x^3 - y \end{cases}$$

has three equilibrium solutions: $(0, 0)$, $(1, 1)$, $(-1, -1)$. ★

15.5 Definition. An equilibrium solution $\bar{\mathbf{u}}$ of the autonomous system $\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u})$ is *stable* if

$$\forall \varepsilon > 0 \exists \delta > 0 : |\mathbf{u}_0 - \bar{\mathbf{u}}| < \delta \implies \forall t \geq 0 : |\mathbf{u}(t; \mathbf{u}_0) - \bar{\mathbf{u}}| < \varepsilon,$$

where $\mathbf{u}(t; \mathbf{u}_0)$ is the solution of the Cauchy problem

$$\begin{cases} \dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}) \\ \mathbf{u}(0) = \mathbf{u}_0. \end{cases}$$

The equilibrium is said to be *unstable* if it is not stable. ★

15.6 Example. It is easy to show that the origin is a stable equilibrium solution for the harmonic oscillator

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases} \quad \star$$

15.7 Definition. An equilibrium solution $\bar{u}$ is *asymptotically stable* if it is stable and

$$\forall u_0 : |u_0 - \bar{u}| < \delta \Rightarrow \lim_{t \rightarrow +\infty} u(t; 0, u_0) = \bar{u}.$$

It is *globally asymptotically stable* if it is stable and the previous limit holds for any $u_0 \in \mathbb{R}^n$.

An equilibrium solution $\bar{u}$ is *simply stable* if it is stable but not asymptotically stable. ★

Among asymptotically stable equilibrium solutions, there exists a special class where the rate of convergence is exponential (so that it is very fast):

15.8 Definition. An equilibrium solution $\bar{u}$ is said to be *exponentially stable* if there exist $\alpha, \lambda > 0$ such that for every u_0 , $|u_0 - \bar{u}| < \delta$:

$$|u(t; 0, u_0) - \bar{u}| \leq \alpha e^{-\lambda t} |u_0 - \bar{u}|.$$

It is said to be *globally exponentially stable* (g.e.s.) if the above property holds for every $u_0 \in \mathbb{R}^n$. ★

15.9 Example. The equilibrium $(0, 0)$

$$\begin{cases} \dot{x} = y - x \\ \dot{y} = -x - y \end{cases}$$

is g.e.s. ★

15.10 Example. All the equilibrium solutions of the *linear damper*

$$\begin{cases} \dot{x} = y \\ \dot{y} = -y \end{cases}$$

are simply stable. ★

16 Higher-order Linear One-dimensional ODEs

A k^{th} -order linear 1D homogeneous ODE can be written in the form

$$\frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + a_2 \frac{d^{k-2} x}{dt^{k-2}} + \dots + a_{k-1} \frac{dx}{dt} + a_k x = 0. \quad (24)$$

There is an explicit formula for its solutions. First of all, replace every $\frac{d^j x}{dt^j}$ with the polynomial term λ^j , obtaining the *characteristic polynomial* associated with the equation:

$$p(\lambda) = \lambda^k + a_1 \lambda^{k-1} + a_2 \lambda^{k-2} + \dots + a_k.$$

It can be proved that this polynomial corresponds to the characteristic polynomial of the matrix of order k

$$A = \begin{bmatrix} -a_1 & -a_2 & -a_3 & \dots & -a_{k-1} \\ 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \quad (25)$$

which gives rise to the linear system of ODEs

$$\dot{\mathbf{u}} = \mathbf{A}\mathbf{u}$$

equivalent to (24).

Now we have to compute the roots of $p(\lambda)$, paying attention to the fact that the roots can be complex-valued and can have multiplicity. If we have real solutions, we distinguish two cases:

- if all the roots $\lambda_1, \dots, \lambda_{k-1}$ are distinct real numbers (that is, the multiplicity of every root is 1), then the general solution of (24) is given by

$$x(t) = c_1 e^{\lambda_1 t} + \dots + c_k e^{\lambda_k t} = \sum_{j=1}^k c_j e^{\lambda_j t} \quad (26)$$

where the constants c_j have to be determined by the initial conditions

$$x(0), \frac{dx}{dt}(0), \dots, \frac{d^{k-1}x}{dt^{k-1}}(0);$$

- if some root λ_j has multiplicity $m \geq 2$, we have to replace the term $c_j e^{\lambda_j t}$ with the expression

$$e^{\lambda_j t} (c_{j,0} + c_{j,1}t + c_{j,2}t^2 + \dots + c_{j,m-1}t^{m-1}). \quad (27)$$

In any case, the solution is given by a linear combination of exactly k terms involving the roots of the characteristic polynomial associated with the equation.

If some root λ_j is complex-valued, that is $\lambda_j = \alpha + i\beta$ for $\beta \neq 0$, then also the conjugate $\bar{\lambda}_j = \alpha - i\beta$ is a root of the characteristic polynomial, and it is possible to combine the two roots and the constants $c_j, \widehat{c}_j$ to get real quantities. One can write

$$\begin{aligned} & c_j e^{(\alpha+i\beta)t} + \widehat{c}_j e^{(\alpha-i\beta)t} \\ &= e^{\alpha t} [(c_j + \widehat{c}_j) \cos(\beta t) + i(c_j - \widehat{c}_j) \sin(\beta t)] \\ &= e^{\alpha t} [c_{j,0} \cos(\beta t) + c_{j,1} \sin(\beta t)]. \end{aligned} \quad (28)$$

Finally, if complex roots have higher multiplicity, one has to multiply the last terms by suitable polynomials of t , as in the real case.

If the ODE is non-homogeneous, it has the form

$$\frac{d^k x}{dt^k} + a_1 \frac{d^{k-1} x}{dt^{k-1}} + a_2 \frac{d^{k-2} x}{dt^{k-2}} + \dots + a_{k-1} \frac{dx}{dt} + a_k x = b$$

and the equilibrium solution writes $\bar{x} = b/a_k$, which exists for $a_k \neq 0$. In this case, the solution is of the form

$$x(t) = \bar{x} + y(t),$$

where $y(t)$ is any solution of the *homogeneous part* of the ODE, that is, the ODE with $b = 0$.

16.1 Stability of the Equilibrium Solution

The stability of the equilibrium solution $\bar{x} = 0$ of a higher-order linear 1D homogeneous ODE depends on the sign of the real part of the roots of the characteristic polynomial. Namely:

- (1) if $\operatorname{Re} \lambda_i < 0$ for every i , then 0 is globally exponentially stable
- (2) if $\operatorname{Re} \lambda_j > 0$ for some j , then 0 is unstable
- (3) if $\operatorname{Re} \lambda_i \leq 0$ for every i but we are not in the first case, then: if all the roots λ_j with $\operatorname{Re} \lambda_j = 0$ are *simple*, that is, the multiplicity is 1, then 0 is simply stable, otherwise it is unstable.

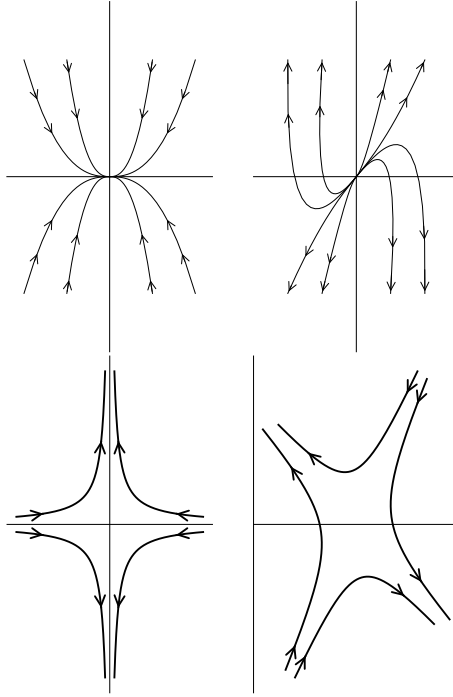


Figure 7: Nodes and saddles.

17 Two-dimensional Linear Systems

Suppose that A is a matrix of order 2 such that $\det A \neq 0$. If the eigenvalues of A are real and distinct, the solutions of the system $\dot{\mathbf{u}} = A\mathbf{u}$ are of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = P \begin{bmatrix} e^{\lambda t} & 0 \\ 0 & e^{\mu t} \end{bmatrix} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

In particular:

- if $\lambda, \mu < 0$ we speak of a *stable node*;
- if $\lambda, \mu > 0$ we speak of an *unstable node*;
- if $\lambda > 0, \mu < 0$ we speak of a *saddle*.

In Figure 7 we see a representation of the phase portrait in these cases.

If there is only an eigenvalue with multiplicity 2, we still speak of a node, stable or unstable depending on the sign of the eigenvalue.

Finally, if the eigenvalues are complex numbers $\lambda = a \pm ib$, the solution is of the form

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = P e^{at} \begin{bmatrix} \cos bt & -\sin bt \\ \sin bt & \cos bt \end{bmatrix} P^{-1} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

If $a \neq 0$ we speak of a *focus*, stable if $a < 0$ and unstable if $a > 0$, while if $a = 0$ we speak of a *center*. In Figure 8 we see a representation of the phase portrait in these cases.

If A is singular, and therefore an eigenvalue is zero, then we have:

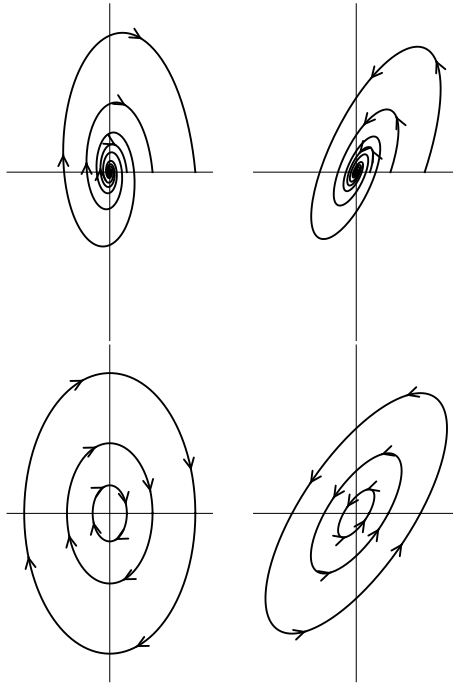


Figure 8: Foci and centers.

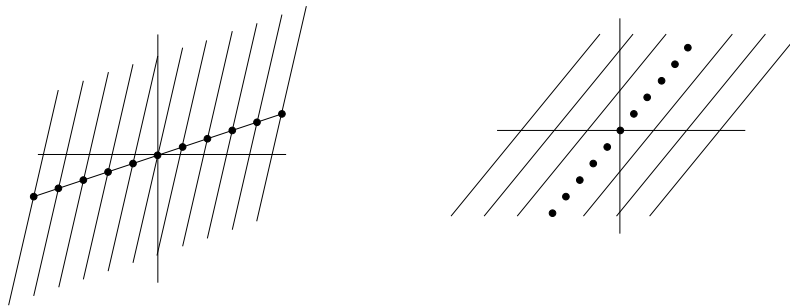


Figure 9: On the left the case in which only one eigenvalue is zero, on the right the case in which both are zero (but the matrix is not null).

- if the other eigenvalue is nonzero all orbits are half-lines, except for a line of equilibrium points;
- if the other eigenvalue is also zero and the matrix is not diagonalizable, then all orbits are parallel lines, except for a line of equilibrium points;
- if the matrix is null, then all orbits are points (indifferent equilibrium).

The following theorem, which summarizes what has been said so far, is easily proved. See also Figure 10.

17.1 Theorem (tau-delta). *Let A be a 2×2 matrix, with trace τ and determinant δ . Consider the differential system $\dot{u} = Au$. Then we have:*

- (1) *if $\delta = 0$, the system is degenerate and the orbits are all half-lines, lines or points;*
- (2) *if $\delta < 0$ we have a saddle;*

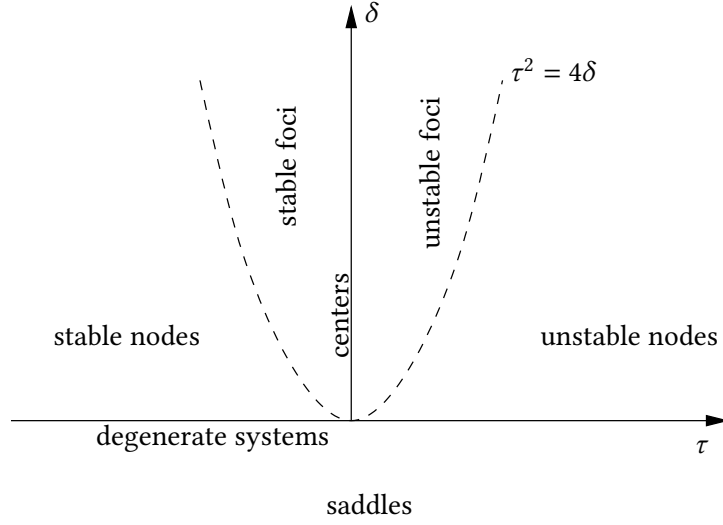


Figure 10: The $\tau - \delta$ plane.

- (3) if $\delta > 0$ and $\tau^2 \geq 4\delta$ we have a node, stable if $\tau < 0$ and unstable if $\tau > 0$;
- (4) if $\delta > 0$ and $\tau^2 < 4\delta$ we have a focus, stable if $\tau < 0$ and unstable if $\tau > 0$;
- (5) if $\delta > 0$ and $\tau = 0$ we have a center.

18 Stability of Linear Systems

The stability of equilibrium points of linear systems is completely determined by the eigenvalues of the matrix A :

18.1 Theorem. Let A be the $n \times n$ matrix of a linear differential system, and let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, $k \leq n$, be its eigenvalues.

- (1) If $\operatorname{Re} \lambda_i < 0$ for every i , then $\mathbf{0}$ is globally exponentially stable.
- (2) If $\operatorname{Re} \lambda_j > 0$ for some j , then $\mathbf{0}$ is unstable.
- (3) If $\operatorname{Re} \lambda_i \leq 0$ for every i but we are not in the first case, then:
 - (a) if for all eigenvalues λ_j with $\operatorname{Re} \lambda_j = 0$, the algebraic and geometric multiplicities coincide, then $\mathbf{0}$ is simply stable;
 - (b) if for at least one eigenvalue λ_j with $\operatorname{Re} \lambda_j = 0$ the multiplicities do not coincide, then $\mathbf{0}$ is unstable.
- (4) If there are infinitely many equilibrium points, they all have the same stability as $\mathbf{0}$ (in this case, $\lambda = 0$ is an eigenvalue, hence we cannot be in the first case).

Proof. The solution is the sum of terms of the form

$$t^k e^{\lambda t} \quad \text{if } \lambda \in \mathbb{R} \quad \text{or} \quad t^k e^{\alpha t} (\cos \beta t + i \sin \beta t) \quad \text{if } \lambda = \alpha \pm i\beta \in \mathbb{C}$$

with $k = 1, \dots, m_a - m_g$ (see Section 16). Thus, the first three cases easily follow.

In the case of multiple equilibrium solutions, let $\bar{\mathbf{u}}$ be another equilibrium solution, and perform the change of variables $\mathbf{v}(t) = \mathbf{u}(t) - \bar{\mathbf{u}}$. Then $\mathbf{v}$ is a solution if and only if $\mathbf{u}$ is a solution; indeed,

$$A\mathbf{v}(t) = A(\mathbf{u}(t) - \bar{\mathbf{u}}) = A\mathbf{u}(t) = \dot{\mathbf{u}}(t) = \dot{\mathbf{v}}(t),$$

so the equilibrium solution $\bar{\mathbf{u}}$ corresponds to the equilibrium solution $\bar{\mathbf{v}} = \mathbf{0}$ of the same differential system, and thus has the same stability. $\square$

18.2 Example. The zero position of the system

$$\begin{cases} \dot{x} = x + z \\ \dot{y} = -y \\ \dot{z} = y + z \end{cases}$$

is unstable, since the eigenvalues of the matrix are $\lambda_1 = -1$, $\lambda_{2,3} = 1$. $\star$

18.3 Example. All positions of the system

$$\begin{cases} \dot{x} = x + 3y + z \\ \dot{y} = -2y + z \\ \dot{z} = -x - y - 2z \end{cases}$$

are simply stable, since the eigenvalues of the matrix are $\lambda_1 = -2$, $\lambda_2 = -1$, $\lambda_3 = 0$. $\star$

18.4 Example. The stability of the equilibrium position $(0, 0)$ of the system

$$\begin{cases} \dot{x} = y \\ \dot{y} = -\omega^2 x - 2py, \end{cases}$$

for $\omega \neq 0$ and $p \in \mathbb{R}$, is the following:

- for $p > 0$, g.e.s. (*damped* harmonic oscillator): a stable node for $p^2 \geq \omega^2$, a stable focus otherwise
- for $p < 0$, unstable (*forced* harmonic oscillator): an unstable node for $p^2 \geq \omega^2$, an unstable focus otherwise
- for $p = 0$ (and $\omega \neq 0$), simply stable (harmonic oscillator): a center. $\star$

For the study of the stability in linear systems, it is important to know if the characteristic polynomial of the system matrix has all roots with strictly negative real parts. Such polynomials are called *stable*. The following property is easily verified:

18.5 Proposition. *If a polynomial is stable, then all its coefficients have the same sign.*

Note that the previous proposition provides only a necessary condition for stability. However, if $n = 2$ this condition is also sufficient (Descartes' rule).

A complete criterion for verifying if a polynomial is stable is the following:

18.6 Theorem (Routh-Hurwitz Criterion). *A polynomial*

$$a_0\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n, \quad a_0 > 0$$

is stable if and only if all the leading principal minors⁽²⁾ of the $n \times n$ Hurwitz matrix

$$H_{ij} = a_{2j-i}, \quad a_k = 0 \text{ for } k < 0 \text{ or } k > n,$$

are strictly positive.

For example, the polynomial $\lambda^3 + 3\lambda^2 + \lambda + 4$ has the Hurwitz matrix

$$\begin{bmatrix} 3 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 3 & 4 \end{bmatrix}$$

and the leading principal minors are 3, -1 , -4 , hence the polynomial is not stable.

18.7 Remark (Liénard-Chipart Criterion). There is a slightly simpler version of the previous criterion, where only certain minors need to be calculated. Indeed, it can be shown that, if $a_0 > 0$, the polynomial is stable if and only if

- (1) $a_i > 0$ for every $i = 1, \dots, n$;
- (2) $\Delta_{n-1} > 0$, $\Delta_{n-3} > 0, \dots$, where Δ_k is the leading principal minor of order k of the Hurwitz matrix.

Thus, for example, if all coefficients are positive, it suffices to verify:

- in the case $n = 3$ that $\Delta_2 > 0$;
- in the case $n = 4$ that $\Delta_3 > 0$;
- in the case $n = 5$ that $\Delta_4 > 0$, $\Delta_2 > 0$;
- in the case $n = 6$ that $\Delta_5 > 0$, $\Delta_3 > 0$.

See [Gandolfo, Ch. 6].

★

19 Phillips' Stabilization Model for a Closed Economy

We now study a model due to A.W. Phillips (1954), see [Gandolfo, Chapters 5 and 7]. In a closed economy, let $Y(t)$ denote the deviation from a desired level of the national output, and $D(t)$ denote the deviation from a desired level of the aggregate demand, so that $Y, D \in \mathbb{R}$.

Assuming that

$$\dot{Y} = \alpha(D - Y), \quad \alpha > 0 \text{ a reaction coefficient} \quad (29)$$

and

$$D = (1 - \ell)Y, \quad 0 < \ell < 1 \text{ the marginal propensity to save,} \quad (30)$$

⁽²⁾A leading principal minor of order k of a matrix is the determinant of the matrix obtained by removing the last $n - k$ rows and $n - k$ columns, for $k = 1, \dots, n$.

we get the linear one-dimensional ODE

$$\dot{Y} = -\alpha\ell Y \quad \Rightarrow \quad Y(t) = Y_0 e^{-\alpha\ell t}$$

so that the equilibrium $\bar{Y} = 0$ is g.e.s.

Suppose now that an exogenous disturbance $u > 0$ lowers the aggregate demand, so that (30) changes in

$$D = (1 - \ell)Y - u.$$

Then we obtain the non-homogeneous ODE

$$\dot{Y} = -\alpha\ell Y - \alpha u,$$

which can be solved as in Section 16:

$$\bar{Y} = -\frac{u}{\ell}, \quad Y(t) = -\frac{u}{\ell} + ce^{-\alpha\ell t}.$$

Imposing that $Y(0) = Y_0$, one has $c = Y_0 + u/\ell$, hence

$$Y(t) = -\frac{u}{\ell} + \left(Y_0 + \frac{u}{\ell}\right) e^{-\alpha\ell t}.$$

Then, the external disturbance provokes a lowering of the equilibrium: the new position is $\bar{Y} = -u/\ell$ and it is g.e.s.

In order to recover the desired equilibrium value $\bar{Y} = 0$, let us assume that the government artificially increases the aggregate demand by a term G :

$$D = (1 - \ell)Y - u + G,$$

where G , the *actual* government expenditure, evolves in time according to the equation

$$\dot{G} = \beta(G^* - G), \quad \beta > 0 \text{ a reaction coefficient,}$$

where G^* is the *theoretical* government expenditure. Then we get the two-dimensional linear system of ODEs

$$\begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = -\beta G + \beta G^*. \end{cases} \quad (31)$$

Phillips proposed three possible choices for G^* , that is, three types of stabilization policies.

- (1) *Proportional* stabilization policy: the theoretical government expenditure is chosen proportional (and opposite) to Y , the deviation of the output, that is,

$$G^* = -f_p Y, \quad f_p > 0.$$

- (2) *Derivative* stabilization policy: the theoretical government expenditure is chosen proportional (and opposite) to the variation of Y , that is,

$$G^* = -f_d \dot{Y}, \quad f_d > 0.$$

- (3) *Integral* stabilization policy: the theoretical government expenditure is chosen proportional (and opposite) to the integral of all the deviations of the output, that is,

$$G^* = -f_i \int_0^t Y(\tau) d\tau, \quad f_i > 0.$$

We now study the effects of the three policies.

(1) With the choice $G^* = -f_p Y$, eq. (31) becomes

$$\begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = -\beta G - \beta f_p Y \end{cases}$$

which is a two-dimensional linear non-homogeneous system of ODEs. The equilibrium is given by

$$\begin{cases} -\alpha\ell Y + \alpha G - \alpha u = 0 \\ -\beta G - \beta f_p Y = 0 \end{cases} \Rightarrow \begin{cases} \bar{Y} = -\frac{u}{\ell + f_p} \\ \bar{G} = \frac{f_p}{\ell + f_p}. \end{cases}$$

Since the matrix of the system is

$$A = \begin{bmatrix} -\alpha\ell & \alpha \\ -\beta f_p & -\beta \end{bmatrix}$$

and $\tau < 0$, $\delta > 0$, we have that $(\bar{Y}, \bar{G})$ is g.e.s. Moreover

$$\Delta = \tau^2 - 4\delta = (\alpha\ell + \beta)^2 - 4\alpha\beta(\ell + f_p) = (\alpha\ell - \beta)^2 - 4\alpha\beta f_p, \quad (32)$$

hence the equilibrium is

- a stable node for $f_p \leq (\alpha\ell - \beta)^2 / 4\alpha\beta$
- a stable focus for $f_p > (\alpha\ell - \beta)^2 / 4\alpha\beta$.

If we compare the new equilibrium value for Y with the old value $-u/\ell$, we have that

$$-\frac{u}{\ell + f_p} > -\frac{u}{\ell},$$

therefore the proportional policy has the effect of reducing the decrease due to the exogenous factor; moreover, the higher f_p , the higher the reduction. However, increasing f_p leads to oscillations, and in any case the desired value $\bar{Y} = 0$ cannot be recovered.

(2) More generally, we study a mixed proportional-derivative policy:

$$G^* = -f_p Y - f_d \dot{Y}.$$

Then, eq. (31) becomes

$$\begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = -\beta G - \beta f_p Y - f_d(-\alpha\ell Y + \alpha G - \alpha u) \end{cases} \Rightarrow \begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = -\beta(f_p - \alpha\ell f_d)Y - \beta(1 + \alpha f_d)G + \alpha\beta u f_d, \end{cases}$$

which is again a two-dimensional linear non-homogeneous system of ODEs. The equilibrium is given by

$$\begin{cases} -\alpha\ell Y + \alpha G - \alpha u = 0 \\ -\beta(f_p - \alpha\ell f_d)Y - \beta(1 + \alpha f_d)G + \alpha\beta u f_d = 0 \end{cases} \Rightarrow \begin{cases} \bar{Y} = -\frac{u}{\ell + f_p} \\ \bar{G} = \frac{u f_p}{\ell + f_p}. \end{cases}$$

In particular, the equilibrium is independent of f_d and the value of $\bar{Y}$ is the same as in case (1). The matrix of the system is

$$A = \begin{bmatrix} -\alpha\ell & \alpha \\ -\beta(f_p - \alpha\ell f_d) & -\beta(1 + \alpha f_d) \end{bmatrix}$$

and, after some computations, $\tau < 0$, $\delta > 0$. Then $(\bar{Y}, \bar{G})$ is again g.e.s. Finally,

$$\Delta = \tau^2 - 4\delta = (\alpha\ell + \beta + \alpha\beta f_d)^2 - 4\alpha\beta(\ell + f_p).$$

Comparing the last equation with (32), one can conclude that it is more likely that $\Delta \geq 0$, hence adding a derivative policy to the proportional one prevents oscillations (and damps them). However, also in this case the desired value $\bar{Y} = 0$ cannot be recovered.

(3) For the integral policy we have the choice

$$G^* = -f_i \int_0^t Y(\tau) d\tau,$$

hence eq. (31) becomes

$$\begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = -\beta G - \beta f_i \int_0^t Y(\tau) d\tau. \end{cases}$$

The second is an integro-differential equation, which can be reduced to a standard ODE by increasing the order of differentiation:

$$\ddot{G} = -\beta\dot{G} - \beta f_i Y.$$

Introducing a new variable $Z = \dot{G}$, we obtain

$$\begin{cases} \dot{Y} = -\alpha\ell Y + \alpha G - \alpha u \\ \dot{G} = Z \\ \dot{Z} = -\beta Z - \beta f_i Y, \end{cases}$$

which is a three-dimensional linear non-homogeneous system of ODEs. The equilibrium is given by

$$\begin{cases} -\alpha\ell Y + \alpha G - \alpha u = 0 \\ Z = 0 \\ -\beta Z - \beta f_i Y = 0 \end{cases} \Rightarrow \begin{cases} \bar{Y} = 0 \\ \bar{Z} = 0 \\ \bar{G} = u. \end{cases}$$

At last, the desired value for $\bar{Y}$ has been recovered. Let us now study the stability: the matrix of the system is

$$A = \begin{bmatrix} -\alpha\ell & \alpha & 0 \\ 0 & 0 & 1 \\ -\beta f_i & 0 & -\beta \end{bmatrix}$$

and the characteristic polynomial writes

$$p(\lambda) = \lambda^3 + (\alpha\ell + \beta)\lambda^2 + \alpha\beta\ell\lambda + \alpha\beta f_i.$$

Notice that the coefficients of $p(\lambda)$ are all strictly positive, so that the necessary condition of Proposition 18.5 is satisfied. The Hurwitz matrix of $p(\lambda)$ is

$$\begin{bmatrix} \alpha\ell + \beta & \alpha\beta f_i & 0 \\ 1 & \alpha\beta\ell & 0 \\ 0 & \alpha\ell + \beta & \alpha\beta f_i \end{bmatrix}$$

and the 2×2 top-left minor is positive if

$$(\alpha\ell + \beta)\alpha\beta\ell > \alpha\beta f_i \Rightarrow f_i < \ell(\alpha\ell + \beta).$$

Hence, the integral policy can eliminate the disturbance u if it is small enough, otherwise it can provoke instability.

20 The Linearization Method for ODEs

As in the case of DDS, under some conditions we can use the Jacobian of the system of ODEs to study the stability of the equilibria.

20.1 Theorem (Linearization theorem). *Let $\bar{\mathbf{u}}$ be an equilibrium point for the autonomous system of ODEs*

$$\dot{\mathbf{u}} = \mathbf{F}(\mathbf{u}).$$

Let $\mathbf{A} = \nabla \mathbf{F}(\bar{\mathbf{u}})$ be the Jacobian matrix in the equilibrium, and let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, $k \leq n$, be its eigenvalues:

- (1) *if $\operatorname{Re} \lambda_i < 0$ for every i , then $\bar{\mathbf{u}}$ is exponentially stable;*
- (2) *if $\operatorname{Re} \lambda_j > 0$ for some j , then $\bar{\mathbf{u}}$ is unstable.*

Proof. We give an idea of the proof for the first point. Apply Taylor expansion to the function $\mathbf{F}$ around $\bar{\mathbf{u}}$:

$$\mathbf{F}(\mathbf{u}) = \mathbf{F}(\bar{\mathbf{u}}) + \nabla(\mathbf{F})(\bar{\mathbf{u}})(\mathbf{u} - \bar{\mathbf{u}}) + \mathbf{R}(\mathbf{u} - \bar{\mathbf{u}}).$$

Since $\bar{\mathbf{u}}$ is an equilibrium, then the expansion writes

$$\mathbf{F}(\mathbf{u}) = \mathbf{A}(\mathbf{u} - \bar{\mathbf{u}}) + \mathbf{R}(\mathbf{u} - \bar{\mathbf{u}})$$

Therefore, by setting $\mathbf{v}(t) = \mathbf{u}(t) - \bar{\mathbf{u}}$, we obtain the differential system

$$\dot{\mathbf{v}}(t) = \mathbf{A}\mathbf{v}(t) + \mathbf{R}(\mathbf{v}) \tag{33}$$

which has $\mathbf{0}$ as an equilibrium position. Clearly, the stability of $\bar{\mathbf{u}}$ in the initial system corresponds to the stability of $\mathbf{0}$ in (33).

Now suppose that all eigenvalues of $\mathbf{A}$ have strictly negative real parts; then it can be proved that there exists an inner product on $\mathbb{R}^n$ and $c > 0$ for which

$$\mathbf{v} \cdot \mathbf{A}\mathbf{v} \leq -c\mathbf{v} \cdot \mathbf{v} =: -c\|\mathbf{v}\|^2$$

for every $\mathbf{v} \in \mathbb{R}^n$.⁽³⁾

Let U be a neighborhood of $\mathbf{0}$ such that $\|\mathbf{R}(\mathbf{v})\| \leq \frac{c}{2}\|\mathbf{v}\|^2$ for every $\mathbf{v} \in U$. Now, for every $\mathbf{v}_0 \in U$, let us study how the distance from the equilibrium $\mathbf{0}$ of $\mathbf{v}(t)$, the solution of (33) starting from $\mathbf{v}_0$, varies over time:

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|^2 = \mathbf{v}(t) \cdot \dot{\mathbf{v}}(t) = \mathbf{v}(t) \cdot (\mathbf{A}\mathbf{v}(t) + \mathbf{R}(\mathbf{v}(t))) \leq -c\|\mathbf{v}(t)\|^2 + \|\mathbf{R}(\mathbf{v}(t))\|\|\mathbf{v}(t)\|.$$

As long as $\mathbf{v}(t) \in U$, we have that $\|\mathbf{R}(\mathbf{v}(t))\| \leq \frac{c}{2}\|\mathbf{v}(t)\|$, thus

$$\frac{1}{2} \frac{d}{dt} \|\mathbf{v}(t)\|^2 \leq -\frac{c}{2} \|\mathbf{v}(t)\|^2,$$

from which

$$\|\mathbf{v}(t)\|^2 \leq \|\mathbf{v}_0\|^2 e^{-ct}.$$

Thus, we conclude that $\mathbf{v}(t)$ cannot leave U , and there is exponential stability. $\square$

⁽³⁾The inequality is easily proved by diagonalization in the case of $\mathbf{A}$ symmetric with the canonical inner product, while the general case is more complicated.

Since in the linearization method it is necessary to determine the sign of the real part of the eigenvalues, the Routh-Hurwitz criterion will be useful also in this case.

20.2 Definition. Let $\bar{u}$ be an equilibrium point for the autonomous system

$$\dot{u} = F(u)$$

and $A = \nabla F(\bar{u})$. Let $\lambda_1, \dots, \lambda_k \in \mathbb{C}$, $k \leq n$, be its eigenvalues. If $\operatorname{Re} \lambda_i \neq 0$ for every $i = 1, \dots, k$, then $\bar{u}$ is an *hyperbolic equilibrium point*. ★

The previous theorem states that the hyperbolic equilibrium points of a nonlinear system have the same stability as those of a linear one; that is, they are either exponentially stable or unstable. In fact, there is an important result, the *Hartman-Grobman Theorem*, which states that the two systems, nonlinear and linearized, are (topologically) equivalent in a neighborhood of the equilibrium point. Thus, in particular, in the two-dimensional case, hyperbolic equilibrium points locally behave like foci, nodes, or saddles.

21 Isocline Method

In this brief section, we explore a graphical method for qualitatively determining the shape of the orbits of a two-dimensional differential system. Consider the system

$$\begin{cases} \dot{x} = f(x, y) \\ \dot{y} = g(x, y); \end{cases}$$

then the slope ϑ of an orbit with respect to the horizontal is given by

$$\tan \vartheta = \frac{g(x, y)}{f(x, y)}.$$

For a fixed ϑ , the previous curve is called the *isocline of angle ϑ* . Drawing several isoclines can help to understand the shape of certain orbits without needing to solve the system. Two particularly important isoclines are the *nullclines*

$g(x, y) = 0$: the set of points where the orbits have a horizontal tangent
 $f(x, y) = 0$: the set of points where the orbits have a vertical tangent.

The signs of f and g indicate the signs of $\dot{x}$ and $\dot{y}$, and thus determine the orientation of the tangent vectors and the orbits. For example, if $f(x, y) = g(x, y) > 0$, then the tangent is parallel to the bisector of the first and third quadrants and oriented toward the northeast; on the other hand, if $f(x, y) = g(x, y) < 0$, the tangent is still parallel to the bisector of the first and third quadrants, but now oriented toward the southwest. Here is a table with the orientations:

	$f > 0$	$f < 0$
$g > 0$		
$g < 0$		

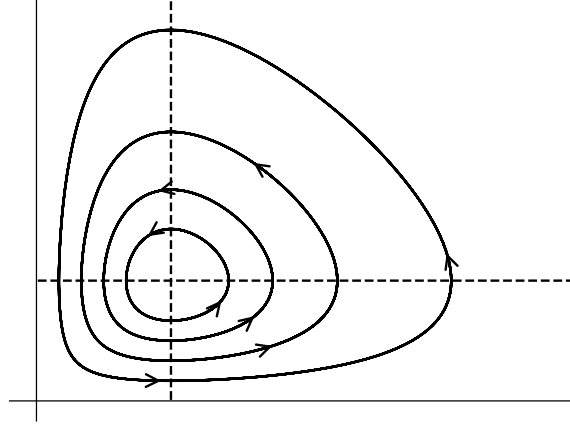


Figure 11: Orbits of the Lotka-Volterra system.

22 A Lotka-Volterra Model for the Price/Supply Interaction

Let us analyze an economic model describing the evolution in time of the market price $P(t) \geq 0$ of a good and the amount $S(t) \geq 0$ of that good which is supplied. Then we can have a model such as

$$\begin{cases} \frac{\dot{P}}{P} = f(P, S) \\ \frac{\dot{S}}{S} = g(P, S) \end{cases}$$

where f is a decreasing function of S , while g is an increasing function of P .

If we make the linear choice $f(P, S) = \alpha - \beta S$ and $g(P, S) = \gamma P - \delta$, with $\alpha, \beta, \gamma, \delta > 0$, we obtain the renowned *Lotka-Volterra system*:

$$\begin{cases} \dot{P} = P(\alpha - \beta S) \\ \dot{S} = S(\gamma P - \delta). \end{cases}$$

This system has equilibrium points at $(0, 0)$ and $(\delta/\gamma, \alpha/\beta)$, both economically feasible. Proceeding via linearization, the null position is unstable (a saddle point), whereas for the other we cannot make a determination, as the eigenvalues are purely imaginary.

For this equilibrium point, however, we can construct a *conserved quantity*:

$$W(P, S) = \gamma P - \delta \ln P + \beta S - \alpha \ln S.$$

Indeed, this function is defined over the entire first quadrant and has a strict minimum at the equilibrium position. Moreover,

$$\frac{d}{dt} W(P(t), S(t)) = \nabla W \cdot F = P \left(\gamma - \frac{\delta}{P} \right) (\alpha - \beta S) + S \left(\beta - \frac{\alpha}{S} \right) (\gamma P - \delta) = 0,$$

hence W is constant on the solutions of the system.

Thus, the position $(\delta/\gamma, \alpha/\beta)$ is simply stable, and the trajectories lie on the level curves of the function W , forming a center.

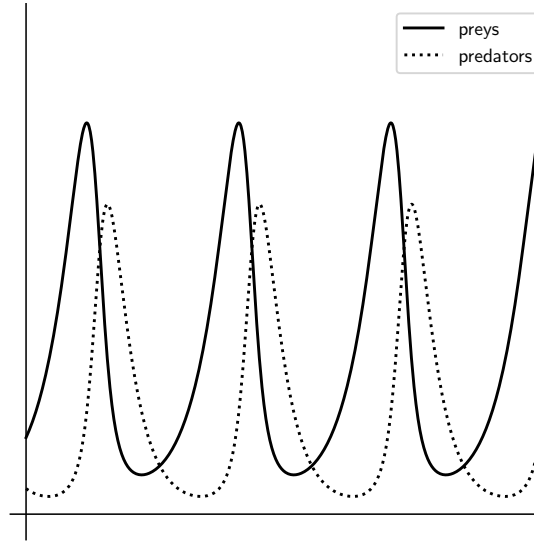


Figure 12: Solutions of a Lotka-Volterra system.

22.1 Average Price and Supply

We have seen that the solutions of the system are periodic. Although the period T depends on the initial conditions and is not easy to calculate,⁽⁴⁾ we can calculate the average price and the average supply. Let us consider the first equation and integrate it from 0 to T :

$$\int_0^T \frac{\dot{P}(t)}{P(t)} dt = \int_0^T (\alpha - \beta y(t)) dt \quad \Rightarrow \quad \ln \frac{P(T)}{P(0)} = \alpha T - \beta \int_0^T S(t) dt.$$

Since $P(0) = P(T)$, the first term vanishes, and we find:

$$\frac{1}{T} \int_0^T S(t) dt = \frac{\alpha}{\beta}.$$

Thus, the average supply over a period coincides with the supply at the equilibrium position. The same holds true for the price.

This means that, regardless of the initial conditions, all solutions of the Lotka-Volterra system have the same averages.

22.2 Gomatam Price/Supply Model

Among models with nonlinear growth rates, the *Gomatam model* (1974) is noteworthy. It is very similar to the Lotka-Volterra model but assumes logarithmic growth rates:

$$\begin{cases} \dot{P} = P(\alpha - \beta \ln S) \\ \dot{S} = S(\gamma \ln P - \delta). \end{cases}$$

Although it appears more complex, the change of variables

$$\xi = \ln P, \quad \eta = \ln S$$

allows it to be transformed into a linear system, making it fully solvable.

⁽⁴⁾By linearizing the system, it is easy to see that for initial conditions near the stable equilibrium point, the trajectories have a period $T = 2\pi/\sqrt{\alpha\delta}$. For initial conditions further away, however, the period increases.

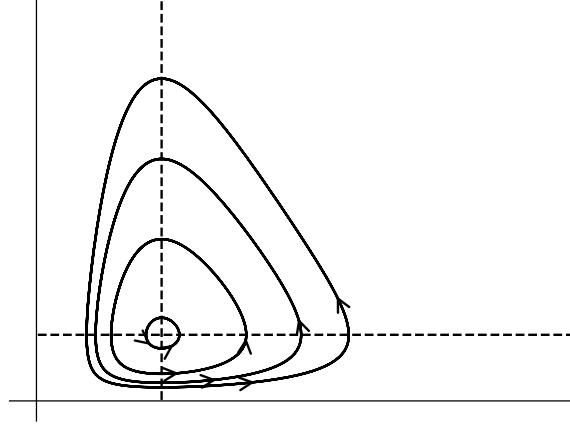


Figure 13: Orbits of the Gomati system.

23 A Model for Asset Prices

We want to describe a market where we can exchange a single asset. Let us make the following assumptions (*efficient market*):

- past history is fully reflected in present price, and nothing else;
- the market responds immediately to new information.

It means that the model which describes the changes in time in an asset price is a *Markov process*.

Let us denote with $S(t)$ the price of the asset at time t . In the (infinitesimal) increment of time dt the price will change of an amount dS . Let us define the *return* of the asset price as dS/S , that is, the change in price divided by its original value.

We want to model the return as a linear process which can be decomposed in two parts:

- (1) a predictable, deterministic, risk-free part

$$\mu dt$$

where μ , the *drift*, is the average rate of growth, and it is assumed constant for simplicity;

- (2) a random part

$$\sigma dX$$

where σ , the *volatility*, is again assumed constant. The quantity dX is a *Wiener process*, that is, a random variable, normally distributed, with zero mean and variance dt . Then,

$$dX \sim \mathcal{N}(0, dt) \sim \sqrt{dt} \mathcal{N}(0, 1).$$

Here Φ denotes the standard Gaussian (or normal) random variable, which has the p.d.f. given by

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2},$$

so that

$$\mu = E[xf(x)] = 0, \quad \sigma^2 = E[x^2 f(x)] = 1.$$

Hence we have the important scaling

$$dX \sim \sqrt{dt} \quad \text{for } dt \rightarrow 0.$$

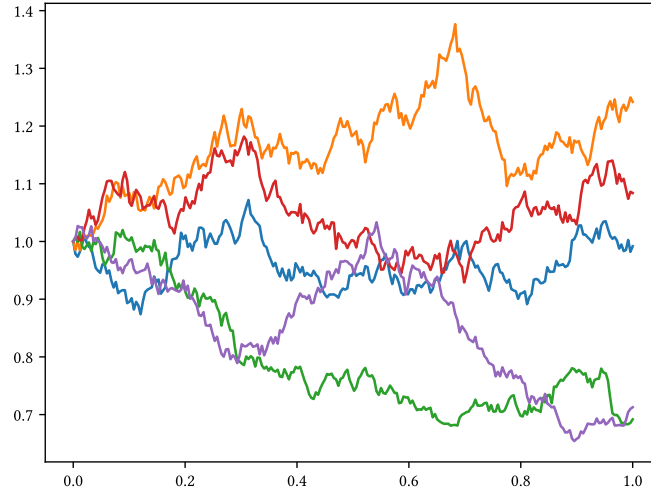


Figure 14: Five realizations of an asset price with the model (34) for $\mu = 0.1$, $\sigma = 0.2$ and $S(0) = 1$.

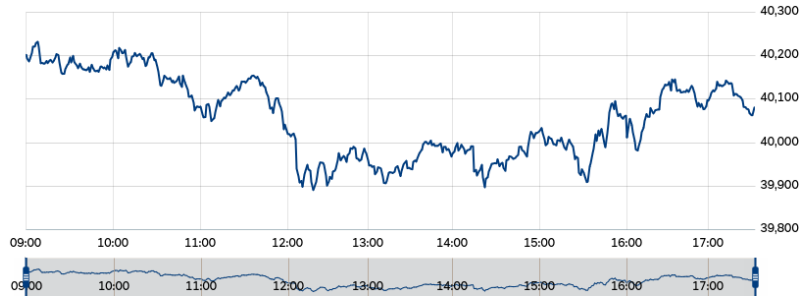


Figure 15: Real data of the FTSE MIB (Milan Stock Exchange), June 4th, 2025 (IlSole24Ore)

We obtain the *stochastic differential equation*

$$dS = \mu S dt + \sigma S dX, \quad (34)$$

which is an example of *random walk* and gives the behavior of S in a probabilistic setting. This model fits real time-series data very well for equities and indices, while it is not so good for currencies. In Figure 14 a numerical simulation of five realizations of the solution of (34) is represented.

23.1 Remark. The stochastic differential equation (34) can be easily discretized, generating a time series (that is, a discrete dynamical system) for the price:

$$S_{h+1} = S_h + dS = S_h(1 + \mu dt + \sigma dX),$$

where dX is a random number drawn from a normal distribution with zero mean and variance dt .

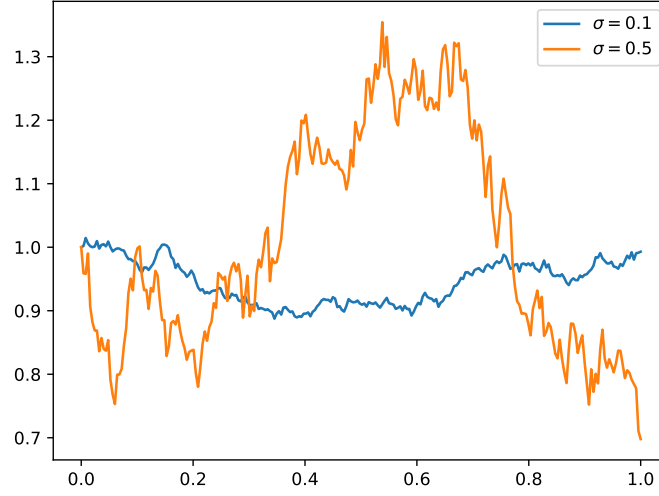


Figure 16: Two realizations of (34) for $\mu = 0.1$ and $S(0) = 1$.

For instance, choosing $\mu = 0.1$, $\sigma = 0.2$, $dt = 1/250$, $S_0 = 1$, the first few values of S are

$$\begin{aligned} dX &= 1.0086 & S_1 &= 1.0086 \\ dX &= 1.0029 & S_2 &= 1.0115 \\ dX &= 0.9945 & S_3 &= 1.0060 \\ dX &= 1.0005 & S_4 &= 1.0064 \\ dX &= 0.9868 & S_5 &= 0.9931 \end{aligned}$$

and we obtain one of the curves of Figure 14. ★

Let us compute the mean value and the variance of the random variable dS :

$$\begin{aligned} E[dS] &= \mu S dt + \sigma SE[dX] = \mu S dt \\ \text{Var}[dS] &= E[(dS)^2] - E[dS]^2 = E[\sigma^2 S^2 (dX)^2] = \sigma^2 S^2 dt, \end{aligned}$$

where we kept into account that $E[dX] = 0$ and $E[(dX)^2] = dt$. Moreover, dS enjoys the *Markov property*: it is independent from the past and depends only on today's price.

The volatility σ is related to the type of equity: for instance, government bonds will generally have a lower volatility than high-tech stocks. The more σ increases, the more the path of S will be jagged (see Figure 16).

23.2 Remark (How to estimate σ from real data). Let $S_0, S_1, \dots, S_n$ the closing prices of an asset at equal time-steps Δt . Then one can compute

$$\begin{aligned} \bar{m} &:= \frac{1}{n\Delta t} \sum_{i=0}^{n-1} \frac{S_{i+1} - S_i}{S_i}, \\ \bar{\sigma}^2 &:= \frac{1}{(n-1)\Delta t} \sum_{i=0}^{n-1} \left(\frac{S_{i+1} - S_i}{S_i} - \bar{m} \right)^2. \end{aligned} \quad \star$$

23.1 Itô's Lemma

We want to study the limit $dt \rightarrow 0$ (it is much more efficient to solve stochastic differential equations than to value options by direct simulations).

23.3 Theorem. *Let $f(S)$ be a smooth function. Then*

$$df = \left(f'(S)\mu S + \frac{1}{2}\sigma^2 S^2 f''(S) \right) dt + f'(S)\sigma S dX \quad \text{as } dt \rightarrow 0.$$

Proof. Let us employ a Taylor expansion around S

$$df = f'(S) dS + \frac{1}{2}f''(S) (dS)^2 + \dots$$

and use (34):

$$(dS)^2 = (\mu S dt + \sigma S dX)^2 = \mu^2 S^2 (dt)^2 + 2\mu\sigma S^2 dt dX + \sigma^2 S^2 (dX)^2.$$

Letting $dt \rightarrow 0$, we have $dX \sim \sqrt{dt}$ and we can neglect $(dt)^2$ and $dt dX$, hence

$$(dS)^2 \sim \sigma^2 S^2 dt$$

and we get

$$df = f'(S)(\mu S dt + \sigma S dX) + \frac{1}{2}f''(S)\sigma^2 S^2 dt \quad \text{as } dt \rightarrow 0.$$

which concludes the proof. □

23.4 Example. Let $f(S) = \log S$.

Then $f'(S) = 1/S$ and $f''(S) = -1/S^2$, so that Itô's formula writes

$$df = \left(\mu - \frac{\sigma^2}{2} \right) dt + \sigma dX \quad \text{as } dt \rightarrow 0.$$

Since the r.h.s. is constant, it means that df is normally distributed. Moreover, since

$$f = f_0 + \int df,$$

also $f = \log S$ is normally distributed, with

$$\text{mean} = \left(\mu - \frac{\sigma^2}{2} \right) t, \quad \text{variance} = \sigma^2 t$$

Hence we say that in this asset price model S is a *log-normal distribution* (meaning that its logarithm is normally distributed). ★

Itô's formula can be generalized to the time-dependent case $f(S, t)$: by taking again the Taylor expansion

$$df = \frac{\partial f}{\partial S} dS + \frac{\partial f}{\partial t} dt + \frac{1}{2} \frac{\partial^2 f}{\partial S^2} (dS)^2 + \dots$$

and, using (34), we get the *generalized Itô's formula*

$$df = \left(\mu S \frac{\partial f}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 f}{\partial S^2} + \frac{\partial f}{\partial t} \right) dt + \sigma S \frac{\partial f}{\partial S} dX \quad \text{as } dt \rightarrow 0. \quad (35)$$

24 Black-Scholes Equation: Option Pricing

Let us make the following assumptions:

- (1) The underlying asset price S follows the dynamics

$$dS = \mu S dt + \sigma S dX.$$

- (2) The drift μ , the risk-free interest rate r and the volatility σ are constant.
- (3) No transaction costs, no dividends.
- (4) No arbitrage opportunities.
- (5) Continuous trading, infinite divisibility of the assets.
- (6) Short selling is allowed.

Assumption (4) (no arbitrage) implies that if risk-free profit opportunities existed, prices would adjust immediately to eliminate them.

24.1 Definition (European Options). A European option is a contract with the following conditions: the owner of the option can buy (*call*) or sell (*put*) a given asset for a prescribed price (*exercise price*), at a prescribed time (*expiration time*) in the future, but they are not obligated to do so. ★

We will use the following notation:

- $V(S, t)$: value of an option (Call $C(S, t)$ or Put $P(S, t)$).
- E : exercise price (or strike price).
- T : expiration time.

Since the owner of the option is not obligated to exercise it, we have the following terminal conditions at $t = T$:

- for a call: $C(S, T) = \max(S(T) - E, 0)$,
- for a put: $P(S, T) = \max(E - S(T), 0)$,

where $S(T)$ denotes the value of the asset at the expiration time T . See Figure 17.

Many portfolios can be constructed by combining calls and puts, both long and short. As an example: going long for a call with strike price E_1 and short for a call with strike price E_2 , $E_1 < E_2$, has the value

$$V(S, T) = \max(S - E_1, 0) - \max(S - E_2, 0),$$

which is represented in Figure 18.

We can also build up a portfolio which is purely deterministic, using the so-called *Put-Call parity*: consider a portfolio which is long for 1 stock, long for 1 put, and short for 1 call (the put and the call having the same strike price and the same expiration time)

$$\Pi = S + P - C.$$

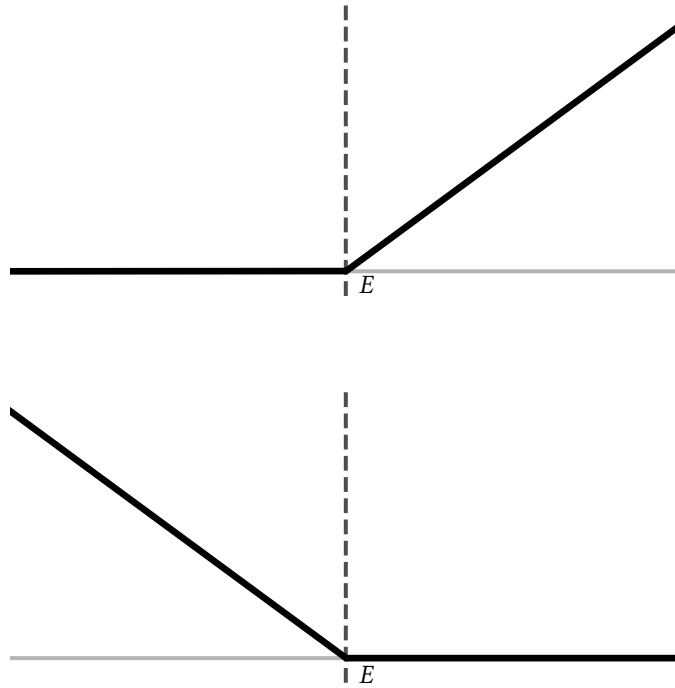


Figure 17: Payoff diagrams (at $t = T$) for a European Call and a European Put.

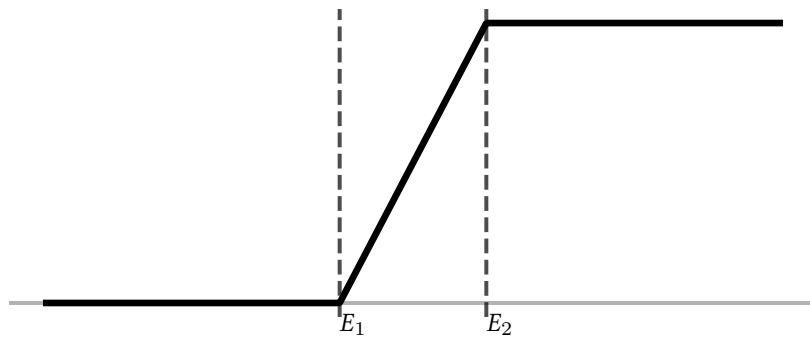


Figure 18: Payoff diagram (at $t = T$) for a portfolio made of a long European Call with strike E_1 and a short European Call with strike E_2 .

Then, at expiration, the value of the portfolio is:

$$\Pi(S, T) = S + \max(E - S, 0) - \max(S - E, 0) = E,$$

which is deterministic! By the assumption of no-arbitrage, for $t < T$ we have

$$\Pi(S, t) = S + P - C = Ee^{-r(T-t)},$$

which is the discounted value at the interest rate r . From this formula one can compute the value of a put option, given the value of a call option with the same exercise price and the same expiration time, in the following way:

$$P(S, t) = C(S, t) - S + Ee^{-r(T-t)}. \quad (36)$$

24.1 Black-Scholes Equation

Now we come to the B&S equation. Let us denote with $V(S, t)$ the value of a European option and apply Itô's lemma to $V(S, t)$:

$$dV = \sigma S \frac{\partial V}{\partial S} dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt.$$

Now construct a portfolio $\Pi = V - \Delta S$ (with Δ to be determined). Then

$$\begin{aligned} d\Pi &= dV - \Delta dS = dV - \Delta(\sigma S dX + \mu S dt) \\ &= \sigma S \left(\frac{\partial V}{\partial S} - \Delta \right) dX + \left(\mu S \frac{\partial V}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} - \mu \Delta S \right) dt. \end{aligned}$$

By choosing $\Delta = \frac{\partial V}{\partial S}$ (delta hedging), we eliminate the stochastic term, hence

$$d\Pi = \left(\frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + \frac{\partial V}{\partial t} \right) dt$$

which is purely deterministic. By no-arbitrage, the portfolio return must equal the risk-free rate:

$$d\Pi = r\Pi dt = r \left(V - \frac{\partial V}{\partial S} S \right) dt.$$

Equating the two expressions for $d\Pi$ gives

$$\frac{\partial V}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 V}{\partial S^2} + rS \frac{\partial V}{\partial S} - rV = 0 \quad (37)$$

This is the Black-Scholes partial differential equation. Note that the drift μ does not appear in the equation.

The B&S PDE (37) has to be complemented by *boundary and initial conditions*. Indeed, being a backward parabolic PDE, it requires a terminal condition (at $t = T$) and two boundary conditions (at $S = 0$ and $S \rightarrow \infty$).

For a European call option:

- terminal condition: $C(S, T) = \max(S - E, 0)$
- at $S = 0$: $C(0, t) = 0$ (if the underlying is worth 0, it remains 0 forever)

- as $S \rightarrow \infty$: $C(S, t)/S \rightarrow 1$ (for large S , the option will be exercised, hence its value will be S)

For a European put option

- terminal condition: $P(S, T) = \max(E - S, 0)$
- at $S = 0$: $P(0, t) = Ee^{-r(T-t)}$ (present value of the strike)
- as $S \rightarrow \infty$: $P(S, t) \rightarrow 0$ (for large S , the option will not be exercised)

By employing standard methods of Calculus, the B&S PDE can be transformed in the so called *heat (or diffusion) equation*, which can be explicitly solved. We will give some mathematical details in the following Section 25; here, we write the final result.

Introducing

$$\Phi(x) := \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-y^2/2} dy,$$

that is, the c.d.f. of the normal distribution $\mathcal{N}(0, 1)$, one has

$$C(S, t) = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2), \quad (38)$$

$$P(S, t) = -S\Phi(-d_1) + Ee^{-r(T-t)}\Phi(-d_2), \quad (39)$$

where

$$d_1 := \frac{\ln \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}},$$

$$d_2 := d_1 - \sigma\sqrt{T-t} = \frac{\ln \frac{S}{E} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}.$$

Notice that eq. (39) can be recovered by (38) using the put-call parity (36). Indeed:

$$\begin{aligned} P(S, t) &= C(S, t) - S + Ee^{-r(T-t)} = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2) - S + Ee^{-r(T-t)} \\ &= -S(1 - \Phi(d_1)) + Ee^{-r(T-t)}(1 - \Phi(d_2)) \\ &= -S\Phi(-d_1) + Ee^{-r(T-t)}\Phi(-d_2), \end{aligned}$$

and we used the fact that $\Phi(x) + \Phi(-x) = 1$.

25 The Mathematical Deduction of the Black-Scholes Formulas

The one-dimensional *heat (or diffusion) equation* is the PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad x \in \mathbb{R}, \tau \geq 0. \quad (40)$$

If the initial condition is $u(x, 0) = u_0(x)$ and u_0 does not grow too fast,⁽⁵⁾ then it can be proved that the solution writes

$$u(x, \tau) = \frac{1}{\sqrt{2\pi\tau}} \int_{-\infty}^{\infty} u_0(s) \exp\left(-\frac{(x-s)^2}{4\tau}\right) ds. \quad (41)$$

⁽⁵⁾For our purposes, it is enough to assume that for some $k, d > 0$

$$\forall x \in \mathbb{R} : |u_0(x)| \leq ke^{d|x|}$$

c We now want to transform the Black-Scholes equation (37) into the heat equation and use the given explicit formula for the solution. Let us write the problem in the case of a European call:

$$\begin{cases} \frac{\partial C}{\partial t} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 C}{\partial S^2} + rS \frac{\partial C}{\partial S} - rC = 0, \\ C(S, T) = \max(S - E, 0). \end{cases} \quad (42)$$

Note that (42) resembles the heat equation, but has some additional terms and coefficients multiplying the derivatives of C with respect to S . Moreover, (42) is a backward equation, while the heat equation is forward, so that we will need to reverse the time flow by a sign change. In this way, the final condition of (42) becomes an initial condition in (40).

First, we try to eliminate the terms S and S^2 that multiply $\frac{\partial C}{\partial S}$ and $\frac{\partial^2 C}{\partial S^2}$, respectively. We set:

$$S = Ee^x, \quad t = T - \frac{\tau}{\frac{1}{2}\sigma^2}, \quad C = Ev(x, \tau).$$

Note that τ moves in the opposite direction to t , so we have switched to a forward equation.

Substituting into (42) we obtain:

$$\begin{cases} \frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (k-1) \frac{\partial v}{\partial x} - kv, \\ v(x, 0) = \max(e^x - 1, 0), \end{cases}$$

where $k = \frac{2r}{\sigma^2}$.

This equation more closely resembles the diffusion equation, but another change of variables is needed to transform it completely. If we set

$$v(x, \tau) = e^{\alpha x + \beta \tau} u(x, \tau),$$

with α and β constants, we obtain:

$$\beta u + \frac{\partial u}{\partial \tau} = \alpha^2 u + 2\alpha \frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2} + (k-1) \left(\alpha u + \frac{\partial u}{\partial x} \right) - ku.$$

Now we choose

$$\beta = \alpha^2 + (k-1)\alpha - k, \quad 2\alpha + (k-1) = 0,$$

so as to obtain an equation without the terms u and $\frac{\partial u}{\partial x}$. This yields

$$\alpha = -\frac{1}{2}(k-1), \quad \beta = -\frac{1}{4}(k+1)^2.$$

Therefore, substituting these values, the change of variable becomes:

$$v(x, \tau) = \exp\left(-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau\right) u(x, \tau),$$

and we obtain

$$\begin{cases} \frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} & x \in \mathbb{R}, \tau \geq 0, \\ u(x, 0) = \max\left(\exp\left(\frac{1}{2}(k+1)x\right) - \exp\left(\frac{1}{2}(k-1)x\right), 0\right). \end{cases}$$

which is the diffusion equation, and has the solution (41) with initial condition

$$u_0(x) = \max \left(\exp \left(\frac{1}{2}(k+1)x \right) - \exp \left(\frac{1}{2}(k-1)x \right), 0 \right).$$

Then we have found an explicit formula for the solution. The only thing left to do is to evaluate the integral in (41). It is convenient to make a further change of variable, setting

$$y = \frac{s-x}{\sqrt{2\tau}},$$

so that we obtain

$$\begin{aligned} u(x, \tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} u_0(s) \exp \left(-\frac{(x-s)^2}{4\tau} \right) ds \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_0(y\sqrt{2\tau} + x) \exp \left(-\frac{1}{2}y^2 \right) dy \\ &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp \left(\frac{1}{2}(k+1)(x+y\sqrt{2\tau}) \right) \exp \left(-\frac{1}{2}y^2 \right) dy \\ &\quad - \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp \left(\frac{1}{2}(k-1)(x+y\sqrt{2\tau}) \right) \exp \left(-\frac{1}{2}y^2 \right) dy \\ &= I_1 - I_2, \end{aligned}$$

We evaluate I_1 by completing the square in the exponential, to obtain a standard integral:

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp \left(\frac{1}{2}(k+1)(x+y\sqrt{2\tau}) - \frac{1}{2}y^2 \right) dy \\ &= \frac{\exp \left(\frac{1}{2}(k+1)x \right)}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau}}^{\infty} \exp \left(\frac{1}{4}(k+1)^2\tau \right) \exp \left(-\frac{1}{2} \left(y - \frac{1}{2}(k+1)\sqrt{2\tau} \right)^2 \right) dy \\ &= \frac{\exp \left(\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau \right)}{\sqrt{2\pi}} \int_{-x/\sqrt{2\tau} - \frac{1}{2}(k+1)\sqrt{2\tau}}^{\infty} e^{-\rho^2/2} d\rho \\ &= \exp \left(\frac{1}{2}(k+1)x + \frac{1}{4}(k+1)^2\tau \right) \Phi(d_1), \end{aligned}$$

where

$$d_1 = \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k+1)\sqrt{2\tau}$$

and Φ is the cumulative distribution function of the normal distribution.

The calculation for I_2 is very similar, one only has to replace $k+1$ with $k-1$, whence

$$\begin{aligned} I_2 &= \exp \left(\frac{1}{2}(k-1)x + \frac{1}{4}(k-1)^2\tau \right) \Phi(d_2), \\ d_2 &= \frac{x}{\sqrt{2\tau}} + \frac{1}{2}(k-1)\sqrt{2\tau}. \end{aligned}$$

Retracing our steps, we recall the change of variables

$$v(x, \tau) = \exp \left(-\frac{1}{2}(k-1)x - \frac{1}{4}(k+1)^2\tau \right) u(x, \tau)$$

together with

$$x = \ln \frac{S}{E}, \quad \tau = \frac{1}{2}\sigma^2(T-t), \quad C = Ev(x, \tau), \quad k = \frac{2r}{\sigma^2},$$

to obtain

$$C(S, t) = S\Phi(d_1) - Ee^{-r(T-t)}\Phi(d_2),$$

where

$$d_1 = \frac{\ln \frac{S}{E} + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}, \quad d_2 = \frac{\ln \frac{S}{E} + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{(T-t)}}.$$

26 The Greeks

Now that we have an explicit formula for $C(S, t)$, the value of a European Call, we can ask how the value depend on the variation of the price of the underlying asset or the value of the parameters of the model. We introduce a definition.

26.1 Definition (Greeks). We denote

$$\Delta = \frac{\partial C}{\partial S}, \quad \Gamma = \frac{\partial^2 C}{\partial S^2}, \quad \rho = \frac{\partial C}{\partial r}, \quad \Theta = \frac{\partial C}{\partial t}, \quad \mathcal{V} = \frac{\partial C}{\partial \sigma}. \quad \star$$

The last quantity, $\mathcal{V}$, is called *vega* for historical reasons.

The meaning of the Greeks is as follows: the larger a Greek is, the more the value of C depends on variations in the corresponding parameter [Björk, Section 9.2].

26.2 Proposition. Let $C(S, t)$ be the value of a European call with strike price E and time of maturity T . Then, using the notation of (38):

$$\begin{aligned} \Delta &= \Phi(d_1), \quad \Gamma = \frac{\varphi(d_1)}{S\sigma\sqrt{T-t}}, \quad \rho = Ee^{-r(T-t)}(T-t)\Phi(d_2), \\ \Theta &= -\frac{S\varphi(d_1)\sigma}{2\sqrt{T-t}} - rEe^{-r(T-t)}\Phi(d_2), \quad \mathcal{V} = S\varphi(d_1)\sqrt{T-t} = Ee^{-r(T-t)}\varphi(d_2)\sqrt{T-t}, \end{aligned}$$

where $\varphi(x)$ denotes the density function of Φ .

27 Perpetual American Put

An *American option* differs from a European option in that it may be exercised at any time prior to the expiration date T . It can be showed that an American call (without dividends) is equivalent to a European call, since it is never optimal to exercise the option before the expiration date. On the other hand, an American put is a different financial instrument and has a higher value than a European put. However, determining the correct price of an American put is a challenging mathematical problem, as it leads to a free-boundary problem.

In this section, we find the correct price of a perpetual American put. This is not a traded option, but it is interesting from a mathematical viewpoint (see for instance [Shreve, Sec. 8.3]). By definition, a *perpetual American put* is an American put which never expires, that is, $T = +\infty$.

So, the unknowns of the problem are P , the value of the put, and s , the threshold value of the stock S for which:

- if $S < s$, it is optimal to exercise the option;
- if $S > s$, it is optimal to wait.

Starting from the Black-Scholes equation for a put, we write

$$\frac{\partial P}{\partial t} + rS \frac{\partial P}{\partial S} + \frac{\sigma^2 S^2}{2} \frac{\partial^2 P}{\partial S^2} - rP = 0.$$

A *perpetual solution* is a solution independent of time t :

$$P = P(S) \quad \Rightarrow \quad \frac{\partial P}{\partial t} = 0 \quad \Rightarrow \quad rSP' + \frac{\sigma^2 S^2}{2} P'' - rP = 0,$$

which is a second-order ODE (linear with non-constant coefficients). We now look for a solution of the form:

$$P(S) = S^n, \quad P'(S) = nS^{n-1}, \quad P''(S) = n(n-1)S^{n-2}.$$

Substituting into the equation:

$$rnS^n + \frac{\sigma^2}{2} n(n-1)S^n - rS^n = 0,$$

that is,

$$(n-1) \left(r + \frac{\sigma^2}{2} n \right) = 0 \quad \Rightarrow \quad n_1 = 1, \quad n_2 = -\frac{2r}{\sigma^2}.$$

Thus:

$$P(S) = c_1 S^{n_1} + c_2 S^{n_2} = c_1 S + c_2 S^{-2r/\sigma^2}. \quad (43)$$

We now impose the boundary conditions for a perpetual put. We know that for $S < s$ the solution is $P(S) = E - S$, since it is optimal to exercise the option. Then, we impose boundary conditions for $S = s$: in particular, we require that P is a C^1 -function at $S = s$, that is, P and P' are continuous at $S = s$:

- $P(s) = E - s$;
- $P'(s) = -1$ (which is the derivative of $E - S$).

Moreover, we impose that $P(S) \rightarrow 0$ as $S \rightarrow +\infty$, so that we have to choose $c_1 = 0$ in (43) and

$$P(S) = c_2 S^{-2r/\sigma^2}.$$

From the condition $P(s) = E - s$ we have:

$$c_2 s^{-2r/\sigma^2} = E - s,$$

and from the condition $P'(s) = -1$:

$$c_2 \left(-\frac{2r}{\sigma^2} \right) s^{-2r/\sigma^2 - 1} = -1.$$

Solving the system:

$$\begin{cases} c_2 s^{-2r/\sigma^2} = E - s \\ c_2 \left(-\frac{2r}{\sigma^2} \right) s^{-2r/\sigma^2 - 1} = -1 \end{cases}$$

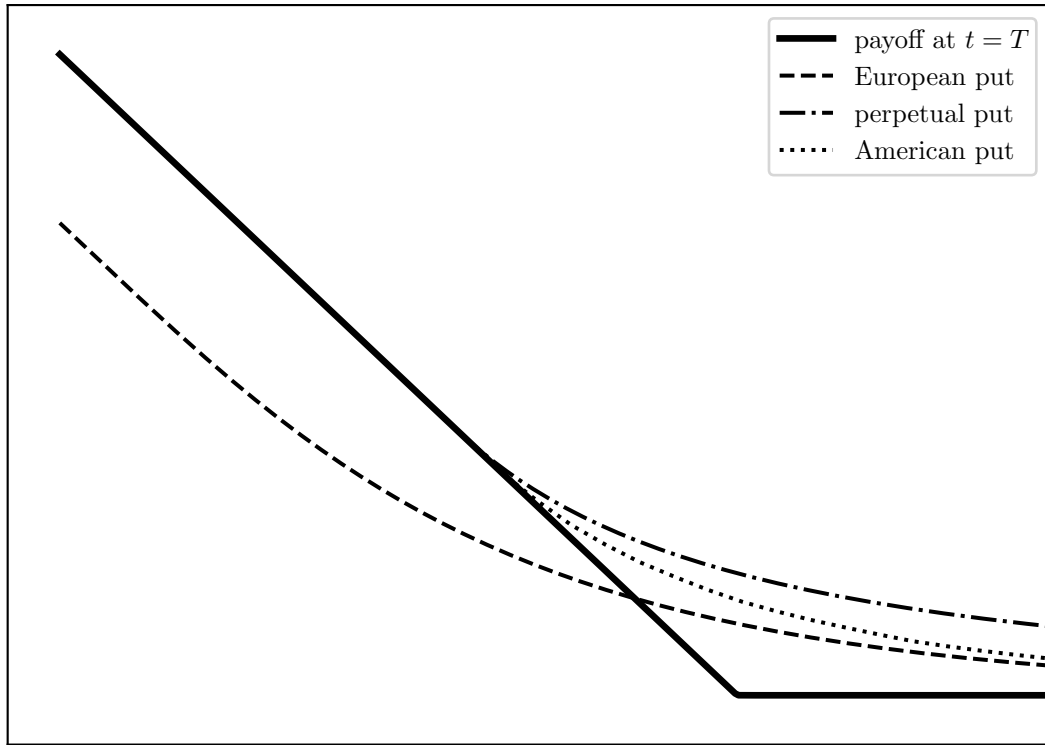


Figure 19: Payoff diagrams for a European Put, an American put and a perpetual put.

we obtain:

$$\left(1 + \frac{2r}{\sigma^2}\right)s = \frac{2r}{\sigma^2}E \quad \Rightarrow \quad s = \frac{\frac{2r}{\sigma^2}}{1 + \frac{2r}{\sigma^2}}E = \frac{2r}{2r + \sigma^2}E$$

and

$$c_2 = (E - s)s^{2r/\sigma^2}.$$

Finally, the value function is:

$$P(S) = \begin{cases} E - S & S < s \\ (E - s)s^{2r/\sigma^2} S^{-2r/\sigma^2} & S \geq s, \end{cases} \quad s = \frac{2r}{2r + \sigma^2}E.$$

In Figure 19 there is a comparison between the value of a European put, an American put and a perpetual put. As one can see, the perpetual put has the highest value, while the European put the lowest.

References

- [Björk] T. Björk, *Arbitrage Theory in Continuous Time*, Oxford University Press, 2009
- [Devaney] R.L. Devaney, *A First Course in Chaotic Dynamical Systems. Theory and Experiment. Second Edition*, CRC Press, 2020
- [Elaydi] S.N. Elaydi, *An introduction to difference equations*, Springer, 2005

- [Gandolfo] G. Gandolfo, *Economic dynamics: methods and models*, North-Holland, 1985
- [Shreve] S. Shreve, *Stochastic Calculus for Finance II*, Springer, 2004
- [Wilmott] P. Wilmott, S. Howison, J. Dewynne, *The Mathematics of Financial Derivatives*, Cambridge University Press, 1995