Three-dimensional nonsimple viscous liquids dragged by one-dimensional immersed bodies

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Abstract

We model the interaction of one-dimensional moving structures with a surrounding three-dimensional fluid, physically close to a Newtonian liquid. The interaction is the adherence of the fluid to the immersed structures, which drag it while moving as rigid bodies. To get solutions of the dynamical problem, we need a model of viscous fluid slightly more general than the Newtonian one, in which the Cauchy stress tensor depends upon higher-order derivatives of the velocity field. Assuming reasonable hypotheses on the motion of the one-dimensional rigid bodies, existence and uniqueness of the solution for the dynamical problem can be proved.

Keywords: Nonsimple fluid, Second-gradient theory, 3D/1D coupling, Fluid/structure interaction

1. Introduction

When a physical system exhibits parts with very different typical scales, the modelling of their interplay is an outstanding challenge for current applied mathematics. Within the realm of multiscale modelling, we want to consider the motion of a three-dimensional viscous fluid dragged by thin immersed rigid bodies, to which the fluid adheres.

Since the dragging structures are one-dimensional, a Newtonian fluid flow may represent these situations only if the velocity field has a regular continuation on the thin bodies. But if weak solutions are considered (as they have to be, since no existence theory in 3D for classical solutions of Navier-Stokes equations is available), the thin bodies would be either invisible or lead to unphysical singularities. Moreover, this feature arises also in very simple geometries, where classical solutions are known, as explained in Sec. 3. Applying to fluids the theory of second-gradient continuous media (see Germain, 1973; Fried and Gurtin, 2006) together with a result of Musesti (2009) on the general form of a linear isotropic tensor field, one can construct a model for linear isotropic viscous fluids of second gradient, which overrides those difficulties.

Linearity and isotropy of the Cauchy stress tensor are the essential physical requirements for Newtonian fluids, and they are shared also by our model. Besides, the present theory allows for additional dissipative requirements which oppose to the growth of the second spatial derivatives of the velocity field. From a mathematical point of view, this produces a regularizing effect which enforces the continuity of the velocity field also on the region occupied by the one-dimensional rigid bodies. This feature is not shared by first-gradient fluids and enables us to impose the adherence of our fluid to the thin structures.

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In this framework, we obtain global existence and uniqueness of the solution for the dynamical problem in the incompressible case employing methods from the theory of Partial Differential Equations on time-varying domains, together with a nontrivial treatment of the motion of the immersed bodies. We stress the importance of such a result both from a theoretical and a computational point of view: it is the basis for the validation of any approximation scheme, and it is still unproven for Newtonian liquids on three-dimensional domains.

2. Second-gradient linear liquids

A good way to set down a thermomechanical theory for continuous media is to specify a model for the virtual power $P_{\rm int}(M, \mathsf{v})$ expended within a region M (contained in a domain Ω) by internal stresses on a test function v, and for the virtual power $P_{\rm ext}(M, \mathsf{v})$ expended by external forces and inertial ones. Real physical processes are characterized by the balance of such power expenditures and then the following fundamental principle should be applied as an axiom (see *e.g.* Germain, 1973; Marzocchi and Musesti, 2003).

Principle of virtual powers. The dynamics is feasible if and only if for every region $M \subset \Omega$ and every virtual velocity field v we have

$$P_{\rm int}(M, \mathsf{v}) = P_{\rm ext}(M, \mathsf{v}) \ , \tag{1}$$

for any instant in a given time interval.

Since our aim is to apply the virtual powers framework to Fluid Mechanics in three-dimensional space, we have chosen an Eulerian point of view, interpreting test functions as velocity fields; moreover, we restrict our attention to a pure mechanical theory, not including thermal effects. Aiming to treat liquids, we impose the incompressibility condition, and for the sake of simplicity we only deal with homogeneous fluids. Using the Eulerian velocity field $\mathsf{u}(t, \mathsf{x})$ as descriptor, the homogeneity allows us to set the mass density $\rho \equiv 1$, and the incompressibility condition takes the usual form

$$\forall t \in [0, T] : \ \operatorname{div} \mathsf{u} = 0 \ . \tag{2}$$

Further prescriptions are of course needed for internal and external powers. The general form of a second-gradient Galilean-invariant internal power is

$$P_{\rm int}(M, \mathsf{v}) = \int_M \mathsf{T} \cdot \nabla \mathsf{v} + \int_M \mathbf{G} \cdot \nabla\nabla \mathsf{v} \ ,$$

where T is a symmetric tensor field of order 2 and $\mathbf{G}$ a tensor field of order 3. Linearity and isotropy of the fluid can be encoded in the dependence of the tensor fields T and $\mathbf{G}$ on u. In the incompressible case, the most general linear isotropic tensor fields, endowed with the symmetries due to Galilean invariance and Schwarz's theorem, take the form (see Musesti, 2009, Theorem 1.1)

$$\begin{aligned} \mathsf{T}_{ij} &= \mu(\mathsf{u}_{i,j} + \mathsf{u}_{j,i}) - p\,\delta_{ij} \ , \\ \mathbf{G}_{ijk} &= \eta_1 \mathsf{u}_{i,jk} + \eta_2(\mathsf{u}_{j,ki} + \mathsf{u}_{k,ij} - \mathsf{u}_{i,ss}\delta_{jk}) \\ &\quad + \eta_3(\mathsf{u}_{j,ss}\delta_{ki} + \mathsf{u}_{k,ss}\delta_{ij} - 4\mathsf{u}_{i,ss}\delta_{jk}) - \mathsf{p}_k\delta_{ij} \ , \end{aligned}$$

where $\mu, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$ and δ_{ij} is the usual Kronecker symbol. The fields p and p, respectively a scalar and a vector one, enter the definition of the *pressure*

$$P := p - \operatorname{div} \mathsf{p} \ ,$$

whose role in incompressible theories reduces to that of a Lagrange multiplier of the constraint (2).

Defining the *symmetric part* of an m-tensor X as

$$\mathrm{Sym}\,\mathsf{X} := \frac{1}{m!}\sum_{\sigma}\mathsf{X}_{\sigma(i_1\ldots i_m)}\ ,$$

where σ runs over the group of permutations of m elements, and setting $\mathsf{I} = (\delta_{ij})$, the previous relations can be written in intrinsic notation as

$$\begin{aligned}
&\mathsf{T} = 2\mu\,\mathrm{Sym}\,\nabla\mathsf{u} - p\,\mathsf{I}\,,\\
&\mathbf{G} = (\eta_1 - \eta_2)\nabla\nabla\mathsf{u} + 3\eta_2\,\mathrm{Sym}\,\nabla\nabla\mathsf{u}\\
&\quad - (\eta_2 + 5\eta_3)\Delta\mathsf{u}\otimes\mathsf{I} + 3\eta_3\,\mathrm{Sym}\,(\Delta\mathsf{u}\otimes\mathsf{I}) - \mathsf{I}\otimes\mathsf{p}\,.
\end{aligned}$$

Following these prescriptions and imposing also on the virtual velocities the constraint (2), we can write the internal power for a linear isotropic incompressible fluid as

$$\begin{aligned}
&P_{\mathrm{int}}(M,\mathsf{v}) =\\
&= 2\mu\int_M \mathrm{Sym}\,\nabla\mathsf{u}\cdot\nabla\mathsf{v} + (\eta_1-\eta_2)\int_M \nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v}\\
&+ 3\eta_2\int_M \mathrm{Sym}\,\nabla\nabla\mathsf{u}\cdot\nabla\nabla\mathsf{v} - (\eta_2+4\eta_3)\int_M \Delta\mathsf{u}\cdot\Delta\mathsf{v}\,.
\end{aligned}$$

We can deduce some constraints on the coefficients $\mu, \eta_1, \eta_2, \eta_3 \in \mathbb{R}$ from thermodynamical considerations. It can be checked that, for the instantaneous dissipation of any flow u to be non-negative, the following are necessary and sufficient conditions:

$$\begin{aligned}
&\mu \ge 0\,, \qquad \eta_1 + 2\eta_2 \ge 0\,, \qquad \eta_1 - \eta_2 \ge 0\,,\\
&\eta_1 - \eta_2 - 6\eta_3 - 2\sqrt{\eta_2^2 + 4\eta_2\eta_3 + 9\eta_3^2} \ge 0\,.
\end{aligned} \tag{3}$$

The only external interaction we consider is the adherence of the fluid to the one-dimensional rigid bodies immersed in it, which will be encoded in the constraints on $\mathsf{u}(t,x)$, hence the external power contains only the inertial nonlinear term:

$$P_{\mathrm{ext}}(M,\mathsf{v}) = -\int_M \left(\frac{\partial\mathsf{u}}{\partial t} + (\mathsf{u}\cdot\nabla)\mathsf{u}\right)\cdot\mathsf{v}\,,$$

namely the power of inertial forces.

3. A simple example

Before getting to the general situation, we want to clarify how, by means of second-gradient fluids, we can manage adherence to parts of the domain with zero surface measure and why the motion of such one-dimensional structures can significantly affect the liquid flow. We do this in a very simple example.

Consider a fluid placed in the cavity between two infinitely long coaxial cylinders of radii $R_1 < R_2$ and the flow is driven imposing a motion with constant velocity U of the inner cylinder along the axial direction e_z. Looking for cylindrically symmetric stationary solutions $u(r)\mathsf{e}_z$, where r is the cylindrical radius, in the case of Newtonian liquids with the usual adherence condition on the walls one easily finds

$$u(r) = U\frac{\log R_2 - \log r}{\log R_2 - \log R_1}\ .$$

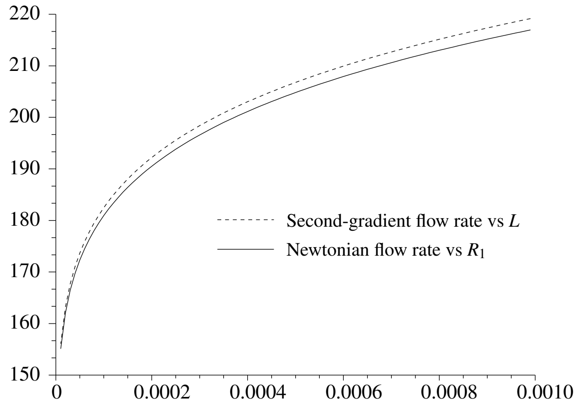

Figure 1: Comparison of the flow rates

It is clear that this solution has no extension to the case $R_1 = 0$.

But with a second-gradient linear isotropic viscous liquid the analogous problem gives the family of solutions

$$u(r) = \alpha_1 + \alpha_2 I_0\left(\frac{r}{L}\right) + \alpha_3 \log\left(\frac{r}{L}\right) + \alpha_4 K_0\left(\frac{r}{L}\right) ,$$

where $\alpha_i, i = 1\dots4$, are constants (depending on R_1, R_2) fixed by the boundary conditions, I_0, K_0 are Bessel functions of imaginary argument (Lebedev, 1972, Sec. 5.7), and the parameter

$$L := \sqrt{(\eta_1 - \eta_2 - 4\eta_3)/\mu}$$

arises from the higher-gradient terms. If we now set $R_1 = 0$ the solution remains bounded provided that $\alpha_3 = \alpha_4$, since

$$\lim_{r\to0}\left(\log\left(\frac{r}{L}\right) + K_0\left(\frac{r}{L}\right)\right) < +\infty ;$$

besides, we verified that one can still meet the prescribed boundary conditions by a suitable choice of the constants.

Moreover, an interesting feature appears. We computed the dependence upon R_1 of the flow rate through an annular section in the Newtonian case. Comparing it with the dependence upon L of the flow rate of our second-gradient fluid when the inner radius is equal to zero (Fig. 1), we observed that the relative difference between the two values computed at $R_1 = L$ is very small, indeed negligible when $R_1, L \ll 1$. This fact suggests that the parameter L represents a sort of *effective thickness* in the limiting case.

4. The dynamical problem

We now want to study the flow of a second-gradient linear liquid dragged by one-dimensional immersed structures in an arbitrary time interval $[0, T]$ when the three-dimensional region occupied by the fluid Ω is a bounded domain with Lipschitz boundary which is fixed in time. For the sake of simplicity we consider only one rigid body Λ_0 immersed in the fluid, where Λ_0 is a connected one-dimensional closed subset of Ω such that $\Lambda_0 \cap \partial\Omega = \varnothing$. Moreover, we denote by $\Lambda(t)$ the image at time t of Λ_0 under the rigid displacement φ, *i.e.* $\Lambda(t) = \varphi(\Lambda_0, t)$ and $\Lambda(0) = \Lambda_0$. We assume the existence of a family of C^2-diffeomorphisms $\psi_t : \Omega \to \Omega$ such that $\psi_t(\Lambda_0) = \Lambda(t)$ and any ψ_t reduces to the identity map in a neighborhood of $\partial\Omega$. This assumption will be fulfilled if the one-dimensional structure never reaches the boundary of the domain, namely $\Lambda(t) \cap \partial\Omega = \varnothing$ for every $t \in [0, T]$. Finally, to avoid technicalities, we assume that φ and ψ_t enjoy a C^1 dependence on time. The introduction of such diffeomorphisms will enable us to map the linearized parabolic problem with a moving domain into an equivalent problem, easier to work out, where $\Lambda(t) = \Lambda_0$ for all the time.

The general statement of the adherence condition for Newtonian fluids would only require the velocity field u on the boundary to be equal to the velocity *of the boundary*, being the traction Tn (where n is the unit outer normal to the boundary) computed after the problem is solved. But within second-gradient theories one is expected to prescribe something also on derivatives of u at the boundary, by fixing a value for the *hypertraction* $(\mathbf{G}\mathsf{n})\mathsf{n}$, as explained in Fried and Gurtin (2006, Sec. 8). We choose the so-called weak adherence condition, setting $\mathsf{u} = 0$ and $(\mathbf{G}\mathsf{n})\mathsf{n} = 0$ on $\partial\Omega$ for every $t \in [0, T]$. Since $\Lambda(t)$ is not a part of $\partial\Omega$, the adherence to the one-dimensional structures does not appear within the boundary conditions: it will be encoded in the definition of an appropriate functional space $\mathfrak{X}$ given below.

The balance principle (1), which is in integral form, leads directly to an interpretation of the functions as defined up to negligible sets (with zero Lebesgue measure). We take the set of virtual velocities as a linear space with the useful topological structure of a Hilbert space, endowing the principle of virtual powers with a natural interpretation as equality of linear forms. Hence we construct for every $t \in [0, T]$ a Hilbert subspace X_t of the Sobolev space $H^2(\Omega; \mathbb{R}^3)$ in the following way: we set

$$V := \left\{ \mathsf{v}|_\Omega \ : \ \mathsf{v} \in C_0^\infty(\mathbb{R}^3; \mathbb{R}^3)\,,\ \operatorname{div} \mathsf{v} = 0 \right\}$$

and denote with H and H^2_d the completions of V in $L^2(\Omega; \mathbb{R}^3)$ and $H^2(\Omega; \mathbb{R}^3)$ respectively. Since $H^2(\Omega; \mathbb{R}^3)$ is continuously embedded in $C^0(\overline{\Omega}; \mathbb{R}^3)$, we can define the closed subspace

$$X_t := \left\{ \mathsf{v} \in H^2(\Omega; \mathbb{R}^3) \ : \ \mathsf{v} = 0 \ \text{ on } \partial\Omega \cup \Lambda(t) \right\} \cap H^2_d$$

endowed with the norm

$$\|\mathsf{v}\|^2_{H^2} := \int_\Omega |\mathsf{v}|^2 + \int_\Omega |\nabla \mathsf{v}|^2 + \int_\Omega |\nabla\nabla \mathsf{v}|^2\,,$$

which encodes the natural regularity requested by the problem.

Now we can define the space

$$\mathfrak{X} := L^2([0, T]; X_t) \cap C^0([0, T]; H) \cap H^1([0, T]; X_t')$$

of divergence-free virtual velocities on Ω which belong to X_t for almost every $t \in [0, T]$, and whose first derivatives with respect to t belong to the dual space X_t' of X_t for a.e. $t \in [0, T]$. Moreover, it is apparent that velocity fields belonging to the space $\mathfrak{X}$ vanish on $[0, T] \times \partial\Omega$ and on the surface $\{ (t, \mathsf{x}) : t \in [0, T], \mathsf{x} \in \Lambda(t) \}$, while their derivatives in general do not.

We then define a continuous bilinear form on $H^2(\Omega; \mathbb{R}^3)$ as

$$\begin{aligned} a_t(\mathsf{u}, \mathsf{v}) := 2\mu \int_\Omega \operatorname{Sym} \nabla \mathsf{u} \cdot \nabla \mathsf{v} \\ + (\eta_1 - \eta_2) \int_\Omega \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} + 3\eta_2 \int_\Omega \operatorname{Sym} \nabla\nabla \mathsf{u} \cdot \nabla\nabla \mathsf{v} \\ - (\eta_2 + 4\eta_3) \int_\Omega \Delta \mathsf{u} \cdot \Delta \mathsf{v}\,. \end{aligned}$$

Since $a_t(\mathsf{u}, \mathsf{u})$ represents the dissipation at time t of the flow u, the classical Korn's inequality together with the constraints (3) on the coefficients (taken with strict inequalities) guarantee the coercivity of this bilinear form on the space X_t.

Consider now the trilinear form on $H^2(\Omega;\mathbb{R}^3)$ given by

$$b(\mathsf{u},\mathsf{v},\mathsf{w}) := \int_\Omega (\mathsf{u}\cdot\nabla)\mathsf{v}\cdot\mathsf{w}\,,$$

which is easily checked to be continuous. Now, for every $\mathsf{u}\in X_t$ and $\mathsf{h}\in H^2_d$ we have

$$b(\mathsf{u},\mathsf{u},\mathsf{u}) = 0 \quad\text{and}\quad b(\mathsf{h},\mathsf{u},\mathsf{u}) = 0\,.$$

Indeed, by standard formulae in tensor calculus we get the equalities for $\mathsf{u},\mathsf{h}\in V$, $\mathsf{u}=0$ on $\partial\Omega$. Then the assertion follows via a density argument.

Now we turn to the existence of a solution for the dynamical problem. Assume to have an *interpolator* $\hat{\mathsf{u}}$, namely a divergence-free velocity field which vanishes on $\partial\Omega$ and equals the velocity of the rigid body on $\Lambda(t)$, and to look for a solution of the type $\mathsf{u}+\hat{\mathsf{u}}$ where $\mathsf{u}\in\mathfrak{X}$. Then, substituting into the balance and using the properties of $\hat{\mathsf{u}}$, it is straightforward to see that u must satisfy the following problem (in variational form):

Problem 1. *Given* $\mathsf{u}_0\in H$ *and* $\hat{\mathsf{u}}$ *as above, find* $\mathsf{u}\in\mathfrak{X}$ *such that* $\mathsf{u}|_{t=0}=\mathsf{u}_0$ *and*

$$\begin{aligned}&\int_0^T\left(\int_\Omega\frac{\partial\mathsf{u}}{\partial t}\cdot\mathsf{v}+a_t(\mathsf{u},\mathsf{v})\right)+\\&\int_0^T\left(b(\mathsf{u},\mathsf{u},\mathsf{v})+b(\hat{\mathsf{u}},\mathsf{u},\mathsf{v})+b(\mathsf{u},\hat{\mathsf{u}},\mathsf{v})\right)+\\&\int_0^T\left(\int_\Omega\frac{\partial\hat{\mathsf{u}}}{\partial t}\cdot\mathsf{v}+a_t(\hat{\mathsf{u}},\mathsf{v})+b(\hat{\mathsf{u}},\hat{\mathsf{u}},\mathsf{v})\right)=0\end{aligned}\tag{4}$$

for every $\mathsf{v}\in\mathfrak{X}$.

In order to show that Problem 1 has a solution $\mathsf{u}\in\mathfrak{X}$ (and that it is the sole one), we need a further 'smallness' condition on the interpolator $\hat{\mathsf{u}}$, that is, for a suitable $\beta>0$,

$$|b(\mathsf{v},\hat{\mathsf{u}},\mathsf{v})|\le\beta\,\|\mathsf{v}\|^2_{H^2}\,,\tag{5}$$

which we suppose, for the time being, to be fulfilled. The problem of finding an interpolator will be considered in a while.

Now we solve Problem 1. The first step is to define the linearized parabolic operator

$$L:\begin{cases}\mathfrak{X} & \to \quad H\times L^2([0,T];X_t')\\ \mathsf{u} & \mapsto \quad \left(\mathsf{u}|_{t=0}\,,\ \dfrac{\partial\mathsf{u}}{\partial t}(t)+a_t(\mathsf{u}(t),\cdot)\right)\end{cases}$$

and to establish that it is a homeomorphism. This can be done in a standard way once we have noticed that the time-varying spaces X_t turn all into X_0 when we apply ψ_t^{-1} to Ω, and the coercivity of the bilinear form a_t is transferred to a new bilinear form $\tilde{a}$, as clearly explained by Límaco et al. (2002). After this transformation, we obtain a parabolic problem on the cylindrical domain $\{\,(t,\psi_t^{-1}(\mathsf{x})) : t\in[0,T],\mathsf{x}\in\Omega\,\}$, which enjoys existence and uniqueness of solution in the space

$$L^2([0,T];X_0)\cap C^0([0,T];H)\cap H^1([0,T];X_0')$$

as an application of the general theory contained *e.g.* in Lions and Magenes (1968). This implies also that L is a homeomorphism, that is our linearized parabolic problem admits a unique solution.

As a second step, one has to deal with the nonlinear term $K(\mathsf{u}) := (\mathsf{u}\cdot\nabla)\mathsf{u}$, which characterizes also the Navier-Stokes equation for Newtonian liquids. Its compactness properties have been successfully exploited to prove the existence of solutions for the Navier-Stokes equation (see *e.g.* Lions, 1969). Since the same arguments apply also to our equation, we need no additional effort to establish the existence of a solution $\mathsf{u} \in \mathfrak{X}$ for Problem 1.

The construction of a good $\hat{\mathsf{u}}$ is not a trivial fact, because in order to prove the existence of solutions we need a family of interpolators of the velocity assigned on $\Lambda(t)$ and $\partial\Omega$ among which, for any $\beta > 0$, a $\hat{\mathsf{u}}$ can be selected such that (5) holds for every $\mathsf{v} \in X_t$. In order to obtain a divergence-free interpolator $\mathsf{g}(t,\mathsf{x})$ we consider at any fixed time t a neighborhood $R_t \subset \Omega$ of $\Lambda(t)$ with ∂R_t smooth. On R_t the field g is taken equal to the velocity of the rigid motion φ; since this is a divergence-free velocity and since we want g to vanish on $\partial\Omega$, we get

$$\int_{\partial(\Omega\setminus R_t)} \mathsf{g}\cdot\mathsf{n} = \int_{\partial\Omega} \mathsf{g}\cdot\mathsf{n} - \int_{\partial R_t} \mathsf{g}\cdot\mathsf{n} = 0\,,$$

where n is the outer unit normal to $\partial(\Omega \setminus R_t)$. Given this property we can adapt Lemma 2.2 of Girault and Raviart (1986, p. 24) to obtain a divergence-free extension of g on Ω belonging to $H^2(\Omega;\mathbb{R}^3)$, carefully considering what happens on ∂R_t. Once g is found, we can construct $\hat{\mathsf{u}}$ satisfying (5) via two steps: we take R_t small enough, and then we apply Lemma 7.1 of Lions (1969, p. 103) on $\Omega \setminus R_t$ thanks to the regularity of ∂R_t, finishing the proof of existence.

We conclude showing that, within this second-gradient theory, one obtains also the uniqueness of the solution, which is still lacking for the Navier-Stokes case, thanks to the regularity of our functions. In fact, let u_1 and u_2 be solutions of Eq. (4) with the same initial datum, and set $\mathsf{w} := \mathsf{u}_1 - \mathsf{u}_2$. We easily obtain $\mathsf{w}|_{t=0} = 0$ and, exploiting the arbitrariness of v in Eq. (4) together with the properties of the trilinear form b,

$$\int_0^t \left(\int_\Omega \frac{\partial \mathsf{w}}{\partial t}\cdot \mathsf{w} + a_t(\mathsf{w},\mathsf{w}) \right) + \\ + \int_0^t \big(b(\mathsf{w},\mathsf{u}_2,\mathsf{w}) + b(\mathsf{w},\hat{\mathsf{u}},\mathsf{w}) \big) = 0$$

for every $t \in [0,T]$. Applying now the coercivity of a_t together with Hölder's and Young's inequalities, we can find $c_1, c_2 > 0$ such that

$$\begin{aligned} \|\mathsf{w}(t)\|_{L^2}^2 + 2c_1 \int_0^t \|\mathsf{w}\|_{H^2}^2 \le \\ &\le 2\int_0^t |b(\mathsf{w},\mathsf{u}_2,\mathsf{w})| + 2\int_0^t |b(\mathsf{w},\hat{\mathsf{u}},\mathsf{w})| \\ &\le 2\int_0^t \|\mathsf{w}\|_{L^\infty} \left(\|\nabla\mathsf{u}_2\|_{L^2} + \|\nabla\hat{\mathsf{u}}\|_{L^2}\right)\|\mathsf{w}\|_{L^2} \\ &\le 2c_1\int_0^t \|\mathsf{w}\|_{H^2}^2 + c_2\int_0^t \left(\|\mathsf{u}_2\|_{H^2} + \|\hat{\mathsf{u}}\|_{H^2}\right)^2 \|\mathsf{w}\|_{L^2}^2\,. \end{aligned}$$

Hence we have the energy estimate

$$\|\mathsf{w}(t)\|_{L^2}^2 \le c_2 \int_0^t \left(\|\mathsf{u}_2(s)\|_{H^2} + \|\hat{\mathsf{u}}(s)\|_{H^2}\right)^2 \|\mathsf{w}(s)\|_{L^2}^2\, ds$$

and by Gronwall's lemma we conclude that $\|\mathsf{w}(t)\|_{L^2}^2 = 0$ for every $t \in [0,T]$, that is $\mathsf{u}_1 = \mathsf{u}_2$.

5. Conclusions

The problem of dragging a three-dimensional liquid by one-dimensional rigid structures can be mathematically solved by the addition of a term to the internal virtual power which dissipates on the second gradient of the velocity field. Since this term satisfies linearity and isotropy conditions, we can consider our model as a perturbation of a Newtonian liquid: it has analogous physical properties, while on the mathematical viewpoint it has a better behavior since we obtain a well-posed Cauchy problem.

The additional term could be considered as a mere mathematical trick; nevertheless, it appears to compensate the loss of information about microscale sizes inherent to the mathematical representation of real fluids. In fact, a length parameter appears in second-gradient theories, and it seems to be related to the actual thickness of the objects whose size is neglected.

Some issues will be the subject of future research: it remains to be clarified whether the appearance of higher-gradient terms is related to approximation and homogenization procedures or it deals with the description of small-scale physics; moreover, due to the second-gradient terms, some difficulties have to be overcome in order to develop efficient schemes for the numerical approximation of our dynamical problem, as it is affected also by the divergence-free constraint.

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