# Review of calculus and introduction to smooth manifolds

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## 1 The differential

Let $U \subseteq \mathbb{R}^n$ be an open set in $\mathbb{R}^n$ endowed with the Euclidean topology. Let $f : U \to \mathbb{R}^m$ be a smooth function. The differential of $f$ is

$$\begin{aligned} \mathrm{d}_x f : \quad & \mathbb{R}^n \to \mathbb{R}^m \\ & v \mapsto \mathrm{d}_x f(v) := f_{*|_x}(v) \end{aligned}$$

and it is explicitly given by

$$v := \begin{pmatrix} v^1 \\ \cdot \\ \cdot \\ v^n \end{pmatrix} \mapsto \begin{pmatrix} \mathrm{d}_x f^1(v) \\ \cdot \\ \cdot \\ \mathrm{d}_x f^m(v) \end{pmatrix} := \begin{pmatrix} \frac{\partial f^1}{\partial x^1}(x) & \dots & \frac{\partial f^1}{\partial x^n}(x) \\ \dots & \dots & \dots \\ \frac{\partial f^m}{\partial x^1}(x) & \dots & \frac{\partial f^m}{\partial x^n}(x) \end{pmatrix} \begin{pmatrix} v^1 \\ \cdot \\ \cdot \\ v^n \end{pmatrix},$$

where the matrix appearing in the last equation is called Jacobian and it is denoted as $Jf(x)$. Denote with $\partial/\partial x_j$ the $j$-th vector of the canonical basis in $\mathbb{R}^n$ and $\partial/\partial y_j$ the $j$-th vector of the canonical basis in $\mathbb{R}^m$. The $j$-th column of the Jacobian matrix is $\partial f/\partial x^j := Jf(x)\,\partial/\partial x_j$. Thus, making use of the Einstein summation convention, we have

$$\frac{\partial f}{\partial x_j} = \frac{\partial f^i}{\partial x_j} \cdot \frac{\partial}{\partial y^i},$$

where $y = f(x)$.

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**Example 1.1** (Geometric interpretation)**.** Let us consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ such that $f(x,y) := x^2+y^2+1$. Fix $P \in \mathbb{R}^2$, the map $\mathrm{d}_P f : \mathbb{R}^2 \to \mathbb{R}$ is given by

$$\mathrm{d}_P f(v) := \begin{pmatrix} \frac{\partial f}{\partial x}(P) & \frac{\partial f}{\partial y}(P) \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} .$$

It has graph the plane tangent in $P$ at the graph of the function $f$ translated in the origin.

**Example 1.2** (Mechanical interpretation)**.** Let us recall that, given a smooth function $g : \mathbb{R}^m \to \mathbb{R}^k$, one has (by the *chain rule*)

$$\mathrm{d}_x(g \circ f) = \mathrm{d}_{f(x)} g \circ \mathrm{d}_x f.$$

Let $\gamma : [a,b] \to \mathbb{R}^n$ be a curve in $\mathbb{R}^n$ with $\gamma(0) = x$ and let $f$ be as before; then the function $F := f \circ \gamma$ is a curve in $\mathbb{R}^m$ with

$$\frac{\mathrm{d}F}{\mathrm{d}t} = \frac{\partial f}{\partial \gamma^j} \cdot \frac{\mathrm{d}\gamma^j}{\mathrm{d}t};$$

and in particular one has

$$\frac{\mathrm{d}F}{\mathrm{d}t}(0) = f_{*|_x}(\dot{\gamma}(0)).$$

In general, a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$ transforms curves in $\mathbb{R}^n$ through $x$ into curves in $\mathbb{R}^m$ through $f(x)$, and $f_{*|_x}$ relates their corresponding velocity vectors.

# 2 The implicit function theorem and its consequences

**Example 2.1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = y - x^2$. Set

$$\mathcal{L} = f^{-1}(0) = \{(x,\, y) \in \mathbb{R}^2 \,:\, y = x^2\} .$$

There exists $\phi : \mathbb{R} \to \mathbb{R}$ such that

$$(x,y) \in f^{-1}(0) \Leftrightarrow y = \phi(x) .$$

**Example 2.2.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be $f(x,y) = x^2 + y^2 - 1$. Set

$$\mathcal{L} = f^{-1}(0) = \{(x,\, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\} .$$

Fix $A := (x_A,\, y_A) = (0,1)$ and $(-\varepsilon, \varepsilon)$ a neighborhood of $x_A$ in $\mathbb{R}$. There exists $\phi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ such that

$$\forall (x,y) \in (-\varepsilon, \varepsilon) \times (\sqrt{1-\varepsilon^2}, 1+\varepsilon) \text{ s.t. } (x,y) \in f^{-1}(0) \Leftrightarrow y = \phi(x)\,.$$

Clearly $\phi(x) = \sqrt{1-x^2}$.

**Example 2.3.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the function $f(x,y) = x^2 - y^2$. Is there $\phi$ as in the previous example?

**Theorem 2.1** (Implicit Function Theorem)**.** *Let $U$ be an open set in $\mathbb{R}^n \times \mathbb{R}^m$, $(x,y) \in U$ and $f : U \to \mathbb{R}^m$ a smooth function. Let us suppose that $f(x,y) = 0$ and that $w \mapsto \mathrm{d}_{(x,y)} f(0,w)$ is an isomorphism.*

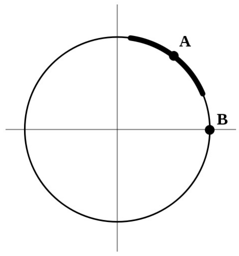
*Then there exist an open neighborhood $V$ of $x$ and an open neighborhood $W$ of $y$ and a smooth function $\phi : V \to \mathbb{R}^n$ such that $\phi(V) \subseteq W$, $\phi(x) = y$ and*

$$\forall \xi \in V,\, \forall \eta \in W \;:\; f(\xi, \eta) = 0 \iff \eta = \phi(\xi).$$

*Proof.* See *Approfondimenti di Analisi Matematica.* □

**Example 2.4.** Let $f(x,y) = x^2 + y^2 - 1$. Consider, for example, the point $A$; we have $\partial f / \partial y\,(A) \neq 0$. The function $\phi : (-\varepsilon, \varepsilon) \to \mathbb{R}$ with $\phi(x) = \sqrt{1-x^2}$ has as graph the circle in a neighborhood of $A$.

At the point $B = (1,0)$ (see the figure below) one has $\partial f / \partial y\,(1,0) = 2 \cdot (0) = 0$ and thus we can not apply Theorem 2.1 (the tangent line to the circle at $B$ is vertical). Notice that $\partial f / \partial x\,(1,0) \neq 0$. In fact, there exists $\phi : U_{y_B} \to \mathbb{R}$ such that $\phi(y) = \sqrt{1-y^2}$.




**Remark 2.1** (Sard's Theorem)**.** Let $f$ be a smooth function. Define $\mathrm{Crit}(f)$ the subset of points in $\mathrm{dom}(f)$ such that the differential is not subjective. Then $f(\mathrm{Crit}(f))$ has Lebesgue measure 0.

**Definition 2.1** (Submersion)**.** Let $U \in \mathbb{R}^n$ be an open set and $n \geq m$. A smooth function $f : U \to \mathbb{R}^m$ is called *submersion* at $\bar{x}$ if $\mathrm{d}_{\bar{x}} f$ is surjective. If it is true for all $x \in U$, we say simply submersion.

**Example 2.5** (Canonical submersion)**.** Let $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^k$ the projection into the first $k$ factors. It is a submersion.

## 3 Constant rank theorem

A necessary condition for $f : U \to \mathbb{R}^n$ to be a local diffeomorphism at $\bar{x}$ is that $\mathrm{d}_{\bar{x}} f$ be an isomorphism. The following theorem states that this linear condition is also sufficient.

**Theorem 3.1** (Inverse Function Theorem)**.** *Let $\bar{x} \in U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^n$ be smooth and let us suppose that $\mathrm{d}_{\bar{x}} f$ is an isomorphism.*

*Then there exists an open neighborhood $V \subset U$ of $x$ such that $f(V)$ is open in $\mathbb{R}^n$ and $f|_V : V \to f(V)$ is a diffeomorphism. Furthermore for each $y \in f(V)$ one has*

$$\mathrm{d}_y(f^{-1}) = \left(\mathrm{d}_{f^{-1}(y)} f\right)^{-1}.$$

*Proof.* See *Approfondimenti di Analisi Matematica.* □

**Remark 3.1.** The differential $\mathrm{d}_{\bar{x}} f$ is simply a single linear map, which we may represent by matrix of numbers and it is nonsingular when its determinant is nonzero. Thus the Inverse Function Theorem tells us that it is sufficient to check if a single number is nonzero to know whether $f$ is a diffeomorphism in a neighborhood of $\bar{x}$.

**Theorem 3.2** (Constant Rank Theorem)**.** *Let $\bar{x} \in U \subseteq \mathbb{R}^n$, $f : U \to \mathbb{R}^m$ be smooth and let us suppose that $\mathrm{d}_x f$ has constant rank $k$ for each $x \in U$.*

*Then there exist two open sets $V \subset \mathbb{R}^n$ and $W \subset \mathbb{R}^m$, and two diffeomorphisms $F : V \to F(V) \subseteq \mathbb{R}^n$ and $G : W \to G(W) \subseteq \mathbb{R}^m$ such that $\bar{x} \in V$ and*

$$(G \circ f \circ F^{-1})(x) = (x_1, \dots, x_k, 0, \dots, 0),$$

*for each $x \in U$.*

*Proof.* Let $\xi \in \mathbb{R}^k$ and $\eta \in \mathbb{R}^{n-k}$ such that $(\xi, \eta) \in U$; there exist $f_1 : U \to \mathbb{R}^{m-k}$ and $f_2 : U \to \mathbb{R}^k$ such that

$$f(\xi, \eta) = (f_1(\xi, \eta), f_2(\xi, \eta))$$

and $\mathrm{d}_{\bar{x}} f_1$ has rank $k$ ($\mathrm{d}_{\bar{x}} f_1$ is represented by a $k \times n$ matrix).

Define $F : U \to \mathbb{R}^n$ as

$$F(\xi, \eta) = (f_1(\xi, \eta), \eta);$$

by Theorem 3.1 there exists a neighborhood $V \subseteq U$ of $\bar{x}$ such that $F|_V : V \to F(V)$ is a diffeomorphism and $F(V)$ can be chosen connected.

There exists a smooth function $g : F(V) \to \mathbb{R}^{m-k}$ such that

$$f(F^{-1}(\xi, \eta)) = (\xi, g(\xi, \eta)).$$

Since $\mathrm{d}(f \circ F^{-1}) = \mathrm{d}f \circ \mathrm{d}F^{-1}$ and $\mathrm{d}F$ is bijective, it follows that $\mathrm{rk}\,(\mathrm{d}(f \circ F^{-1})) = \mathrm{rk}\,(\mathrm{d}f) = k$ on $F(V)$. Explicitly one has

$$J(f \circ F^{-1}) = \begin{pmatrix} I_k & 0 \\ * & (g^r_{\eta_s})_{\substack{r=k+1,\dots,n \\ s=k+1,\dots,m}} \end{pmatrix} \quad \text{on } F(V),$$

then necessarily $g^r_{\eta_s} = 0$ on $F(V)$ and thus $g$ does not depend on $\eta$. We write $\tilde{g} : \mathbb{R}^k \to \mathbb{R}^{m-k}$ for the restriction of $g$ to the first $k$ components. It is now sufficient to define $G : W \to G(W)$ where $G(u, v) := (u, v - \tilde{g}(u))$.

□

**Definition 3.1.** Let $U \subseteq \mathbb{R}^n$ be an open set. The set

$$\mathcal{S} := \{(x_1, \dots, x_k, x_{k+1}, \dots, x_n) \in U \ : \ x_{k+1} = c_{k+1}, \dots, x_n = c_n\},$$

for some constants $c_i \in \mathbb{R}$, is called $k$-slice.

**Remark 3.2.** Instead of saying *"there exist two open sets $V \subset \mathbb{R}^m$ and $W \subset \mathbb{R}^n$, and two diffeomorphisms $\phi : V \to \phi(V) \subseteq \mathbb{R}^n$ and $\psi : W \to \psi(W) \subseteq \mathbb{R}^m$"* we will write *"there exist coordinates $(x_1, \dots, x_n)$ centered in $x$ and $(y_1, \dots, y_n)$ centered in $y$"*.

**Example 3.1** (Curves in $\mathbb{R}^2$)**.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth function. We would like to study the cases when the 0-level set of $f$ is a curve in $\mathbb{R}^2$. Assume $\partial f / \partial y \neq 0$. Then locally there exists a unique $\phi : I \to \mathbb{R}$ such that $f(t, \phi(t)) = 0$.

Let us consider the function $\varphi : I \to \mathbb{R}^2$, $\varphi(t) = (t, \phi(t))$. The function

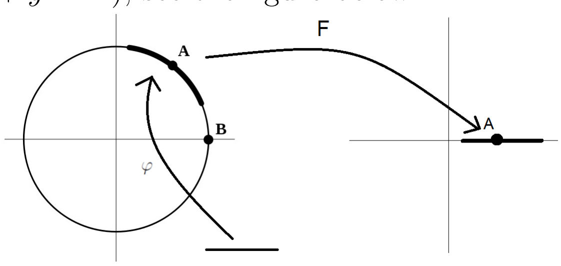
$$\begin{aligned} F \ : & \varphi(U) \to \mathbb{R}^2 \\ & (x_1, x_2) \mapsto (x_1, f(x_1, x_2)) \end{aligned}$$

is a diffeomorphism on an open set $V \subseteq \varphi(U)$, since

$$\det(JF) = \det \begin{pmatrix} 1 & 0 \\ f_x & f_y \end{pmatrix} \neq 0.$$

Notice that $F \circ \varphi(t) = (t, 0)$, and thus we found coordinates as in Theorem 3.2.

For example, if $f$ is as in Example 2.4, the function $F$ is given by $F(x, y) := (x, x^2 + y^2 - 1)$, see the figure below.



**Example 3.2** (Surfaces in $\mathbb{R}^3$)**.** In a similar way as in the previous example let us consider the level set of a smooth function $f : \mathbb{R}^3 \to \mathbb{R}$ with $f_z(x) \neq 0$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is such that $f(x) = 0$. Then, locally around $x$, there exists a function $\phi : \mathbb{R}^2 \to \mathbb{R}$ such that $f(\xi_1, \xi_2, \phi(\xi_1, \xi_2)) = 0$.

The function

$$\begin{aligned} F \, : & \varphi(U) \to \mathbb{R}^3 \\ & (\xi_1, \xi_2, \xi_3) \mapsto (\xi_1, \xi_2, f(\xi_1, \xi_2, \xi_3)) \end{aligned}$$

has non-singular Jacobian in $x$, and thus locally (in a neighborhood of $x$) is a diffeomorphism. If we set $\varphi : \mathbb{R}^2 \to \mathbb{R}^3$ with $\varphi((x, y)) = (x, y, \phi(x, y))$, we obtain the same result as in Theorem 3.2:

$$F\left(\varphi(\xi_1, \xi_2)\right) = (\xi_1, \xi_2, 0).$$

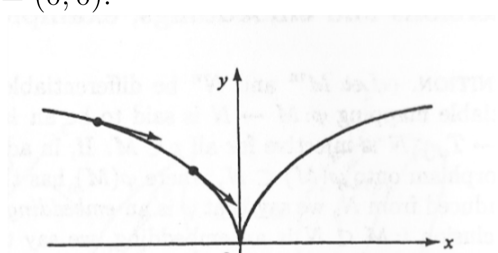
# 4 Immersion and Embedding

**Definition 4.1** (Immersion)**.** Let $U \in \mathbb{R}^n$ be an open set and $m \geq n$. A smooth function $f : U \to \mathbb{R}^m$ is called *immersion* at $\bar{x}$ if $\mathrm{d}_{\bar{x}} f$ is injective. If it is true for all $x \in U$, we say simply immersion.

**Example 4.1** (Canonical Immersion)**.** Let us consider $\iota : \mathbb{R}^n \to \mathbb{R}^{n+k}$, $\iota(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0, \ldots, 0)$ is an immersion. The Theorem 3.2 states that an immersion is equivalent (in the sense of Remark 3.2) to the canonical immersion.

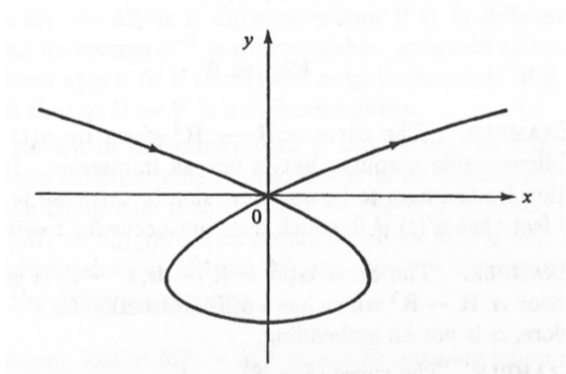
**Example 4.2.** The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t, |t|)$ is not smooth in $t = 0$ and thus it is not an immersion.

**Example 4.3.** The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^3, t^2)$ is not an immersion at $t = 0$. Indeed $\gamma'(0) = (0, 0)$.




**Definition 4.2** (Embedding)**.** An *embedding* $f : U \to \mathbb{R}^m$ is an immersion which is also an homeomorphism onto its image.

**Example 4.4.** The curve $\gamma : \mathbb{R} \to \mathbb{R}^2$, $t \mapsto (t^3 - 4t, t^2 - 4)$ is an immersion, since its derivative $\gamma'(t) = (3t^2 - 4, 2t) \neq 0$ for each $t \in \mathbb{R}$; but it is not an embedding since $\gamma(2) = \gamma(-2)$.

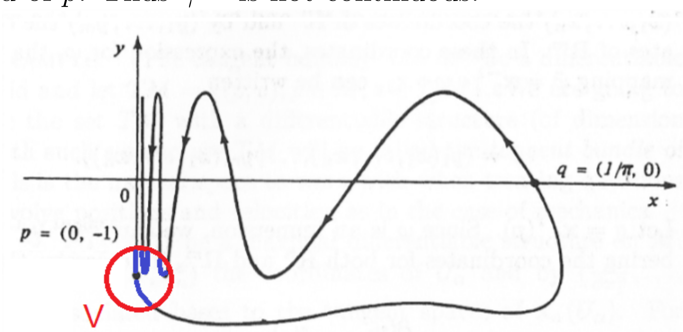



**Example 4.5.** The curve $\gamma : (-3, 0) \to \mathbb{R}^2$,

$$\gamma(t) = \begin{cases} (0, -(t+2)) & \text{if } t \in (-3, -1) \\ \text{regular curve} & \text{if } t \in (-1, -\frac{1}{\pi}) \\ (-t, -\sin\frac{1}{t}) & \text{if } t \in (-\frac{1}{\pi}, 0) \end{cases}$$

is an immersion without self intersections (see the figure below). Nevertheless, $\gamma$ is not an embedding. Let us consider a neighborhood $V$ of the point $p$ in $\mathbb{R}^2$, see the figure below; the blue set $V \cap \gamma((-3, 0))$ is a neighborhood of the point $p$ in the topology on $\gamma(I)$, induced from $\mathbb{R}^2$ and it is not connected.

Now, notice that $\gamma(-1) = p$. Take the interval $(-1 - \epsilon, -1 + \epsilon)$ in $I$, an open interval containing $-1$. The pre-image via $\gamma^{-1}$ of this set is not a neighborhood of $p$. Thus $\gamma^{-1}$ is not continuous.




**Example 4.6.** Let us consider the map $\exp : \mathbb{R} \to \mathbb{R}$. Its differential is given by

$$\mathrm{d}_x f \ : v \mapsto e^x \cdot v$$

and it is an isomorphism. Thus exp is an embedding and a submersion.

# 5 Submanifolds of $\mathbb{R}^n$

**Definition 5.1.** A subset $M \subseteq \mathbb{R}^n$ is a *submanifold* if for each $x \in M$ there exist an open neighborhood $U$ of $x$ in $X$, $m > 0$ and a smooth map $g : U \to \mathbb{R}^m$ such that

- $M \cap U = \{\xi \in U \, : \, g(\xi) = 0\}$,
- $g$ is a submersion in $p$, for each $p \in M \cap U$.

**Example 5.1** (Surfaces in $\mathbb{R}^3$)**.** We can define a surface $\mathcal{S}$ in $\mathbb{R}^3$ as the zero level set of a function $f \in \mathcal{C}^\infty(\mathbb{R}^3, \mathbb{R})$ which is a submersion for each $x \in \mathcal{S}$.

For example, let us consider

$$f((x_1, x_2, x_3)) = x_1^2 + x_2^2 + x_3^2 - 1 \,.$$

The zero level set of this function is the sphere $S^2$. The differential of $f$ is given by

$$\mathrm{d}_x f(v) := 2\, x_1 v_1 + 2\, x_2 v_2 + 2\, x_3 v_3 = 2 \, \langle x, v \rangle \,.$$

For each $x \in S^2$ we have $\mathrm{d}_x f(x) = 2\, ||x||^2 \neq 0$.

**Definition 5.2.** Let $M$ be a submanifold. For each $x \in M$ we denote with $T_x M$ the set of $v \in T_x\mathbb{R}^n$ such that there exist $\delta > 0$ and a smooth map $\gamma : (-\delta, \delta) \to \mathbb{R}^n$ such that

$$\forall t \in (-\delta, \delta) : \gamma(t) \in M, \quad \gamma(0) = x \quad \text{and} \quad \gamma'(0) = v.$$

**Theorem 5.1.** *Let $M$ be a submanifold of $\mathbb{R}^n$ and $g$ be a smooth map. We have*

$$T_x M = \ker\left(\mathrm{d}_x g\right) \,.$$

*Proof.* First let us prove

$$T_x M \subseteq \ker\left(\mathrm{d}_x g\right) \,.$$

Let $v \in T_x M$ and $\gamma$ as in the previous definition. Since $g \circ \gamma \equiv 0$, one has

$$\mathrm{d}_x g(v) = \mathrm{d}_{\gamma(0)} g(\gamma'(0)) = (g \circ \gamma)'(0) = 0 \,.$$

Vice-versa let us prove the inclusion

$$\ker\left(\mathrm{d}_x g\right) \subseteq T_x M.$$

There exists $V$ such that $\mathbb{R}^n = T_x\mathbb{R}^n = \ker(\mathrm{d}_x g) \oplus V$, $x = x^{(1)} + x^{(2)}$ where $x^{(1)} \in \ker(\mathrm{d}_x g)$ and $x^{(2)} \in V$. Thus

$$\mathrm{d}_x g_{|V} \,:\, V \to \mathbb{R}^m$$

is bijective. We need the following Lemma, which is a consequence of the Dini's theory.

**Lemma 5.1.** *Let $V, W$ two vector subspace of $\mathbb{R}^n$ such that $\mathbb{R}^n = W \oplus V$. We write $x = x^{(1)} + x^{(2)}$ where $x^{(1)} \in W$ and $x^{(2)} \in V$. Let us suppose that $g(x) = 0$ and that the linear map*

$$\mathrm{d}_x g_{|V} \,:\, V \to \mathbb{R}^m$$

*is bijective.*

*Then there exist an open set $U_1$ of $x^{(1)}$ in $W$, $U_2$ of $x^{(2)}$ in $V$ and a smooth map $\phi \,:\, U_1 \to W$ such that $\phi(x^{(1)}) = x^{(2)}$, $\phi(U_1) \subseteq U_2$ and*

$$\forall \xi^{(1)} \in U_1, \forall \xi^{(2)} \in U_2 \,:\, g(\xi^{(1)} + \xi^{(2)}) = 0 \iff \xi^{(2)} = \phi(\xi^{(1)}).$$

*Furthermore, for each $\xi^{(1)} \in U_1$ and for each $v^{(1)} \in W$*

$$\mathrm{d}_{\xi^{(1)}+\phi(\xi^{(1)})} g\left(\mathrm{d}_{\xi^{(1)}}\phi(v^{(1)})\right) = -\mathrm{d}_{\xi^{(1)}+\phi(\xi^{(1)})} g\left(v^{(1)}\right)$$

Let $v \in \ker(\mathrm{d}_x g) = W$, let $\delta > 0$ such that $(x^{(1)} + tv) \in U_1$ for each $t \in (-\delta, \delta)$. Then

$$\gamma(t) = x^{(1)} + tv + \phi(x^{(1)} + tv)$$

defines a curve with $\gamma(0) = x$, $\gamma(t) \in M$ and

$$\gamma'(0) = v + \mathrm{d}_{x^{(1)}}\phi\,(v) = v\,.$$

□

**Example 5.2** (Curves in $\mathbb{R}^2$)**.** We have already seen how to define a curve in $\mathbb{R}^2$ as the zero level set of a function $f \in \mathcal{C}^\infty(\mathbb{R}^2, \mathbb{R})$ with $f_y \neq 0$. This last condition ensures that $f$ is a submersion.

The tangent space at a point $p = (x_0, y_0)$ is given by those vectors $v = (x, y)$ such that

$$f_x(p)(x - x_0) + f_y(p)(y - y_0) = 0,$$

which means $v \in \ker(\mathrm{d}_p f)$.

# References


[D] M. Degiovanni, *Approfondimenti di Analisi Matematica*

[S] M. Spera, *diffgeotopo v2 I-II-III-IV*