Antonio DiCarlo
Doctoral Program in Physics
Università « Roma Tre »

Lamé's Elastic Solid Ether Rekindled

An aside to the crash course on

Basic mathematical structures of neo-classical continuum mechanics

Dip. di Matematica e Fisica «Niccolò Tartaglia»
Università Cattolica del Sacro Cuore
Brescia, January 29–31, 2007

[Rev. February 20, 2008]

Energies through the **ether** flow,
Waves travel to and fro,
And with a ratio
Their speed you measure.
Colours yield their secret hue,
And Saturn's rings subdued by you
Suggest that gases
Might be measured too.

Science you freed
From cramping mechanistic creed,
And by your theory brought
The *elastic solid ether* to naught,
And changed the axiomatic basis
Of scientific thought.

Oh Maxwell! How can I declaim
On such a genius, such a fame,
And speak of one so very wise
Who saw the world through splendid eyes,
And though of such a subtle mind
Was yet so humorous and kind?
Yours was a mind unique and rare
That, nurtured in a northern air,
Struck out new paths in many ways
Through all too short, yet fruitful days.
How can one capture in a line
Something so great, so pure, so fine?

Give thanks,
That such a man drew breath,
And lament with all the world
His early death.

Eulogy in verse on display in Maxwell's House at 14 India Street, Edinburgh

Dieses Wort **Äther** hat in der Entwicklung der Wissenschaft viele Male seine Bedeutung geändert. [...] Seine Geschichte ist aber noch keineswegs beendet [...].

A. Einstein & L. Infeld, Die Evolution der Physik (1950)

In speaking of the Energy of the field, however, I wish to be understood literally. **All energy is the same as mechanical energy**, whether it exists in the form of motion or in that of elasticity, or in any other form. The energy in electromagnetic phenomena is mechanical energy. The only question is, Where does it reside?

J. C. Maxwell (1876)

Background

This note is a revised and augmented version of my essay *G. Lamé vs. J.C. Maxwell: how to reconcile them*, as published in Scientific Computing in Electrical Engineering, W. H. A. Schilders *et al.* eds., Mathematics in Industry Series, Vol. 4, pp. 1–13, Springer, Heidelberg, 2004. Most of it was composed during summer 2002, to write down the lecture I had given at the 4th Conference on Scientific Computing in Electrical Engineering (SCEE-2002, Eindhoven, June 23–28, 2002). An abridged version was accepted into the above book of proceedings in July 2003. Giving a talk (*Electromagnetic and mechanical field theories set on the same stage*) at the MACSI-net Workshop on Optimization and Coupled Problems in Electromagnetism (Naples, September 22–23, 2003) gave me the opportunity to detect and correct a few errors prior to publication. Later reworking was mostly prompted by two courses I gave for the Doctoral Program in Physics at Università "Roma Tre": *Mathematical structures of neo-classical physics* (May 31–June 10, 2004), and *The geometry of neo-classical continuum mechanics* (June 5–12, 2006). Also a couple of talks I gave in the USA were instrumental in this respect: *Electromagnetism and continuum mechanics: what makes them different* (College of Engineering, University of Wisconsin–Madison, June 18, 2004), and *Basic mathematical structures of neo-classical continuum mechanics* (Department of Mathematics, University of California, Berkeley, December 8, 2005). The final impetus to gather and smooth out my scribbled notes, while summoning unwritten thoughts, came from the present course in Brescia, paired with an invitation to the Workshop on Advanced Computational Electromagnetics (ACE '07) to be held in Aachen next June—a late aftermath of my participation in the 2002 SCEE-conference at the Technische Universiteit Eindhoven, where it all began.

Roma–Milano, December 20, 2006–January 28, 2007

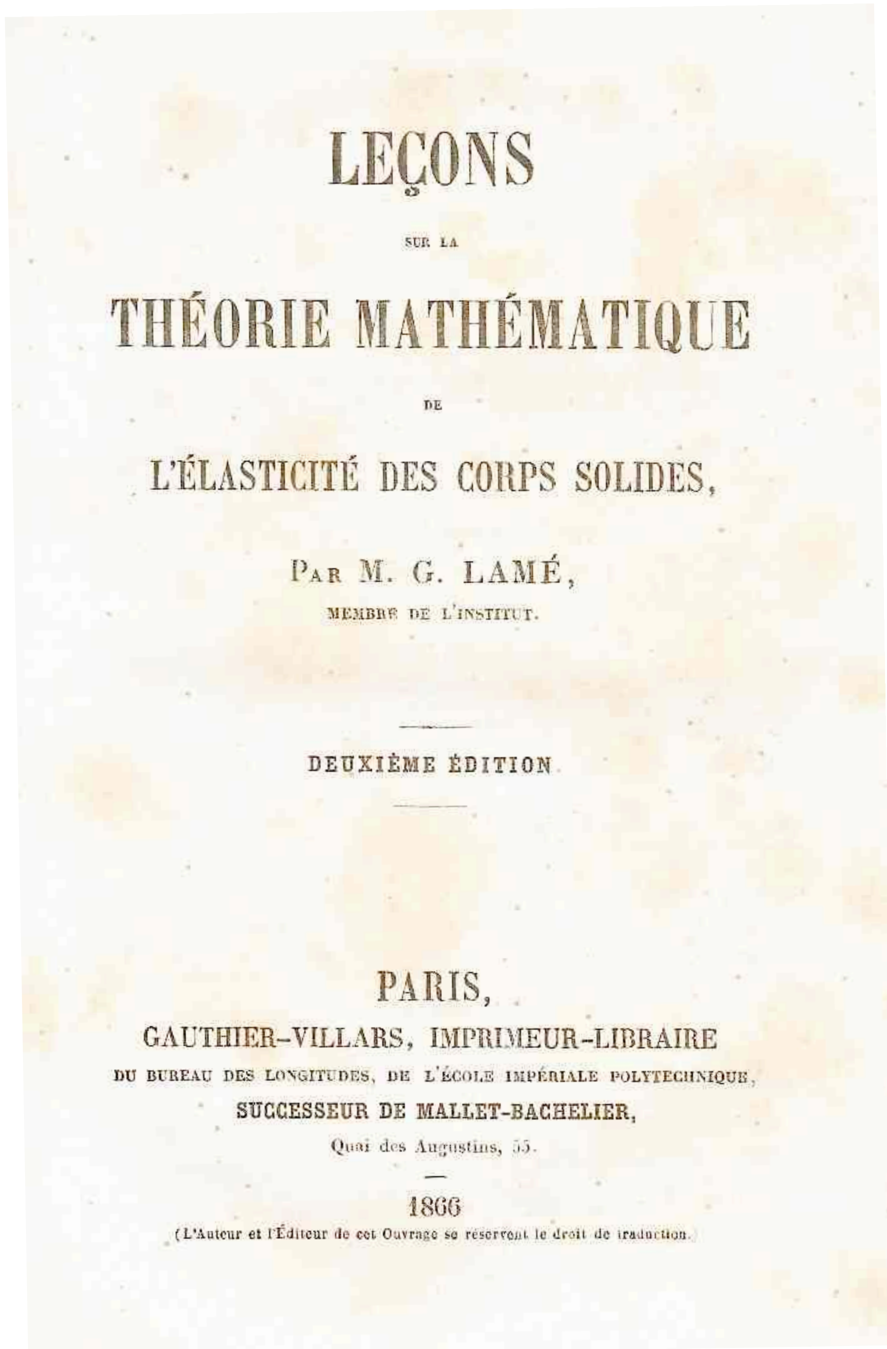
Lamé's elastic solid ether rekindled

The electromagnetic and the mechanical response of a medium cannot be characterized independently of each other, in general. Therefore, a unified formulation of both theories is basic to a successful approach to strongly coupled electro-mechanical problems. Nowadays, after more than a century of inconsiderate divergence between electromagnetic and mechanical field theories, we find it hard to bring them together. This is best exemplified by the problematic status of the electrodynamics of deformable media. The blame can be laid mainly on the limitations of the underlying theoretical frameworks and on the practitioners' education, too narrow to bridge the gap between them. I would like to concentrate here on the first problem—even though I am convinced that the second one carries more weight.

Here I transcribe the two field theories into the same unifying language, as is needed both to contrast and to couple them. In electromagnetic field theory there are plenty of real-valued physical quantities associated with geometrical objects (cells) of various dimensions, embedded in four-dimensional space-time. On the contrary, in continuum mechanics there is only one true real-valued quantity, namely work, associated with the cells of highest dimension in the body-time manifold (beware: body-time, not space-time). Mechanical work is backed up by a composite team of vector-valued and covector-valued physical quantities, associated with low- or high-dimensional cells. In electromagnetics, on the contrary, there are no true vector quantities—despite appearances and ingrained habits.

Lamé's treatise on three-dimensional elastic solids

I happen to have a copy of the second edition of Gabriel Lamé's *Leçons sur la Théorie Mathématique de l'Elasticité des Corps Solides* [1] on my bookshelf. It was published in 1866 (Fig. 1). Its first edition had been published in 1852. Compare with the dates of the electromagnetic trilogy by Maxwell: *On*



LEÇONS

SUR LA

THÉORIE MATHÉMATIQUE

DE

L'ÉLASTICITÉ DES CORPS SOLIDES,

PAR M. G. LAMÉ,

MEMBRE DE L'INSTITUT.

DEUXIÈME ÉDITION.

PARIS,

GAUTHIER-VILLARS, IMPRIMEUR-LIBRAIRE

DU BUREAU DES LONGITUDES, DE L'ÉCOLE IMPÉRIALE POLYTECHNIQUE,

SUCCESSEUR DE MALLET-BACHELIER,

Quai des Augustins, 55.

1866

(L'Auteur et l'Éditeur de cet Ouvrage se réservent le droit de traduction.)

Fig. 1. The title page of the second edition of Lamé's *Leçons* [1].

Faraday's lines of force [2] appeared in 1856, *On physical lines of force* [3] in 1861–2, and *A dynamical theory of the electromagnetic field* [4] in 1865.

Lamé's lecture notes on the mathematical theory of elastic solids differ strongly from any present-day book with a similar title. I do not mean by this simply that its notions and notations are somewhat outdated, or its mathematics obsolete, which would be quite trivial. The main difference is that Lamé was much bolder and more oriented towards fundamental physics than any of his modern followers in elasticity or solid mechanics. Secondly, and as a consequence, his main aim was to study the vibrations and waves of a peculiarly thin and outlandish solid medium: the all-pervading elastic ether.

Mathematical physics as a new science

Let me quote *verbatim* from the terse preface to the first edition, interspersing commented translations of its key points. Lamé writes:

> La Physique mathématique, proprement dite, est une création toute moderne, qui appartient exclusivement aux Géomètres de notre siècle. Aujourd'hui, cette science ne comprend en réalité que trois chapitres, diversement étendus, qui soient traités rationnellement ; c'est-à-dire qui ne s'appuient que sur des principes ou sur des lois incontestables. Ces chapitres sont : la théorie de l'électricité statique à la surface des corps conducteurs ; la théorie analytique de la chaleur ; enfin la théorie mathématique de l'élasticité des corps solides. Le dernier est le plus difficile, le moins complet ; il est aussi le plus *utile*, à une époque où l'on veut apprécier l'importance d'une théorie mathématique par les résultats qu'elle peut fournir immédiatement à la pratique industrielle.

Lamé held the opinion that—at the moment of his writing—the only well-founded chapters of proper Mathematical Physics were the electrostatics of conducting bodies, the theory of heat conduction and the mathematical theory of three-dimensional elasticity. Insisting that these creations—contrary to Rational Mechanics[1] of old—belonged exclusively to his own century, Lamé was unfair to the "Geometers" of the Baroque period (Euler and the Bernoullis, to name the most prominent ones), at least as much as a modern quantum physicist could be to him and his contemporaries in elasticity.

Elasticity—the most difficult and the least developed of the three chapters of Mathematical Physics—was also the most *useful*, he emphasized. It is thought-provoking to read that Lamé already viewed that his own time was willing to assess the importance of a mathematical theory through the immediate benefits it could provide industrial practice. We could with all the more reason appropriate his judgment, even though we would not put elasticity at the top of our list of most useful theories.

[1] More on this terminology in the next subsection on elasticity and electrodynamics.

However, Lamé did not focus on the engineering side of elasticity—the most prominent nowadays, mainstream physics having repudiated it long ago. His preface continues as follows:

> L'Analyse ne tardera pas, sans doute, à embrasser d'autres parties de la Physique générale, telles que la théorie de la lumière, et celle des phénomènes électrodynamiques. Mais, on ne saurait trop le répéter, la véritable Physique mathématique est une science aussi rigoureuse, aussi exacte que la Mécanique rationnelle. Elle se distingue, par là, de toutes les applications qui s'appuient sur des principes douteux, sur des hypothèses gratuites ou commodes, sur des formules empiriques ; le plus souvent ce ne sont là que des essais, que des calculs numériques au service d'une classification factice.

Elasticity and electrodynamics

No mention of light or electrodynamics is to be expected from a modern author in elasticity, who typically relates Lamé's name only to his celebrated elastic moduli (shortly introduced in no more than two pages out of the three hundred and thirty-five of the whole treatise). The big divide between Maxwell's and Lamé's descendants does *not* extend to the ascendants themselves, as we shall see even better on Maxwell's side (see, in particular, the subsection on Maxwell and the theory of elasticity on page 24).

In Lamé's eyes, novel Mathematical Physics was far deeper and wider in scope than older Rational Mechanics, which he saw confined to Celestial Mechanics,[2] rigid-body mechanics, hydrostatics, a primitive hydrodynamics, and the mechanics of flexible low-dimensional bodies. In Chaps. 8–10 Lamé touches upon elastic curves and surfaces with unjust disdain. He proclaims:

> Ces deux questions ont été traitées, à l'aide de principes particuliers, longtemps avant la création de la théorie mathématique de l'élasticité. [. . .] Les anciens géomètres ont pu croire qu'avant d'étudier les corps élastiques à trois dimensions finies, il convenait d'essayer d'abord les fils minces et les membranes peu épaisses, c'est-à-dire les lignes et les surfaces avant les solides. Mais cette marche, qui paraissait naturelle et logique, a complétement manqué son but, car la vrai théorie de l'élasticité n'a rien emprunté à ces premiers essais ; elle est née tout à fait en dehors de ce champ d'exploration.

It is an irony that Lamé foreran both the reductionistic fundamentalism of twentieth century physics and its most illustrious scapegoat: contemporary continuum physics.

[2] Elsewhere, Lamé designates Mathematical Physics as Terrestrial Mechanics. In the closing section of his treatise on curvilinear coordinates [5], published in 1859, he writes: « Mais quand il aura transformé et complété toutes les solutions de la Mécanique céleste, il faudra s'occuper sérieusement de la Physique mathématique, ou de la Mécanique terrestre ». See also footnote 5 on page 9.

However, he insisted that genuine Mathematical Physics shared the standards of rigour and precision of Rational Mechanics. This quality differentiated it from the host of empirical treatments based on doubtful principles and *ad hoc* hypotheses, whose only merit was practicality, and whose function was essentially provisional:

> Cependant, la lenteur des progrès de la vraie science oblige d'avoir recours à ce genre d'applications, pour coordonner les théories physiques, pour étudier et comparer le moteurs, les machines, les projets de constructions de toute sorte, pour jauger les cours d'eau, les conduites de gaz, etc. Malgré leur utilité actuelle, qui est incontestable, toutes ces théories empiriques et partielles ne sont que des sciences d'attente. Leur règne est essentiellement passager, intérimaire. Il durera jusqu'à ce que la Physique rationnelle puisse envahir leur domaine. Elles n'auront plus alors qu'une importance historique.

Surrogate sciences and engineering education

I cannot be as confident as Lamé in the final victory of the one Rational Physics over the myriad of special, empirical theories devised to cope with practical problems for which a truly scientific treatment is not yet available. For two reasons: first, I know that make-do sciences actively reproduce, after swallowing morsels of true science; second, I doubt whether there is only one Rational Physics.

However, I find his lucid depiction of the subtle rivalry between technical and scientific thinking[3] in the age of technology a great contribution from him, and I stand on his side in this confrontation. I am also strongly in favour of his impassioned call for a balanced and open-minded engineering education:

> Jusqu'à cette époque, peut-être plus voisine qu'on ne le croit généralement, enseignons avec soin ces sciences d'attente, que d'habiles praticiens ont édifiées, afin de répondre aux besoins incessants des arts industriels. Mais ne les enseignons pas seules : tenons les élèves-ingénieurs au courant des progrès lents, mais sûrs, de la véritable Physique mathématique ; et, pour qu'ils puissent eux-mêmes accélérer ces progrès, faiçons en sorte qu'ils connaissent toutes les ressources actuelles de l'Analyse.
>
> C'est ce dernier but que je me propose, en publiant des Leçons sur la Théorie mathématique de l'élasticité, considérée dans les corps solides.

Let us teach these surrogate sciences with care—I proclaim after him—developed by skilful practitioners in response to the unceasing needs of industrial crafts. On the other hand, let us not teach only them. Let our engineering students keep abreast of the gradual, but positive, progress of genuine

[3] *Cf.* the opposition "techne" *vs.* "episteme" in the superb essay [6] by Koyré.

Mathematical Physics. In order that they themselves be able to speed up its progress, let them know all the resources of modern Analysis.

It seems to me that such an aim is even worthier—but more difficult—to attain today than in Lamé's time. I would incite any good engineering school to make it its blazon. Not being in a position to do so with my home institution, I would humbly address the Technische Universiteit Eindhoven[4], rated as "the best university of technology in Europe" by Der Spiegel in 1998.

Elasticity and molecular mechanics

At variance with modern textbooks in continuum mechanics, elasticity is defined by Lamé in terms of molecular interactions, even though in a rather vague way. The very first paragraph of Chap. 1, Sect. 1 runs as follows:

> 1. *Définition de l'élasticité.* — Lorsque les molécules de la matière constituent un corps ou un milieu, limité ou indéfini, les causes qui ont assigné à ces molécules leurs positions relatives sont en quelque sorte persistentes, ou agissent continuellement ; car, si quelque effort extérieur change un peu et momentanément ces positions, les mêmes causes tendent à ramener les molécules à leur places primitives. C'est cette tendance ou cette action continue que l'on désigne sous le nom d'*élasticité*.

Roughly speaking, Lamé calls elastic the restoring forces which tend to bring molecules back to their equilibrium positions. Notice that Lamé did not use his naïve molecular picture as a convenient pedagogic cartoon. Strange as it may appear to us, he and other founding fathers of *continuum* physics strongly believed in the *necessity* of an underlying *discrete* structure of matter.[5] Lamé criticized Navier's method for establishing the general equations of three-dimensional elasticity, on the grounds that it presupposed matter to be continuous, which he considered flatly absurd and inadmissible. In Sect. 15 he writes:

> Telle est la méthode suivi par Navier et autres géomètres, pour obtenir les équations générales de l'élasticité dans les milieux solides. Mais cette méthode suppose évidemment la continuité de la matière, hypothèse inadmissible.

[4] The Technische Universiteit Eindhoven was the venue of the conference where I gave the lecture preliminary to this note (see Background on page 3).

[5] Recall the dichotomy between Celestial and Terrestrial Mechanics (footnote 2 on page 7). In Sect. 134—the closing section of [1]—Lamé contrasts Celestial Mechanics with Molecular Physics, thereby identifying Molecular and Mathematical Physics: « Nous terminons cette Leçon, et le Cours que nous avons entrepris, par quelques réflexions sur la constitution intérieure des corps solides. [...T]outes les questions relatives à la Physique moléculaire ont été rétardées, plutôt qu'avancées, par l'extension, au moins prématurée sinon fausse, des principes et des lois de la Mécanique céleste. »

And again, in Sect. 30:

> On trouvera peut-être longue et minutieuse la marche que nous avons adoptée, en la comparant à celle qu'ont suivie Navier, Poisson et d'autres savants [. . .]. Chez nous, plus d'intégrations autour d'un point, lesquelles supposent évidemment la continuité de la matière, hypothèse absurde et complétement inadmissible, même par abstraction ; mais, au lieu de cette continuité imaginaire, existe la continuité réelle des déplacements géométriques, n^{o} 11 [*Projections du déplacement moléculaire*].

The subtle role played in the nineteenth century by the molecular interpretation of elasticity is best expounded in the masterly historical account of structural mechanics by Benvenuto [7]. Let me quote from Sect. 14.2 (but see also Sects. 6.7–8):

> Elasticity represented the most promising line of inquiry, not only because of its extraordinary practical usefulness and the accuracy of the theoretical synthesis that it permitted, but also because of the implications of its general principles and equations. The molecular interpretation of elastic behavior that Navier, Cauchy and Poisson promoted led many scientists to attempt finally to unify and explain all forces operating in Nature in the light of a universal law of attraction and interatomic repulsion, like that foreseen by Boscovich. From the masses of stars, subject to grand Newtonian forces, to the minute attractions between atoms in a molecule—all seemed to be governed by the same principle.
>
> In this area, [. . .] Mossotti's work is of great interest. His essay « Sur les forces qui régissent la constitution intérieure des corps, aperçu pour servir à la détermination de la cause et des lois de l'action moléculaire » was published in Turin in 1836; Faraday presented it to the Royal Institution in 1837. Mossotti aims (rather ambitiously) to unite all natural phenomena—gravity, molecular interactions, the "repulsive force of the caloric," electricity, light—by studying the "combined action of the attraction and the repulsion between two of more substances." He enriches Boscovich's model, taking into account as "an important contribution to the stable equilibrium" of bodies the action of ether, that "imponderable fluid . . . to which the phenomena of elasticity and of heat can be attributed." From this perspective, the laws of elasticity are by no means restricted to a specific class of bodies, but express an inherent property of matter itself.

Lamé shared that same perspective. Indeed, his definition of elasticity concludes along very similar lines (third paragraph of Chap. 1, Sect. 1):

> L'élasticité est donc une des propriétés générales de la matière. Elle est, en effet, l'origine réelle ou l'intermédiaire indispensable des phénomènes physiques les plus importants de l'univers. C'est par elle que

> la lumière se répand, que la chaleur rayonne, que le son se forme, se propage et se perçoit, que notre corps agit et se déplace, que nos machines se meuvent, travaillent et se conservent, que nos constructions, nos instruments échappent à mille causes de destruction. En un mot, le rôle de l'élasticité, dans la nature, est au moins aussi important que celui de la pesanteur universelle. D'ailleurs la gravitation et l'élasticité doivent être considérées comme les effets d'une même cause, qui rend dépendantes ou solidaires toutes les parties matérielles de l'univers, la première manifestant cette dépendance à des distances considérables, la seconde à des distances très-petites.

Lamé's view of ether

The main focus of Lamé's treatise on elasticity is undoubtedly the study of small vibrations and linear waves. Out of one hundred and thirty-four sections, grouped in twenty-four chapters, only the eight sections of Chaps. 12 and 16—making up twenty-eight pages in all—are devoted to equilibrium problems of three-dimensional elasticity. In fact, Lamé closes Chap. 10 (devoted to the study of transversal vibrations of a flat membrane as a two-dimensional preliminary to the three-dimensional problems treated afterwards) with a note of apology to the practical-minded reader for such a bias:

> L'object de cette Leçon paraitra sans doute fort peu important aux ingénieurs qui s'intéressent spécialement à l'équilibre d'élasticité.

He then asks a provocative question we are in a better position to appreciate, with the benefit of hindsight:

> Mais, outre qu'il est souvent nécessaire d'étudier l'effet des vibrations sur certaines constructions, le temps n'est-il pas venu de se demander si l'état moléculaire des corps dont le repos nous paraît le mieux établi est bien réellement un état statique ; s'il n'est pas, au contraire, le résultat de vibrations très-rapides, et qui ne s'arrêtent jamais ? Tout porte à penser, en effet, que le repos relatif des molécules d'un corps n'est qu'un cas très-exceptionnel, une pure abstraction, une chimère peut-être.

The notion of wave speed, with emphasis on the classification into longitudinal and transversal waves, is introduced in Chap. 11. From Chap. 17 on—one hundred and eleven pages in all—Lamé strives to explain light waves through elasticity theory, starting from Fresnel's birefringence. The first section of Chap. 17 starts with the following resolution:

> 91. *Application de la théorie de l'élasticité à la double réfraction.* — Jusqu'ici nous avons traité la théorie de l'élasticité comme une science rationnelle, donnant l'explication complète et les lois exactes de faits que ne peuvent pas évidemment avoir une autre origine. Nous allons maintenant la présenter comme un instrument de recherches [...].

> La théorie physique des ondes lumineuses porte certainement en elle l'explication future de tous les phénomènes de l'optique ; mais cette explication complète ne peut être atteinte par le seul secours de l'analyse mathématique, il faudra revenir, et souvent, aux phénomènes, à l'expérience.

This endeavour leads him to the following conclusion: the phenomena of light propagation in space, diffraction, and birefringence prove the ubiquitous existence of ether beyond all conceivable doubt. In Sect. 131 he writes:

> Il ne peut plus exister de doute sur cette question, car il résulte clairement de notre analyse que la matière pondérable, seule, est complétement incapable de produire les ondes progressives qui expliquent les phénomènes optiques des corps biréfringents [...]. Les ondes lumineuses sont donc produites et propagées, dans les corps diaphanes, par les vibrations d'un fluide impondérable, lequel ne peut être que l'éther. Or, ces conséquences importantes ne pouvaient être déduites, d'une manière certaine et rigoureuse, qu'à l'aide du calcul et en partant de la théorie de l'élasticité.

While crediting the mathematical theory of elasticity for this important and rigorous result, Lamé was confident that accounting properly for the interaction between ethereal and ponderable matter would have disclosed the secrets of a host of mysterious and incomprehensible beings, ranging from caloric, electricity, magnetism, universal attraction, cohesion, to chemical affinities. He closes his treatise with the following optimistic conjecture, which sounds odd to us:

> Il n'est donc plus possible d'arriver à une explication rationnelle et complète des phénomènes de la nature physique, sans faire intervenir [le fluide éthéré], dont la présence est inévitable. On n'en saurait douter, cette intervention, sagement conduite, trouvera le secret, ou la véritable cause des effets qu'on attribue au calorique, à l'électricité, au magnétisme, à l'attraction universelle, à la cohésion, aux affinités chimiques ; car tous ces êtres mystérieux et incompréhensibles ne sont, au fond, que des hypothèses de coordination, utiles sans doute à notre ignorance actuelle, mais que les progrès de la véritable science finiront par détrôner.

Maxwell's electromagnetism *vs.* Lamé's elasticity

Lamé was wrong with his elastic ether. As is clear to us now, Maxwell was right with his electromagnetic field theory, shaped on Faraday's unorthodox ideas. This makes me appreciate Maxwell's penchant for understatement and the "absurd and infuriating modesty" which Dyson reproached him with in a witty and enlightening short essay [8]. But was Maxwell against ether?

Maxwell on ether

By no means. Consider the following quote from the presidential address[6] he gave at the annual meeting of the British Association for the Advancement of Science in 1870:

> Another theory of electricity which I prefer denies action at a distance and attributes electric action to tensions and pressures in an all-pervading medium, these stresses being the same in kind with those familiar to engineers, and the medium being identical with that in which light is supposed to be propagated.

Such was the passing mention of *his own* theory that Maxwell uttered, after praising the vortex theory of matter by Thompson as a wonderful example of recent advances on the frontier between mathematics and physics. It should be noted that at the moment of this speech Maxwell was concentrating on his monumental treatise [9], having resigned from his professorship at King's College five years earlier. However, despite the kinship of "stresses" vaguely advocated by Maxwell, his electromagnetic ether deeply differs from Lamé's.[7]

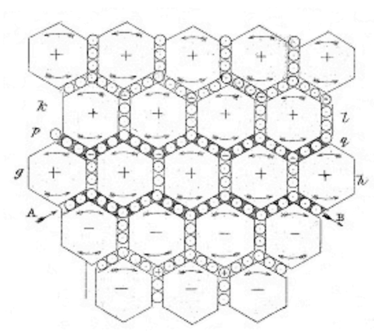


Fig. 2. Spinning molecular vortices from Maxwell's *On physical lines of force* [3].

[6] Published in Nature, Vol. 2. See also the entry "Ether" in the ninth edition of the Encyclopædia Britannica, written by Maxwell himself.

[7] The role played by mechanical modelling (see Fig. 2) in earlier Maxwell's conceptualizations of his electromagnetic field theory is well appreciated in [10] (but see [11] for an opposing argument). In his 1865 paper [4], Maxwell himself drew a clear distinction between his equations, which he firmly believed to be correct, and the mechanical model that supposedly accounted for them, which he did not defend any longer. For a strong view on this paradigm shift see [8], where Dyson writes: "Maxwell's theory becomes simple and intelligible only when you give up thinking in terms of mechanical models. [...] Maxwell theory had to wait for the next generation of physicists, Hertz and Lorentz and Einstein, to reveal its power and clarify its concepts. [...] The primacy of fields was as natural to Einstein as the primacy of mechanical structures had been to Maxwell."

The key difference between Maxwell's and Lamé's theories

As we have just seen, this is not in being for or against the existence of ether.[8] Also their different invariance properties are not as discriminating as is commonly adduced. After all, "classical" (*i.e.*, non-relativistic and non-quantistic) field theories of mechanics behave well under the action of *slow*[9] Lorentz changes in observer—which is all to be expected.

The distinguished feature that makes the real difference between Lamé's ether and Maxwell–Faraday's ether is *topological* in nature.[10] In electromagnetic field theory there are plenty of *real*-valued physical quantities associated with geometric objects (*cells*) of dimension ranging from 1 to 3, embedded in 4-dimensional *space*-time (*cf.* the discussion by Tonti in [16, Sect. 5.1]; see also the concise account by Vilasi in [18, Chap. 11]). On the contrary, in continuum mechanics there is only one *bona fide* real-valued quantity, namely *work*, associated with the cells of highest dimension in the *body*-time manifold (pay attention: body-time, *in lieu* of space-time!). Mechanical work is backed up by a dual team comprised of *vector*- and *covector*-valued physical quantities, associated with cells of small (≤ 1) dimension and *co*-dimension, respectively. In electromagnetics, on the contrary, there are no *bona fide* vector quantities (distrust Euclidean/Riemannian appearances!).

Space-time cells

Let me explain myself, starting from the electromagnetic side. In the following, I consider space-time as the product of a 3-dimensional space manifold times a 1-dimensional time line, thus adopting the viewpoint of an observer. Space-time vectors decompose accordingly into space and time components. A *space vector* is, by definition, a space-time vector with null time component (and viceversa). The dichotomy (space, time) is invariant under *slow* Lorentz changes in observer. It should *not* be confused with the Minkowskian trichotomy (spacelike, lightlike, timelike), which is invariant under a *general* Lorentz transformation [19]. All space vectors are spacelike, and time vectors timelike—but the converse does not hold true.

Consider the hierarchy of *parallelepipedal cells* in space-time, ranging from 0-dimensional (an event, *i.e.*, a place times an instant) to (3+1)-dimensional (a chunk of space times a time lapse). A nondegenerate k-*cell* (*i.e.*, a k-dimensional parallelepipedal cell) has k independent edges, which are space-time vectors. More precisely, a k-cell with $k > 0$ (synonymic to a k-*vector*) is

[8] While changing meaning many times, this term keeps its place in physics, as Einstein and Infeld put it: "Dieses Wort Äther hat in der Entwicklung der Wissenschaft viele Male seine Bedeutung geändert. [...] Seine Geschichte ist aber noch keineswegs beendet [...]."

[9] Right after my lecture, Alain Bossavit assured me that this notion could be made precise using well-established mathematical tools [12, 13].

[10] I espouse the viewpoint pionereed by Tonti [14–16] and expounded by Mattiussi [17], among others. However, they completely miss the issue I think is crucial here.

the equivalence class of all k-parallelepipeds sitting on the same 0-cell, lying in the same k-plane and having the same (signed) k-volume (as a basic reference for this gadgetry, see [20, Chap. 4]). When talking about cell edges, I really mean the edges of *at least one* of its representative parallelepipeds.[11]

Let me call *plumb* all nondegenerate cells whose edges are either space or time vectors; *slant* the other ones. Most cells are slant, but all of them can be obtained as linear combinations of plumb cells. Among plumb cells, I single out *time-dipped cells*, which have one time edge (they cannot have more); the remaining ones, having no time edges, I call simply *space cells*. All cells decompose (in a unique way) into space and time-dipped components.

The distribution of real-valued quantities additively associated with k-dimensional space-time objects is properly gauged by (real-valued) *k-forms*, *i.e.*, fields of *k-covectors*, which are—by construction—the integrands that makes sense to integrate on patches of k-cells, yielding their content of the gauged quantity. The *co*-space and *co*-time components[12] of a k-form are singled out by integrating it on space and time-dipped k-cells, respectively.

The basic structure of electromagnetic field theory

Maxwell's play—when staged in this transcription—has two leading characters: the even *electromagnetic* *2-form* F and the odd *charge-current* *3-form* $\widetilde{\mathsf{J}}$.[13] Maxwell's equations establish the existence of their companion

[11] Parallelepipeds are *affine* constructs *par excellence*. Space and time (and, *a fortiori*, deformable media) should be thought of basically as mere *differentiable* manifolds. Therefore, space-time k-cells sitting on $\{(\boldsymbol{x},\mathtt{i})\}$, where $\boldsymbol{x}$ is a place and $\mathtt{i}$ an instant, lie in the *tangent space* to the space-time manifold at $(\boldsymbol{x},\mathtt{i})$. They can only be approximately realized on the manifold itself: roughly speaking, only vanishingly small parallelepipedal cells belong in a differentiable manifold.

[12] To be read space-conjugate and time-conjugate components, respectively.

[13] The physically relevant distinction between *even* and *odd* k-forms is rather tricky to explain. This notwithstanding, I am not prone to take it for granted, because of the deficient attention odd forms and related notions usually get. For an introduction to these notions with a view to application to classical electromagnetism, the reader is referred to [21,22] and [15,16], with a warning: most often, odd forms are called *twisted*—a misnomer, in my opinion; see also the very nice book [23].

An even k-form is just a k-form; an odd k-form is a k-form paired with a collection of *local orientations* of the m-dimensional ambient manifold $\mathcal{M}$ (space-time, in the case under consideration). At each $\boldsymbol{e} \in \mathcal{M}$, the orientation of the tangent space $\mathsf{T}_{\boldsymbol{e}}\mathcal{M}$ may be characterized by a *nonzero* m-covector $\omega_{\boldsymbol{e}}$, any *positive* multiple of it providing the same orientation; since m-covectors have only one strict component, each tangent space may be oriented in two distinct ways. On each coordinate patch $\mathcal{U} \subset \mathcal{M}$, it is also possible to associate a coherent orientation to all tangent spaces in $\mathsf{T}\mathcal{U}$ through a smooth local m-form $\omega_{\mathcal{U}}$ which vanishes nowhere in $\mathcal{U}$. Therefore, all manifolds are *locally* orientable; however, patching together local orientations into a global one may be precluded by topological obstructions. The (local or global) orientation imparted to a k-submanifold $\mathcal{S}$ of $\mathcal{M}$

potentials: the *electromagnetic potential* A (an even 1-form), and the *charge-current potential* $\widetilde{\mathsf{G}}$ (an odd 2-form).[14] To sum up, we have a nice symmetric panoply, centered on dimension 2: one even 1-form (A), two 2-forms—one even (F), one odd ($\widetilde{\mathsf{G}}$)—, and one odd 3-form ($\widetilde{\mathsf{J}}$). All of them are real-valued, none of them sit at the extremes (0 and 4).

Down-to-earth electric and magnetic quantities are elicited by evaluating the co-space and co-time components of the above k-forms: the scalar-valued *charge density* ρ and the vector-valued *current density* $\mathbf{J}$ represent respectively the co-space and co-time components of the odd charge-current 3-form $\widetilde{\mathsf{J}}$; the vector-valued *electric flux density* $\mathbf{D}$ and *magnetic field intensity* $\mathbf{H}$ represent

---

by attaching a nonzero k-covector to each submanifold tangent space does not depend on the way $\mathcal{S}$ is embedded into $\mathcal{M}$. It is labelled as *inner*, to tell it apart from the *outer* orientation $\mathcal{S}$ may have as a submanifold of $\mathcal{M}$. At each $\boldsymbol{e} \in \mathcal{S}$, the outer orientation of the tangent space $\mathsf{T}_{\boldsymbol{e}}\mathcal{S}$ may be characterized by a *normal* $(m-k)$-covector $\nu_{\boldsymbol{e}}$, any *positive* multiple of it providing the same orientation. A normal $(m-k)$-covector is any *nonzero* $(m-k)$-covector $\nu_{\boldsymbol{e}}$ such that

$$\langle\, \mathsf{u} \lrcorner \nu_{\boldsymbol{e}}, \mathsf{w} \,\rangle_{(m-k-1)} \equiv \langle\, \nu_{\boldsymbol{e}}, \mathsf{u} \wedge \mathsf{w} \,\rangle_{(m-k)} = 0$$

for each $(m-k-1)$-cell w at $\boldsymbol{e}$ and for all 1-cell (*i.e.*, vector) $\mathsf{u} \in \mathsf{T}_{\boldsymbol{e}}\mathcal{S}$. The duality pairing between r-covectors and r-vectors (r-cells) is the basic algebraic structure underlying integration; the *interior* ("hook" or "edge") multiplication $\lrcorner$ is dual to the *exterior* ("wedge") multiplication $\wedge$ (which I take for granted). Since the $(m-k)$-covector $\nu_{\boldsymbol{e}}$ vanishes on all $(m-k)$-cells that are not transversal to the k-plane $\mathsf{T}_{\boldsymbol{e}}\mathcal{S}$, it has only one strict component (nonnull by hypothesis); hence, $\mathsf{T}_{\boldsymbol{e}}\mathcal{S}$ may be outer-oriented in two distinct ways. Local (and conditional global) outer-orientability of $\mathcal{S}$ follows as for inner-orientability.

Odd k-forms are tailored to be integrated on k-submanifolds endowed with *outer* orientation, irrespective of whether they are inner-oriented or not. In particular, odd m-forms—also called *densities*—can be integrated on the whole of $\mathcal{M}$ also when $\mathcal{M}$ is not inner-orientable (any manifold is outer-orientable as a submanifold of itself). Let $\widetilde{\varphi} := (\varphi, \Omega)$ be an odd k-form on $\mathcal{M}$, with φ an even k-form and Ω a collection of local orientations of $\mathcal{M}$; let c be a nondegenerate k-cell lying in $\mathsf{T}_{\boldsymbol{e}}\mathcal{S}$, $\omega_{\boldsymbol{e}}$ the orientation imparted by Ω to $\mathsf{T}_{\boldsymbol{e}}\mathcal{M}$, and $\nu_{\boldsymbol{e}}$ the normal $(m-k)$-covector providing $\mathsf{T}_{\boldsymbol{e}}\mathcal{S}$ with an outer orientation. Then, the integration of $\widetilde{\varphi}$ on $\mathcal{S}$ is based on the following building block:

$$\langle\, \widetilde{\varphi}|_{e}, \mathsf{c} \,\rangle_k := \operatorname{sgn}(\nu_{\boldsymbol{e}}/(\mathsf{c} \lrcorner \omega_{\boldsymbol{e}}))\, \langle\, \varphi|_{e}, \mathsf{c} \,\rangle_k \,.$$

The ratio between $\nu_{\boldsymbol{e}}$ and the interior product $\mathsf{c} \lrcorner \omega_{\boldsymbol{e}}$ makes sense, since $\mathsf{c} \lrcorner \omega_{\boldsymbol{e}}$ is a normal $(m-k)$-covector to $\mathcal{S}$ at $\boldsymbol{e}$, hence a nonnull multiple of $\nu_{\boldsymbol{e}}$. Clearly, $\langle\, \widetilde{\varphi}|_{e}, -\mathsf{c} \,\rangle_k = \langle\, \widetilde{\varphi}|_{e}, \mathsf{c} \,\rangle_k$, while $\langle\, \varphi|_{e}, -\mathsf{c} \,\rangle_k = -\langle\, \varphi|_{e}, \mathsf{c} \,\rangle_k$. The normal covector $\nu_{\boldsymbol{e}}$ may be conveniently normalized to the *scaled* normal covector $\hat{\nu}_{\boldsymbol{e}}$ such that

$$\operatorname{sgn}(\nu_{\boldsymbol{e}}/(\mathsf{c} \lrcorner \omega_{\boldsymbol{e}})) = \operatorname{sgn}(\hat{\nu}_{\boldsymbol{e}}/(\mathsf{c} \lrcorner \omega_{\boldsymbol{e}})) = \hat{\nu}_{\boldsymbol{e}}/(\mathsf{c} \lrcorner \omega_{\boldsymbol{e}}) \,.$$

[14] I am not especially satisfied with my nomenclature (inspired by [17,24]). In fact, it seems that everybody in electromagnetism is unhappy with its terminology [25].

respectively the co-space and co-time components of the odd charge-current potential 2-form $\widetilde{\mathsf{G}}$; the vector-valued *electric field intensity* **E** and *magnetic flux density* **B** represent respectively the co-time and co-space components of the even electromagnetic 2-form F; the *scalar potential* V and the *vector potential* **A** represent respectively the co-time and co-space components of the even electromagnetic potential 1-form A.

It should be stressed that all of the above scalar fields and (space) vector fields are but proxies of the thing-in-itself. Much structural information is obliterated when those pallid substitutes are taken at face value: in fact, only **J** and **D** should be regarded as true vector fields: **A** and **E** are *co*vector fields, **B** is a *pseudo*vector field, and **H** is a *pseudoco*vector field in Riemannian disguise.[15] These distinctions were well known to Maxwell [26]. However, they never entered the physics vulgate—or evaporated early on.

The electric and magnetic fields **E**, **B**, *et cetera* can obviously be extended to any space dimension—as bare vector fields. This is a futile exercise, however, since the delicate underlying structure can *not* be exported, as I now formally state.

Proposition 1. *Let take for granted that the time manifold is 1-dimensional. Then, the only alternating multilinear forms having the same number of strict* co-time *and strict* co-space *components are those sitting exactly halfway from the extremes: if space has even dimension, there are none; if space has dimension* $2k-1$ (*with* k *a positive integer*), k*-forms (and no others) will do.*

Proof. A k-form in m dimensions has $N = m!/((m-k)!\,k!\,)$ strict components. If space has dimension $m-1$, k-forms have therefore N strict components (co-time plus co-space: $N = N_t + N_s$) and $N_s = (m-1)!/((m-1-k)!\,k!\,)$ strict co-space components. Now, $N/N_s = m/(m-k)$; hence, $N_t = N_s \Leftrightarrow m = 2\,k$. □

Remark 1. Prop. 1 proves that the crucial *electric-magnetic duality*—the key experimental finding by Faraday—can only exist in an odd-dimensional space manifold (paired with a one-dimensional time manifold). Such a duality has far-reaching extensions beyond classical electromagnetism (see, *e.g.*, [27–30]).

[15] In fact, the representation of the co-space component of the electromagnetic potential form via the vector potential **A** and of the co-time component of the electromagnetic form via the electric field intensity **E** depends upon the identification between vector and covector fields brought about by the Riemannian structure imparted to the space manifold by a metric g. The co-space component of the electromagnetic form is represented by the magnetic flux density **B** in a way mediated by the introduction of (local) volume forms μ on the space manifold (see note 13 on page 15); whence the *pseudo-* prefix. Finally, to represent the co-time component of the charge-current potential form through the magnetic field intensity **H** both volume and metric structures are needed, whence the compound *pseudoco-* prefix. The dichotomy *true/pseudo-* is the same as *polar/axial*.

Proposition 2. *Let space have odd dimension $n_s = 2k-1$ (with k a positive integer). Then, k-forms have exactly n_s strict co-space components (and n_s strict co-time components, because of Prop. 1) if and only if $k < 3$.*

Proof. A k-form has $N_s = n_s!/(n_s - k)!\,k! = (2k-1)!/(k-1)!\,k!$ strict components. Hence, $N_s = n_s$ if and only if

$$\frac{(2k-1)!}{(k-1)!\,k!} = 2k-1 \quad \Leftrightarrow \quad \frac{(2k-2)!}{(k-1)!\,k!} = 1\,.$$

After letting $j = k-1$, the problem may be conveniently rephrased as follows:

$$\text{Find} \quad j \geq 0 \quad \text{such that} \quad \frac{(2j)!}{(j+1)!} = j!\,. \tag{1}$$

Checking that 0 and 1 solve Prob. (1) is straightforward. For $j > 1$,

$$\frac{(2j)!}{(j+1)!} = \prod_{i=2}^{j}(j+i)\,, \qquad j! = \prod_{i=2}^{j} i\,,$$

and the two products (each of which has $j-1$ factors) cannot be equal, since each factor of the first is larger than the corresponding factor of the second. This proves that Prob. (1) has no solution greater than 1. □

Remark 2. Prop. 2, together with Prop. 1, proves that the co-time component of the electromagnetic form can be represented by a single (space) vector field (the electric field intensity **E**), and the co-space component by another one (the magnetic flux density **B**), if and only if the space dimension is odd and small, namely, 1 or 3. If space had dimension 5, $2+2$ (space) vector fields would be needed to represent the electromagnetic 3-form; if $n_s = 7$, the electromagnetic 4-form would have as counterpart a host of $5+5$ (space) vector fields, and so on.

Proposition 3. *An electromagnetic theory possessing the properties stated in Remark 2 is physically unsound if $k = 1$.*

Proof. Because of Prop. 2, $k = 1$ implies $n_s = 1$. If space dimension were 1, electricity and magnetism would not couple in empty space, electromagnetic waves would not travel through it, and the *Poynting vector field* $\mathbf{E} \times \mathbf{H}$ (another true vector field, like **J** and **D**) would be identically null. □

Remark 3. Summing up, space dimension has to be odd and small, but not too small: Maxwell's play can only be staged on a (3+1)-dimensional space-time.

The basic structure of mechanical field theory

Let me now transcribe Lamé's play into the same unifying language. The first major change in the stage setting is that mechanical quantities are associated with cells embedded in an$(n+1)$-dimensional *body-time* manifold, defined as the product of an n-dimensional *body* manifold $\mathscr{B}$ ($0 \leq n \leq n_s = 3$) times a 1-dimensional time line $\mathscr{T}$. It should be noted that introducing a body manifold does not privilege any observer. Singling out a time line from the space-time continuum does call for an observer, however.[16] The cases $n = 1, 2$ describe respectively corporeal curves and surfaces (Lamé's « fils minces et membranes peu épaisses » quoted on page 7). Following Lamé, I concentrate here on the mechanics of space-filling bodies ($n = n_s = 3$). Even in this case it is of the essence to keep body quite distinct from space. Not to confuse the issue, I distinguish carefully between elements of the space manifold $\mathscr{S}$—which I call *places*—and elements of the body manifold $\mathscr{B}$—which I call *points*.

Body-time cells are isomorphic to space-time cells, so only terminology needs adapting. In particular, space cells translate into *body cells*. A 0-cell is now a *point* times an instant, a body 1-cell a *line element*, a body $(n-1)$-cell a *facet*, and a body n-cell a *bulk element*. At the top of the cell hierarchy sit body-time lumps (bulk elements times time lapses). Distributions of physical quantities associated with k-dimensional body-time cells are gauged by k-forms, endowed with *co*-body and *co*-time components.[17] The important difference with respect to electromagnetic field theory is that most mechanical quantities—and hence the corresponding forms—are *not* real-valued.[18]

The fundamental mechanical descriptor is the (even) *placement* 0-*form* $\boldsymbol{p}$, a *place*-valued field attaching to each point at each instant—*i.e.*, to each 0-cell—a place in space:

$$\boldsymbol{p} : \mathscr{B} \times \mathscr{T} \to \mathscr{S} . \tag{2}$$

At variance with electromagnetic forms, the very definition of placement calls for an observer: a different observer sees a *corresponding* placement, as decreed by the action of the group of changes in observer on space-time. The restriction of a placement to all *simultaneous* 0-cells is required to be an *embedding*.[19] The (exterior) differential of the placement 0-form is the (*space*)

[16] A *proper*-time line may be attached to each body point independently of any observer. But an observer is required to trivialize the proper-time bundle, *i.e.*, to equate time lines attached to different body points [19, Sect. 1.4].

[17] Body-conjugate and time-conjugate components (*cf.* footnote 12 on page 15).

[18] The notion of vector-valued forms surfaces in a passing remark—entitled "A glimpse of other physical theories"—in [31]. Vector- and covector-valued forms are explicitly introduced in [32, 33], where their use is rightly advocated for the evaluation of electromagnetic forces. The way these papers treat mechanics is, however, far from satisfactory.

[19] Any such embedding may be adopted to pull spatial structures back onto the body, or to push body structures forward into space. The ingrained habit to grant undue privileges to one embedding or another is cause of perennial confusion.

tangent bundle-valued *displacement* 1-*form* $\mathbf{d}\boldsymbol{p}$, such that

$$\langle \mathbf{d}\boldsymbol{p}\,, \boldsymbol{X}\rangle = \mathcal{L}_{\boldsymbol{X}}\, \boldsymbol{p} \tag{3}$$

for all vector field $\boldsymbol{X}$ tangent to the body-time manifold $\mathscr{B}\times\mathscr{T}$, $\mathcal{L}_{\boldsymbol{X}}$ denoting *Lie differentiation* along the vector field $\boldsymbol{X}$. The co-time and co-body components of the (even) 1-form $\mathbf{d}\boldsymbol{p}$ are represented respectively by the *velocity* $\dot{\boldsymbol{p}}$ and the *body gradient* of placement $\boldsymbol{\nabla}\boldsymbol{p}$.

To focus on the essentials, in the following I stick to an *affine* space manifold $\mathscr{S}$ (and, where needed, to an affine space-time $\mathscr{S}\times\mathscr{T}$), whose tangent bundle has a canonical global connection, whereby all tangent spaces are trivially identified with each other. If the manifold $\mathscr{S}$ is affine, then $\mathsf{T}\mathscr{S}\simeq\mathscr{S}\times\mathsf{V}\mathscr{S}$, $\mathsf{V}\mathscr{S}$ being the *translation space* of $\mathscr{S}$, *i.e.*, the vector space that acts freely and transitively on $\mathscr{S}$. Handling *vector*-valued forms requires an easy extension of the rules valid for real-valued forms (see [34, Def. 6.3.11]). This is not sufficiently general, however: think of the curved space-time of general relativity or, in a lower key, of shallow waters flowing on a smooth but uneven rocky bed. The general notions which are needed are that of *bundle*-valued forms and *twisted* (or *covariant*) exterior differentiation (see [35, Sect. II-1]). While such refinements may be set aside in the present discussion, the moral to be drawn from them does belong to the essentials: no vector-valued *global* quantity should be allowed in the *basic* structure of any physical theory, since bundle-valued forms cannot be integrated—unless their target bundle is trivial. This criterion dooms such "basic" notions as that of *resultant force*, as I tried to explain—with mixed success—since the nineties [36], advocating and enlarging Germain's viewpoint [37,38]. See also [39] for a later contribution from Segev and Rodnay along the same lines.

In dynamics, a key role is played by *test velocities*, $\mathsf{V}\mathscr{S}$-valued 0-forms sharing the physical dimensions of $\dot{\boldsymbol{p}}$. No differential compatibility is required between a test velocity $\boldsymbol{v}$ and the placement $\boldsymbol{p}$, the equality $\boldsymbol{v}=\dot{\boldsymbol{p}}$ selecting the one velocity *realized* along $\boldsymbol{p}$. Let me emphasize that test velocities are *zero*-forms: a test velocity should be interpreted as attaching to each 0-cell $\{\mathtt{b}\}\times\{\mathtt{i}\}$ ($\mathtt{b}$ a point, $\mathtt{i}$ an instant) the difference $\boldsymbol{p}_\varepsilon(\mathtt{b},\mathtt{i})-\boldsymbol{p}(\mathtt{b},\mathtt{i})$ between the place assigned to it by a *juxtaposed* placement $\boldsymbol{p}_\varepsilon$, for vanishingly small ε, and that assigned by $\boldsymbol{p}$, it being intended that $\lim_{\varepsilon\to 0}\boldsymbol{p}_\varepsilon=\boldsymbol{p}$ (*cf.* [40]):

$$\boldsymbol{v} := \lim_{\varepsilon\to 0}\varepsilon^{-1}(\boldsymbol{p}_\varepsilon-\boldsymbol{p}) \tag{4}$$

In other words, test displacements develop in an extra subsidiary time dimension, parameterized by the *pseudo-time* ε.[20]

Work is the chief integral quantity in mechanics, associated with body-time lumps.[21] It is gauged by an *odd* $\mathbb{R}$-valued $(n{+}1)$-form, obtained as the

[20] Therefore, the equality $\boldsymbol{v}=\dot{\boldsymbol{p}}$ is trickier than it seems.

[21] Playing with the real-valued time coordinate parameterizing $\mathscr{T}$, other real-valued work-related ancillary quantities may be introduced, as the time rate of work, namely the *power* (or *working*), or its time integral, namely the *action*.

sum of exterior products of even $\mathsf{V}\mathscr{S}$-valued and odd $\mathsf{V}^*\mathscr{S}$-valued forms.[22] The basic dynamical features of each individual mechanical theory are encoded in the structure of this distinguished bilinear functional. The standard model of continuum mechanics—encompassing, in particular, Lamé's theory of three-dimensional elasticity—is founded on the assumptions I present in the following paragraphs.

The (so-called "virtual") work done on a test velocity $\boldsymbol{v}$ over an $(n+1)$-dimensional body-time cell c (provided with *external* orientation) is the sum of two contributions: an integral over the cell itself, and another over its *boundary* $\boldsymbol{\partial}\mathsf{c}$.[23] The odd real-valued $(n+1)$-form $\widetilde{w}_{\mathtt{lmp}}$ to be integrated over the lump c is the sum of two exterior products: the $\mathsf{V}\mathscr{S}$-valued 0-form $\boldsymbol{v}$

---

[22] The exterior product of an even and an odd form is odd. The exterior multiplication between vector- and covector-valued forms makes use of the duality pairing $\langle\!\langle \cdot\,, \cdot \rangle\!\rangle$ between $\mathsf{V}\mathscr{S}$ and $\mathsf{V}^*\mathscr{S}$. If $\boldsymbol{\alpha}$ is a $\mathsf{V}\mathscr{S}$-valued r-form and $\boldsymbol{\beta}$ a $\mathsf{V}^*\mathscr{S}$-valued s-form, $\boldsymbol{\alpha} \wedge \boldsymbol{\beta}$ is the $\mathbb{R}$-valued $(r+s)$-form such that

$$\langle\, \boldsymbol{\alpha} \wedge \boldsymbol{\beta}\,, \boldsymbol{X}_1 \wedge \cdots \wedge \boldsymbol{X}_{r+s} \,\rangle_{(r+s)} =$$
$$\frac{1}{r!\,s!} \sum_{\pi \in S} \epsilon(\pi)\, \langle\!\langle\, \langle\, \boldsymbol{\alpha}\,, \boldsymbol{X}_{\pi(1)} \wedge \cdots \wedge \boldsymbol{X}_{\pi(r)} \,\rangle_{(r)}\,, \langle\, \boldsymbol{\beta}\,, \boldsymbol{X}_{\pi(r+1)} \wedge \cdots \wedge \boldsymbol{X}_{\pi(r+s)} \,\rangle_{(s)} \,\rangle\!\rangle$$

for all (body-time)-vector fields $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{r+s}$ (*i.e.*, for all sections of the tangent bundle $\mathsf{T}(\mathscr{B} \times \mathscr{T}) \simeq (\mathsf{T}\mathscr{B}) \times (\mathsf{T}\mathscr{T})$). The symmetric group of degree $r+s$ is denoted S; the parity of the permutation π is denoted $\epsilon(\pi)$: $\epsilon(\pi) = +1$ if π is even, $\epsilon(\pi) = -1$ if π is odd.

[23] I consider the notion of *boundary operator* $\boldsymbol{\partial}$ as primitive, and that of *exterior derivative* $\mathbf{d}$ as derived from it by duality: the first applies to multivector fields, the second to differential forms, and they are dual to each other with respect to the duality pairing between r-forms and r-vector fields (*cf.* note 13 on page 15):

$$\langle\, \mathbf{d}\varphi, \mathsf{c} \,\rangle_{(k)} = \langle\, \varphi, \boldsymbol{\partial}\mathsf{c} \,\rangle_{(k-1)}$$

for each $(k-1)$-form φ and k-vector field c. The boundary of a parallelepipedal k-cell, *i.e.*, a cell whose edges are defined by *commuting* vector fields $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_k$, consists of k pairs of $(k-1)$-cells: by definition,

$$\langle\, \varphi, \boldsymbol{\partial}(\boldsymbol{X}_1 \wedge \cdots \wedge \boldsymbol{X}_k) \rangle = \sum_{\ell=1}^{k} (-1)^{\ell-1} \mathcal{L}_{X_\ell} \langle\, \varphi, \boldsymbol{X}_1 \wedge \cdots \wedge \hat{\boldsymbol{X}}_\ell \wedge \cdots \wedge \boldsymbol{X}_k \rangle$$

for each $(k-1)$-form φ ($\hat{\boldsymbol{X}}_\ell$ denotes that $\boldsymbol{X}_\ell$ is deleted). The boundary of a quasi-parallelepipedal k-cell, built out of k *non* commuting vector fields, includes also $\binom{k}{2}$ "smaller"$(k-1)$-cells, whose contribution to the boundary integral is accounted for by the sum

$$\sum_{1 \le i < j \le k} (-1)^{i+j} \langle\, \varphi, (\mathcal{L}_{X_i} \boldsymbol{X}_j) \wedge \boldsymbol{X}_1 \wedge \cdots \wedge \hat{\boldsymbol{X}}_i \wedge \cdots \wedge \hat{\boldsymbol{X}}_j \wedge \cdots \wedge \boldsymbol{X}_k \rangle\,,$$

to be added to the right side of the previous equality. This is Palais' definition of $\mathbf{d}$ in terms of $\mathcal{L}$ [41], translated by duality into a definition of $\boldsymbol{\partial}$; see [34, Sect. 7.4].

times the *impulse-supply form* $\widetilde{\boldsymbol{\varsigma}}$ (an odd $\mathsf{V}^*\mathscr{S}$-valued $(n+1)$-form) plus the *opposite*[24] of the differential $\mathbf{d}\boldsymbol{v}$ (a $\mathsf{V}\mathscr{S}$-valued 1-form) times the *impulse-flux form* $\widetilde{\boldsymbol{\varphi}}$ (an odd $\mathsf{V}^*\mathscr{S}$-valued n-form):

$$\widetilde{w}_{\mathtt{lmp}} = \boldsymbol{v} \wedge \widetilde{\boldsymbol{\varsigma}} - (\mathbf{d}\boldsymbol{v}) \wedge \widetilde{\boldsymbol{\varphi}} \,. \tag{5}$$

The odd real-valued n-form $\widetilde{w}_{\mathtt{bry}}$ to be integrated over the cell boundary $\boldsymbol{\partial}\mathsf{c}$ is the exterior product of $\boldsymbol{v}$ times the *boundary-impulse form* $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}}$ (an odd $\mathsf{V}^*\mathscr{S}$-valued n-form living on $\boldsymbol{\partial}\mathsf{c}$):

$$\widetilde{w}_{\mathtt{bry}} = \boldsymbol{v} \wedge \widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}} \,. \tag{6}$$

The impulse-supply form $\widetilde{\boldsymbol{\varsigma}}$ has a single ($\mathsf{V}^*\mathscr{S}$-valued) strict component, the *bulk force* per unit body volume $\mathbf{b}$; the co-body and co-time components of the impulse-flux form $\widetilde{\boldsymbol{\varphi}}$ are represented respectively by the *momentum* per unit body volume $\mathbf{m}$ (a $\mathsf{V}^*\mathscr{S}$-valued field) and the *stress* $\mathbf{S}$ (a $\mathsf{T}\mathscr{B} \otimes \mathsf{V}^*\mathscr{S}$-valued field, *i.e.*, a section of the tensor product bundle of body (pseudo-)vectors times space covectors).[25] The boundary-impulse form $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}}$, being an n-form on an n-manifold, has a single strict component. On a time-dipped n-cell (*i.e.*, a facet f times a time lapse $\mathtt{l}$) the strict component of $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}}$ is the ($\mathsf{V}^*\mathscr{S}$-valued) *traction* $\mathbf{t}_{\mathsf{f}}$; on a body n-cell (*i.e.*, a bulk element b times an instant $\mathtt{i}$) the strict component of $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}}$ is the ($\mathsf{V}^*\mathscr{S}$-valued) *impulse* per unit body volume $\mathbf{i}$. All of the above vector- or tensor-valued densities are representations mediated by the choice of a collection of local volume forms on the body manifold $\mathscr{B}$ and local area forms on each body surface (*i.e.*, on all 1-codimensional submanifolds of $\mathscr{B}$).[26] Different choices affect densities in such a way that the forms they represent (namely, $\widetilde{\boldsymbol{\varsigma}}, \widetilde{\boldsymbol{\varphi}}$, and $\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\partial}\mathsf{c}}$) stay the same.

An overriding *balance principle* [36–38, 40, 43] commands that the total ("virtual") work done on any test velocity over any body-time lump should be zero: for each test velocity $\boldsymbol{v}$ and for all body-time cell c,

$$\langle\, \widetilde{w}_{\mathtt{lmp}}, \mathsf{c} \,\rangle_{(n+1)} + \langle\, \widetilde{w}_{\mathtt{bry}}, \boldsymbol{\partial}\mathsf{c} \,\rangle_{(n)} = 0 \;. \tag{7}$$

Since body-vector fields (*i.e.*, sections of $\mathsf{T}\mathscr{B}$) commute with time-vector fields (*i.e.*, sections of $\mathsf{T}\mathscr{T}$), the boundary $\boldsymbol{\partial}\mathsf{c}$ of a body-time cell $\mathsf{c} = \mathtt{l} \wedge \mathsf{b}$ ($\mathtt{l}$ a time lapse, b a bulk element) consists of the boundary of $\mathtt{l}$ (two oriented instants) times b minus $\mathtt{l}$ times the boundary of b:

$$\boldsymbol{\partial}(\mathtt{l} \wedge \mathsf{b}) = (\boldsymbol{\partial}\mathtt{l}) \wedge \mathsf{b} - \mathtt{l} \wedge \boldsymbol{\partial}\mathsf{b} \,.$$

If the bulk element b is parallelepipedal, its boundary $\boldsymbol{\partial}\mathsf{b}$ consists of $2n$ facets.

[24] There is nothing deep here: the minus sign in (5) is simply the most convenient—and most common—choice. The rationale behind it is best caught in the context of higher grade theories: see footnote 27 on page 23.

[25] More details and motivations for a treatment of stress dispensing with all "reference configuration" may be found in [36,42]. See also note 19 on page 19 and [39].

[26] Volume forms on $(n-1)$-dimensional submanifolds are aptly called *area* forms. Note also that I take for granted a parameterization of the time line, as usual.

Since

$$\mathbf{d}(\boldsymbol{v}\wedge\widetilde{\boldsymbol{\varphi}}) = (\mathbf{d}\boldsymbol{v})\wedge\widetilde{\boldsymbol{\varphi}} + (-1)^0\,\boldsymbol{v}\wedge(\mathbf{d}\widetilde{\boldsymbol{\varphi}}) \tag{8}$$

and

$$\langle\,\mathbf{d}(\boldsymbol{v}\wedge\widetilde{\boldsymbol{\varphi}}), \mathsf{c}\,\rangle_{(n+1)} = \langle\,\boldsymbol{v}\wedge\widetilde{\boldsymbol{\varphi}}\,, \boldsymbol{\partial}\mathsf{c}\,\rangle_{(n)}\,, \tag{9}$$

standard localization arguments prove that the balance principle (7) is equivalent to the following two conditions: (i) the impulse-supply form and the exterior derivative of the impulse-flux form should add up to the null $\mathsf{V}^*\mathscr{S}$-valued $(n+1)$-form:

$$\widetilde{\boldsymbol{\varsigma}} + \mathbf{d}\widetilde{\boldsymbol{\varphi}} = \mathbf{0} \tag{10}$$

in $\mathscr{B}\times\mathscr{T}$; and (ii) the boundary-impulse form on any n-cell should match with the unique impulse-flux form:

$$\widetilde{\boldsymbol{\beta}}_{\boldsymbol{\iota}(\mathscr{F})} = \boldsymbol{\iota}^*\,\widetilde{\boldsymbol{\varphi}} \tag{11}$$

for each embedding $\boldsymbol{\iota} : \mathscr{F} \hookrightarrow \mathscr{B}\times\mathscr{T}$ of a prototype n-manifold $\mathscr{F}$ into the $(n+1)$-dimensional body-time manifold. In other words, the boundary-impulse form on $\boldsymbol{\iota}(\mathscr{F})$ should be the pull-back of the impulse-flux form $\widetilde{\boldsymbol{\varphi}}$ by the embedding $\boldsymbol{\iota}$. Equation (11) constitutes a body-time version of the celebrated *Cauchy stress theorem.*

In comparison with Maxwell–Faraday's gossamer edifice, continuum mechanics has much more robustness than fineness in it: granted that time is one-dimensional, its basic structure can easily accommodate for any space dimension. Also body dimension may be chosen freely—provided it does not exceed space dimension.[27]

[27] *Extended placements* are often introduced in neo-classical continuum mechanics, by adding an *order parameter* of some sort to mere place in space:

$$\boldsymbol{p} : \mathscr{B}\times\mathscr{T} \to \mathscr{S}\times\mathscr{O}\,,$$

where $\mathscr{O}$ is the manifold spanned by the selected order parameter. Test velocities are extended accordingly. The requirement to be an embedding now applies to the *space projection* of the restriction of $\boldsymbol{p}$ to all simultaneous 0-cells $(\boldsymbol{\pi}_{\mathscr{S}}\circ\boldsymbol{p})(\,\cdot\,,\mathtt{i})$, for all $\mathtt{i}\in\mathscr{T}$. In other terms, the topological properties of the body manifold $\mathscr{B}$ are modelled after those of the space manifold $\mathscr{S}$, independently of $\mathscr{O}$.

A distinct extension—possibly allied with the previous one—is to *higher grade* theories, where the work density at $(\mathsf{b},\mathtt{i})$ is allowed to depend on the value at $(\mathsf{b},\mathtt{i})$ of higher body-time gradients of $\boldsymbol{v}$, namely, of all $\mathbf{D}^j\boldsymbol{v}$ $(0\le j\le k>1)$, not only of $\mathbf{D}^0\boldsymbol{v}=\boldsymbol{v}$ and $\mathbf{D}^1\boldsymbol{v}=\mathbf{d}\boldsymbol{v}$, as in (5). Since $\mathbf{D}=\mathbf{d}$ only on 0-forms, the theory is extended by using recursively the following elementary properties: (i) $\mathbf{D}^j\boldsymbol{v} := \mathbf{D}\,(\mathbf{D}^{j-1}\boldsymbol{v})$, and (2) $\mathbf{D}\,(\mathbf{D}^{j-1}\boldsymbol{v}) = \mathbf{d}\,(\mathbf{D}^{j-1}\boldsymbol{v})$ if $\mathbf{D}^{j-1}\boldsymbol{v}$ is regarded as a 0-form taking values in the tensor bundle $\mathfrak{T}_j := \big(\bigotimes^j\mathsf{T}^*(\mathscr{B}\times\mathscr{T})\big)\otimes\mathsf{V}\mathscr{S}$. Eq. (5) generalizes to

$$\widetilde{w}_{\mathtt{imp}} = \boldsymbol{v}\wedge\widetilde{\boldsymbol{\varsigma}} + \sum_{j=1}^{k}(-1)^j\,(\mathbf{d}\,(\mathbf{D}^{j-1}\boldsymbol{v}))\wedge\widetilde{\boldsymbol{\varphi}}_j\,,$$

where the j-th *hyper*-impulse-flux form $\widetilde{\boldsymbol{\varphi}}_j$ is an odd $\mathfrak{T}^*_j$-valued 1-form. The assumption (6) for $\widetilde{w}_{\mathtt{bry}}$ is generalized accordingly, but I won't go into that here.

Constitutive issues

Neither the mechanical nor the electromagnetic field theories given above are complete. To complete them, the response of the medium has to be characterized. Luckily enough, the electromagnetic response of ether—or empty space, if you prefer—is stupendously simpler than the mechanical response of the simplest earthly materials.[28] Things get suddenly complicated—and surprisingly unexplored—when the "classical" electromagnetic response of material media comes into play.

Electromagnetic-mechanical coupling

On the surface, coupled electromagnetic-mechanical problems are mostly taken as issues in computer-aided design—of a quite difficult nature, admittedly.[29] This is no surprise, thanks to the lasting division between Maxwell's and Lamé's disciples, further aggravated in their pet computer codes, laboriously developed by niche experts.

However, the heart of the matter lies much deeper. First of all, lumping electromagnetic and mechanical effects into *distinct* components of the same device is not always possible. Therefore, loose coupling between preexistent specialized simulators may not suffice. Second, and more important: the electromagnetic and the mechanical response of a medium cannot in general be characterized independently of each other. Therefore, a unified understanding of both disciplines cannot be dispensed with—at least in principle.

Before discussing the gist of this issue, let us revisit our two heroes of old.

Did Maxwell understand Lamé's theory of elasticity?

Unquestionably he did. He has an explicit—and vaguely critical—reference to Lamé's treatise in his fundamental paper [46] on linearly elastic frames:

> [T]he theory of the equilibrium and deflections of frameworks subjected to the action of forces is sometimes considered as more complicated than it really is, especially in cases in which the framework is not simply stiff, but is strengthened (or weakened as it may be)

[28] But somewhat less special than it may seem at first, since it determines the geometry of space-time—namely, its conformal Lorentzian structure (see [44] and [18, Sect. 11.3.2]).

[29] Such difficulties were vividly reported at the conference. Unfortunately, none of the talks most relevant to this issue ("Continuous simulation of coupled systems" by P. Schwarz and "Coupled simulation of electromagnetic fields and mechanical deformation" by K. Rothemund and U. van Rienen) are documented in the proceedings. However, one of these authors has a nearly identical title [45] published in the proceedings of the next SCEE-conference.

> by additional connecting pieces.[30] I have therefore stated a general method of solving all such questions in the least complicated manner. The method is derived from the principle of the Conservation of Energy, and is referred to in Lamé's *Leçons sur l'Elasticité, leçon 7ème*, as Clapeyron's Theorem; but I have not yet seen any detailed application of it.

A modicum of chronology is in order here: Maxwell's paper [46] was published in 1864, that is, twelve years after the first edition of Lamé's treatise [1], one year before the culminating paper [4] of his own electromagnetic trilogy.

In the terse pages of this paper—six in all—Maxwell made two major contributions: the celebrated reciprocity theorem bearing his name (together with that of Betti, who improved its formulation in 1872), and the so-called "method of forces"—the "general method" he mentions in the quote above—which is most often related to the names of Castigliano, Müller-Breslau and Mohr. The paper received little attention, and later authors struggled to reach the same conclusions without the benefit of Maxwell's work [7, Sects. 15.2-3].

Ten years earlier, in 1854, Maxwell had already published a juvenile paper on elasticity [47], where he cites, among others, Lamé's *Cours de Physique* [48] and the memoir by Lamé and Clapeyron that had appeared on Crelle's Mathematical Journal in 1852. The introductory paragraphs are worth quoting:

> There are few parts of mechanics in which theory had differed more from experiment than in the theory of elastic solids.
>
> Mathematicians, setting out from very plausible assumptions with respect to the constitution of bodies, and the laws of molecular action, came to conclusions which were shewn to be erroneous by the observations of experimental philosophers. The experiments of Œrsted proved to be at variance with the mathematical theories of Navier, Poisson, and Lamé and Clapeyron, and apparently deprived this practically important branch of mechanics of all assistance from mathematics.
>
> The assumption on which these theories were founded may be stated thus:—
>
> *Solid bodies are composed of distinct molecules, which are kept at a certain distance from each other by the opposing principles of attraction and heat. When the distance between two molecules is changed, they act on each other with a force whose direction is in the line joining the centres of the molecules, and whose magnitude is equal to the change of distance multiplied into a function of the distance which vanishes when that distance becomes sensible.*
>
> The equations of elasticity deduced from this assumption contain only *one* coefficient, which varies with the nature of the substance.
>
> The insufficiency of one coefficient may be proved from the existence of bodies with different degree of solidity.

[30] That is to say *redundant*, in the current terminology of structural mechanics.

Did Lamé understand Maxwell's electromagnetic theory?

I do not know whether Lamé was aware of Maxwell's work on electromagnetism—but I suppose he was not. In any event, if he had heard of it, he would have shared the poor opinion held by most contemporary scientists. In the words of Dyson [8]:

> To his contemporaries, Maxwell's theory was only one of many theories of electricity and magnetism. It was difficult to visualise, and it did not have any clear advantage over other theories that described electric and magnetic forces in Newtonian style as direct action at a distance between charges and magnets. It is no wonder that few of Maxwell's contemporaries made the effort to learn it.
>
> [... T]he importance of Maxwell's work was not obvious to his contemporaries. For more than twenty years, his theory of electromagnetism was largely ignored. [...] It was regarded as an obscure speculation without much experimental evidence to support it. The physicist Michael Pupin[31] [...] travelled from America to Europe in 1883 in search of somebody who understood Maxwell. [...] Pupin went first to Cambridge and enrolled as a student, hoping to learn the theory from Maxwell himself. He did not know that Maxwell had died four years earlier. After learning that Maxwell was dead, he stayed on in Cambridge and was assigned to a college tutor. But his tutor knew less about the Maxwell theory than he did, and was only interested in training him to solve mathematical tripos problems. He was amazed to discover, as he says, "how few were the physicists who had caught the meaning of the theory, even twenty years after it was stated by Maxwell in 1865." Finally he escaped from Cambridge to Berlin and enrolled as a student with Hermann von Helmholtz. Helmholtz understood the theory and taught Pupin what he knew.

As implicitly recalled in the above account, Maxwell died in 1879. He was forty-eight years old. Lamé had died much older—nearly seventy-five—nine years earlier, in 1870 on May Day.[32]

---

[31] Michael Idvorsky Pupin, an epitomic personality of the making of America: born of a poor Serbian family in Idvor, Banat (in Hungary at that time), at the age of sixteen he emigrated to America, where he worked at odd jobs, studying at night to prepare himself for admission to Columbia, where he graduated in 1883. After being awarded a doctorate in Berlin in 1899, he returned to America and served as professor of electro-mechanics at Columbia until 1931, achieving positions of influence and power with captains of industry by his inventions and discoveries.

[32] « La mort lui épargna du moins les angoisses qu'une guerre funeste réservait au cœur des pères. » (from the obituary published on the Annales des Mines in 1872).

A sound basis for the electrodynamics of deformable media

"[T]he issue of force densities in material media is the most controversial, the least investigated, and the least understood topic of classical electromagnetism." This is how it is put by the editor of the Academic Press Electromagnetism series, Isaak Mayergoyz, in his foreword to the book [49] by Bobbio, where several specific materials are detailed: polarized dielectric fluids, amorphous and crystalline solid dielectrics (including piezo- and pyroelectric crystals), and magnetically polarized (nonhysteretic) materials. In all cases, the key point is that sound constitutive assumptions for the free-energy density—ascribed to matter—and for the electromagnetic energy—ascribed to ether[33]—can *not* be laid down independently of each other.[34]

In the case of an electrically linear fluid dielectric, it is shown that Helmholtz's and Kelvin's formulas for the *electric* force density—which notoriously disagree—correspond to a different splitting of the same total energy density into ether- and matter-related terms—hence, to different prescriptions for the *mechanical* force density, *i.e.*, for pressure. A long-lasting—and quite idle—controversy between supporters of either formula is dispelled this way.[35]

The discussion of permanent magnets by Henrotte and Hameyer [51, Sect. 3] is also a case in point. Assuming that the magnetization vector field is the Euclidean proxy for a 1-form or for a 2-form does not change the **H**–**B** relation, nor the expression for the magnetic energy density. However, the two assumptions do lead to different expressions for the *Maxwell stress tensor*, hence to different electromagnetic forces. The physical interpretation by the authors reads as follows:

> There is [. . .] no mathematical reason to favour one of these expressions. The first one might be better, for instance, when the magnetisation is actually due to the presence of microscopic magnetic dipoles, whereas the second one might better fit a magnetization due to microscopic flux carriers (such as Abrikosov vortices in HTc Type II superconductors).

Another surprisingly persistent disagreement, still unresolved, is the *Abraham-Minkowski controversy*, concerning electromagnetic momentum within dielectric media. The debate is whether the Abraham expression of electromagnetic momentum $\mathbf{E} \times \mathbf{H} / c^2$ (with c the speed of light) or the Minkowsky expression $\mathbf{D} \times \mathbf{B}$ is appropriate. Of course, the German mathematician

---

33 This is my own wording, inspired also by the recent mind-teasing contribution [50] by Ericksen. Bobbio uses the neuter term "field".

34 This point is touched upon also in [51, Sect. 2], where Henrotte and Hameyer write: "In order to define specifically *electromagnetic forces*, we have to use *electromagnetic* energy functionals instead of *total* energy functionals." Then they admit that defining "such restricted functionals" is not at all obvious, *e.g.*, for a magnetostrictive material.

35 See [49, Sect. 4.7] and [52] for a review of the experimental side of the dispute.

Hermann Minkowsky (1864–1909) and the German physicist Max Abraham (1875–1922) agreed perfectly well in vacuum. It is interesting—and somewhat ironic—to read the following paragraphs from the conclusions of a recent NASA inquiry into emerging prospects for future spaceflight [53]:[36]

> *3.2.1. Slepian-Drive.* Funded through a Congressional earmark, the West Virginia Institute for Scientific Research (ISR) is conducting experimental and theoretical assessments of the propulsive implications of electromagnetic momentum in dielectric media. The equations that describe electromagnetic momentum in vacuum are well established (photon radiation pressure), but there is still scientific debate concerning momentum within dielectric media, specifically the "Abraham-Minkowski controversy." [. . .] An independent assessment by the Air Force Academy concluded that no *net* propulsive forces are expected with this approach [. . .].
>
> Separate from the ISR work, independent research published by Dr. Hector Brito details a propulsive device along with experimental data [*reference omitted*]. The signal levels are not sufficiently above the noise as to be conclusive proof of a propulsive effect.
>
> While not specifically related to propulsion, a recent journal article assessed the Abraham-Minkowski controversy from a quantum physics perspective, suggesting it might be useful for micro-fluidic or other applications [*see* [55] *in my list of references*].
>
> In all of these approaches, the anticipated forces are relatively small, and critical issues remain unresolved. In particular, the conversion of *oscillatory* forces to *net* forces (Slepian-Drive) remains questionable, and the issue of generating *external* forces from different *internal* momenta remains questionable. Even if not proven suitable for propulsion, these approaches provide empirical tools for further exploring the Abraham-Minkowski controversy of electromagnetic momentum. This topic is considered unresolved.

The basic structure underlying Maxwell's stress tensor

The Maxwell stress tensor—*i.e.*, the (opposite of) the cospace-cospace component of the electromagnetic *energy-momentum tensor*—represents the fundamental coupling between electromagnetism and mechanics.[37] As a matter of fact, it represents even more than that. As Eshelby wrote in his recapitulation paper [56],

[36] See also a very recent paper by Ido [54], where the equations for a non-conducting magnetic fluid are obtained under the Abraham assumption, and compared with those obtained in previous papers by the same author and others under the Minkowski assumption. The outcome is that the two assumptions entail negligible differences unless the magnetic field fluctuates at high frequency.

[37] I need not belabour this point, which is so neatly expounded by Henrotte and Hameyer in [51]. I agree with them and dissent from [43, Sect. 4.2].

> [t]he archetypal energy-momentum tensor is Maxwell's stress tensor in electrostatics [...]. The writer, having looked at a book on field theory, felt that the force on a defect ought to be given by a similar expression involving the energy-momentum tensor appropriate to the elastic field.

More recently, this tensor—now currently labelled with Eshelby's name—has been given a status well beyond hyperelasticity and other merely energetic theories (see my own papers [42, 57] and the tract [58] by Gurtin).

The basic structure underlying Maxwell's stress tensor is readily accessible via the mathematical apparatus developed so far. Let us consider a placement $\boldsymbol{p}$ as defined in (2). I find now convenient to associate with $\boldsymbol{p}$ the mapping

$$\begin{array}{ccc} \mathscr{B} \times \mathscr{T} & \hookrightarrow & \mathscr{S} \times \mathscr{T} \\ (\mathsf{b}, \mathtt{i}) & \mapsto & (\boldsymbol{p}(\mathsf{b}, \mathtt{i}), \mathtt{i}) \,, \end{array} \tag{12}$$

which embeds body-time into space-time. By a slight abuse of language, I will denote also this trivial extension by $\boldsymbol{p}$. Let c be a *body*-time k-cell, and ϑ a *space*time-conjugate k-form (Fig. 3). Of course, ϑ cannot be integrated on c, but it makes sense to integrate it on its *push-forward* by $\boldsymbol{p}$, namely $\boldsymbol{p}_\star \mathsf{c}$. By definition, the same result is obtained integrating on c the *pull-back* of ϑ by $\boldsymbol{p}$, namely $\boldsymbol{p}^\star \vartheta$:

$$\langle \boldsymbol{p}^\star \vartheta, \mathsf{c} \rangle = \langle \vartheta, \boldsymbol{p}_\star \mathsf{c} \rangle \,. \tag{13}$$

Given c, the value of the integral depends on both ϑ and $\boldsymbol{p}$, which may be varied independently of each other. In particular, a test dispacement (*cf.* footnote 20 on page 20) may be added to the placement $\boldsymbol{p}$ and the spacetime-conjugate form ϑ changed, *so as to keep its body-time pull-back* $\boldsymbol{p}^\star \vartheta$ *fixed.* This kind of operation is central to a proper definition—and an efficient computation—of Maxwell's stress tensor and related force densities, as clearly pointed out by Henrotte and Hameyer [51] (*cf.* my Fig. 3 with their Fig. 1).

What is to be done?

The heuristic role played by Maxwell's electromagnetic constructs in the pioneering work by Eshelby on continuum mechanics of defective solids has just been pointed out. It is fair to add that—until today—continuum mechanics as a whole has felt little influence from those quarters. Eshelby's paper [56] starts in a coy tone:

> In that corner of the theory of solids which deals with lattice defects(dislocations, impurity and interstitial atoms, vacant lattice sites and so forth) which are capable of altering their position or configuration in a crystal, it has been found useful to introduce the concept of the force acting on a defect. [...] The normal theory of elasticity recognizes nothing which corresponds with the force on a defect. (It has nothing to do with the ordinary body force, of course.) [...]

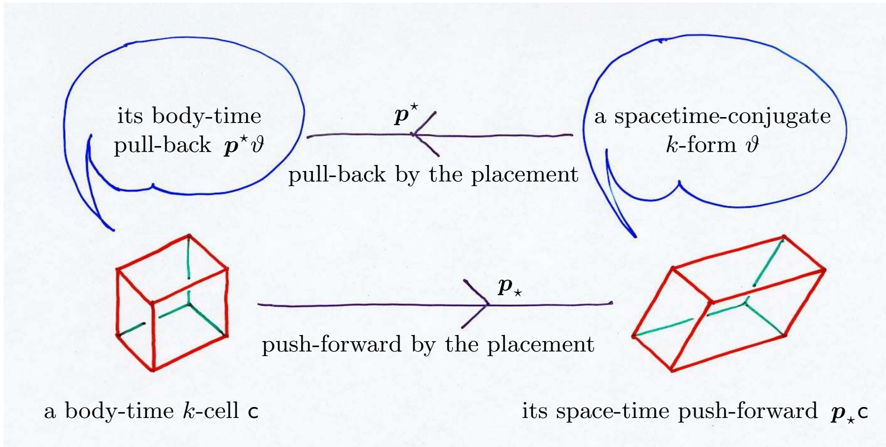


Fig. 3. Placement-related k-cells and k-forms

> Apart from its connection with the theory of lattice defects the energy-momentum tensor and kindred concepts associated with elastic and other material media are of interest for their own sake, but they have received scarcely any attention from applied mathematicians, even during the intensive re-examination and estension of continuum mechanics which has been under way for the last couple of decades, perhaps because of the artificial separation whis has grown up between applied mathematicians and theoretical physics. [. . .]
>
> It is hoped that the present paper may perhaps help to dispel this lack of interest.

The self-mocking understatement by Eshelby contrasts acutely with the pæan sung by Dyson in the closing paragraph of [8] in praise of Maxwell and mainstream twentieth-century physics:[38]

> The ultimate importance of the Maxwell theory is far greater than its immediate achievement in explaining and unifying the phenomena of electricity and magnetism. Its ultimate importance is to be the prototype of all the great triumphs of twentieth-century physics. It is the prototype for Einstein's theory of relativity, for quantum mechanics, for the Yang–Mills theory of generalised gauge invariance, and for the unified theory of fields and particles that is known as the Standard

[38] Dyson's panegyric is well-founded. As you may expect, the 175th anniversary of Maxwell's birth in 2006 has triggered a splurge of extravagant exaggerations in popular physics journals, such as: "The greatest equations ever", or "Could Maxwell have deciphered quantum mechanics?", and even "Had he not died so young, Maxwell would almost certainly have developed special relativity a decade or more before Einstein." Have a look at the December 2006 issue of Physics World for these and other more or less ludicrous claims [59].

Model of particle physics. All these theories are based on the concept of dynamical fields, introduced by Maxwell in 1865.

My personal view is that Maxwell's theory, while paving the way to quantum mechanics, interacted early on with the truly Newtonian chapters of classical mechanics, *i.e.*, the classical theories of inertia and gravitation. This conceptual integration produced the special and general theories of relativity, well within the first two decades of the twentieth century. On the contrary, the integration of Maxwell's electromagnetic theory with the main body of classical mechanics, *i.e.*, the mechanics of deformable continuous media,[39] still remains to be done, nearly a century later.

Should we succeed in this endeavour, we could ascribe our accomplishment exclusively to the "Geometers" of our own century—as justly (or as unjustly) as Lamé did with his « Physique mathématique, proprement dite. » While skeptical with regard to our ability to counter parochialism in science and education, I am quite confident that the present strong trend towards miniaturization—down to the nano-scale—could make a unified electromechanics of deformable media one of the most *useful* physical theories in the near future.

Acknowledgements

My work was partly supported by GNFM-INdAM (the Italian Group for Mathematical Physics), MIUR (the Italian Ministry of University and Research) through the Project "Mathematical Models for Materials Science", and the Fifth European Community Framework Programme through the Project HPRN-CT-2002-00284 ("Smart Systems"). I feel indebted to many people, but I am especially grateful to the organizers of the 2002 SCEE-conference, who had the heart to invite an outsider and the patience to bear with his slowness and longwindedness; to my old professor, the late Giulio Ceradini, who donated the collection of books where I picked out Lamé's *Leçons* [1, 5]; to my personal copy-editor, Christine Mackie, who redressed my English when faltering or simply wrong and made me more confident; to my colleagues in the Doctoral Program in Physics at "Roma Tre"—in particular, to its President, Orlando Ragnisco; to the organizers of the ACE'07 workshop; and, last but not least, to my friends in Brescia who made arrangements for my short course "Basic mathematical structures of neo-classical continuum mechanics" and the students who volunteered. Without incentives or help from all of them, this note would have never reached its present state.

[39] As starkly put by Truesdell and Toupin in the exordium of [24]: "Only pædagogical custom has hindered general realization that *as a physical theory, continuum mechanics is better than mass-point mechanics.* [. . .] As Hamel stated, *classical mechanics is the mechanics of extended bodies.*"

References

1. Lamé, G.: Leçons sur la théorie mathématique de l'élasticité des corps solides, deuxième édition, Gauthier-Villars, Paris, 1866.
2. Maxwell, J. C.: On Faraday's lines of force. Cambridge Philosophical Transactions **10** (1856) part I, art. 3.
3. Maxwell, J. C.: On physical lines of force. Philosophical Magazine (Series 4) **21** (1861) pp. 161–175, 281–291, 338–348 & **23** (1862) pp. 12–24, 85–95.
4. Maxwell, J. C.: A dynamical theory of the electromagnetic field. Philosophical Transactions of the Royal Society of London **155** (1865) pp. 459–512.
5. Lamé, G.: Leçons sur les coordonnées curvilignes et leur diverses applications, Mallet-Bachelier, Paris, 1859.
6. Koiré, A.: Les philosophes et la machine. Critique **23** & **26** (1948). Reprinted in: Etudes d'histoire de la pensée scientifique, P.U.F., Paris, 1966.
7. Benvenuto, E.: An introduction to the history of structural mechanics, Springer, New York, NY, 1991.
8. Dyson, F. J.: Why is Maxwell's theory so hard to understand? In D. O. Forfar ed.: James Clerk Maxwell Commemorative Booklet, produced by the James Clerk Maxwell Foundation on the occasion of the Fourth International Congress on Industrial and Applied Mathematics (ICIAM 99), Edinburgh, 1999; `[http://www.sonnetusa.com/news/DMLfiles/dysonessay.pdf]`.
9. Maxwell, J. C.: A treatise on electricity and magnetism, Clarendon Press, Oxford, 1873.
10. Siegel, D. M.: Innovation in Maxwell's electromagnetic theory: molecular vortices, displacement current, and light, Cambridge Univ. Press, Cambridge, 2003.
11. Chalmers, A.: Maxwell, mechanism, and the nature of electricity. Physics in Perspective **3** (2001) pp. 425–438.
12. Saletan, E. J.: Contraction of Lie groups. J. Mathematical Physics **2** (1961) pp. 1–21.
13. Le Bellac, M., Levy-Leblond, J.-M.: Galilean electromagnetism. Il Nuovo Cimento **14 B** (1973) pp. 217–232.
14. Tonti, E.: A mathematical model for physical theories. Rend. Accademia dei Lincei **52** (1972) pp. 175–181 (1st part) & pp. 350–356 (2nd part).
15. Tonti, E.: On the geometrical structure of electromagnetism. In: Gravitation, Electromagnetism and Geometrical Structures (Proc. of a Conference held for the 80th birthday of A. Lichnerowicz at Villa Tuscolana – Frascati (Roma) on October 19–23, 1995; G. Ferrarese ed.), Pitagora, Bologna (1996) pp. 281–308.
16. Tonti, E.: Finite formulation of the electromagnetic field. In: PIER 32 – Geometric Methods for Computational Electromagnetics (F. L. Teixeira ed.), EMW Publishing, Cambridge, MA (2001) pp. 1–44.
17. Mattiussi, C.: The finite volume, finite element, and finite difference methods as numerical methods for physical field problems. Advances in Imaging and Electron Physics **113** (2000) pp. 1–146.
18. Vilasi, G.: Hamiltonian dynamics, World Scientific, Singapore, 2001.

19. Naber, G. L.: The geometry of Minkowsky spacetime: an introduction to the mathematics of the special theory of relativity, Springer, New York, NY, 1992.
20. Crampin M., Pirani F. A. E.: Applicable differential geometry, Cambridge University Press, Cambridge, 1986.
21. Burke, W. L.: Applied differential geometry, Cambridge University Press, Cambridge, 1985.
22. Bossavit, A.: Applied differential geometry (a compendium), 2002 [http://www.icm.edu.pl/edukacja/mat/Compendium.php].
23. Weeks, J. R. : The shape of space (2nd edition), Dekker, New York, NY, 2002.
24. Truesdell, C., Toupin, R. A.: The classical field theories. In: Handbuch der Physik, Vol. 3/1 (S. Flügge ed.), Springer, Berlin (1960) pp. 226–793.
25. Field, J. C. G.: Electromagnetism and semantics. IEE Proceedings **135 A** (1988) pp. 523–527.
26. Maxwell, J. C.: Remarks on the mathematical classification of physical quantities. Proceedings of the London Mathematical Society **3** (1871) pp. 224–232.
27. Sen, A.: Electric magnetic duality in string theory. Nucl.Phys. **B404** (1993) pp. 109–126; [arXiv:hep-th/9207053].
28. Chan, H.-M., Tsou S. T.: Nonabelian generalization of electricmagnetic duality - a brief review. International J. Mod. Phys. **A14** (1999) pp. 21392172; [arXiv:hep-th/9904102].
29. Tsou S. T.: Concepts in gauge theory leading to electric magnetic duality. In: Geometric Methods for Quantum Field Theory (Proceedings of the Summer School, Villa De Leyva, Colombia, July 12–30, 1999; H. Ocampo *et al.* eds.), World Publishing, Singapore (2001) pp. 361–420; [arXiv:hep-th/0006178].
30. Julia, B. L.: Electric-magnetic duality beyond four dimensions and in general relativity. LPT-ENS/05-43 (Presented at the 23rd International Conference on Differential Geometric Methods in Theoretical Physics, Tianjin, China, August 20–26, 2005); [arXiv:hep-th/0512320].
31. Mattiussi, C.: Edge elements and cochain-based field function approximation. In: Proceedings of the 4th International Workshop on Electric and Magnetic Fields (Marseille, May 12–15, 1998), A.I.M., Liège (1998) pp. 301–306.
32. Henrotte, F., Legros, W.: Elasticity and electromagnetic force calculation with differential forms. In: Proc. of the 4th International Workshop on Electric and Magnetic Fields (Marseille, May 12–15, 1998), A.I.M., Liège (1998) pp. 295–300.
33. Henrotte, F., Hameyer, K.: A mathematical framework for the finite element modelling of electromechanical problems. In: Scientific Computing in Electrical Engineering (Proc. of the 3rd International Workshop, Warnemünde, August 20–23, 2000, U. van Rienen *et al.* eds.), Springer, Heidelberg (2001) pp. 359–366.
34. Abraham, R., Marsden J. E., Ratiu, T.: Manifolds, tensor analysis, and applications, Addison-Wesley, Reading, MA, 1983.
35. Tóth, G.: Harmonic and minimal maps: with applications in geometry and physics, Ellis Horwood, Chichester, 1984.
36. DiCarlo, A.: A non-standard format for continuum mechanics. In: Contemporary Research in the Mechanics and Mathematics of Materials (Papers presented at the Symposium on Recent Developments in Elasticity held to commemorate Jerald L. Ericksen 70th birthday at the Johns Hopkins University on June 12–15, 1996; R. C. Batra, M. F. Beatty eds.), CIMNE, Barcelona (1996) pp. 263–268.
37. Germain, P.: Sur l'application de la méthode des puissances virtuelles en mécanique des milieux continus. Comptes Rendus de l'Académie des Sciences de Paris **274** Série A (1972) pp. 1051–1055.

38. Germain, P.: The method of virtual power in continuum mechanics. Part 2: Microstructure. SIAM Journal on Applied Mathematics **25** (1973) pp. 556–575.
39. Segev, R., Rodnay, G.: Cauchy's theorem on manifolds. Journal of Elasticity **56** (1999) pp. 129–144.
40. Nardinocchi, P., Teresi, L., Tiero, A.: A direct theory of affine rods. European Journal of Mechanics A/Solids **21** (2002) pp. 653–667.
41. Palais, R.: Definition of the exterior derivative in terms of Lie derivatives. Proc. Am. Math. Soc. **5** (1954) pp. 902–908.
42. DiCarlo, A., Quiligotti, S.: Growth and balance. Mechanics Research Communications **29** (2002) pp. 449–456.
43. Maugin, G. A.: The method of virtual power in continuum mechanics: application to coupled fields. Acta Mechanica **35** (1980) pp. 1–70.
44. Bossavit, A.: On the notion of anisotropy in constitutive laws: some implications of the 'Hodge implies metric' result. COMPEL **20** (2001) pp. 233–239.
45. Schreiber, U., van Rienen, U.: Coupled calculation of electromagnetic fields and mechanical deformation. In: Scientific Computing in Electrical Engineering (Proceedings of the SCEE-2004 Conference held in Capo d'Orlando, A. M. Anile *et al.* eds.), Springer, Berlin (2006) pp. 63-69.
46. Maxwell, J. C.: On the calculation of the equilibrium and stiffness of frames. Philosophical Magazine **27** (1864) pp. 294–299.
47. Maxwell, J.C.: On the equilibrium of elastic solids. Transactions of the Royal Society of Edinburgh **20** (1854) part 1.
48. Lamé, G.: Cours de physique de l'École Polytechnique, Bachelier, Paris, 1840.
49. Bobbio, S.: Electrodynamics of materials: forces, stresses, and energies in solids and fluids, Academic Press, San Diego, CA, 2000.
50. Ericksen, J. L.: Electromagnetic effects in thermoelastic materials. Mathematics and Mechanics of Solids **7** (2002) pp. 165–189.
51. Henrotte, F., Hameyer, K.: Electromagnetic force densities in a continuous medium. In: Scientific Computing in Electrical Engineering (Proc. of the SCEE-2002 Conference held in Eindhoven, W. H. A. Schilders *et al.* eds.), Springer, Berlin (2004) pp. 214–222.
52. Brevik, I.: Experiments in phenomenological electrodynamics and electromagnetic energy-momentum tensor. Physics Reports **52** (1979) pp. 133–201.
53. Millis, M. G.: Prospects for breakthrough propulsion from physics. NASA/TM2004-213082.
54. Ido, Y.: Basic equations and constitutive equations of micropolar magnetic fluids with *E-B* analogy and the Abraham expression of the electromagnetic momentum. JSME International Journal, Series B **48** (2005) pp. 488–493.
55. Feigel, A.: Quantuum vacuum contribution to the momentum of dielectric media. Phys. Rev. Lett. **92** (2004) 020404.
56. Eshelby, J. D.: The elastic energy-momentum tensor. Journal of Elasticity **5** (1975) pp. 321–335.
57. DiCarlo, A.: Surface and bulk growth unified. In: Mechanics of Material Forces (Proc. EUROMECH Colloquium 445 held in Kaiserslautern, 2003; P. Steinmann & G. A. Maugin eds.), Springer, New York, NY (2005) pp. 53–64.
58. Gurtin, M. E.: Configurational forces as basic concepts of continuum physics, Springer, Berlin, 2000.
59. `http://physicsweb.org/articles/world/19/12/2/1`.