Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture I

PROLOGUE: REVIEW OF CALCULUS IN SEVERAL VARIABLES I (in a nutshell)

or an open subset U thereof

$$f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

smooth function

$y = f(x)$

$y = y(x)$

$$p = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix} \longmapsto \begin{pmatrix} y^1 = f^1(x^1 .. x^n) \\ \vdots \\ y^m = f^m(x^1 .. x^n) \end{pmatrix} = f(p)$$

$\varphi \to \mathbb{R}$ smooth function

$$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{\varphi} \mathbb{R}, \quad \varphi \circ f$$

[tangent space to $\mathbb{R}^n$ at p] a copy of $\mathbb{R}^n$: vectors "applied" at p $\equiv$ directional derivatives

$$df|_p \;:\; T_p\mathbb{R}^n \longrightarrow T_{f(p)}\mathbb{R}^m$$

$$\text{III} \qquad \Psi \qquad\qquad \Psi$$

$$f_*|_p \qquad h \longmapsto df|_p \cdot h$$

push-forward

$\left(h = (h^i) \leftrightarrow h^i \frac{\partial}{\partial x^i} \right)$

Jacobian matrix $(m \times n)$

★ differential of f at p

explicitly

$$h = \begin{pmatrix} h^1 \\ \vdots \\ h^n \end{pmatrix} \longmapsto \begin{pmatrix} df^1(h) \\ \vdots \\ df^m(h) \end{pmatrix} = \begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix} \begin{pmatrix} h^1 \\ \vdots \\ h^n \end{pmatrix}$$

In particular

$$\frac{\partial}{\partial x^j} \longleftrightarrow \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \leftarrow j \longmapsto \begin{pmatrix} \frac{\partial f^1}{\partial x^j} \\ \vdots \\ \frac{\partial f^m}{\partial x^j} \end{pmatrix} \longleftrightarrow \frac{\partial f^\ell}{\partial x^j} \cdot \frac{\partial}{\partial y^\ell}$$

Einstein's convention
sum over repeated indices

$f^\ell = y^\ell$

Chain rule:

$$\frac{\partial}{\partial x^j}(\varphi \circ f) = \frac{\partial f^\ell}{\partial x^j} \frac{\partial \varphi}{\partial y^\ell}$$

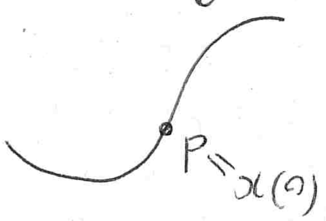
stenographically

$$\frac{\partial}{\partial x} \longmapsto \frac{\partial y}{\partial x} \frac{\partial}{\partial y}$$

★ Complementary description of $f_*|_p$ (geometrical / mechanical)

(for a function $f: \mathbb{R}^n \to \mathbb{R}$)

Let $\underset{\mathcal{C}}{x} = x(t) = \begin{pmatrix} x^1(t) \\ \vdots \\ x^n(t) \end{pmatrix}$ $\quad t \in I$ interval containing 0, to fix ideas



$x(0) = p$ be a smooth curve $\mathcal{C}$ issuing from p

Let $\boxed{F(t) := f(x(t)) \qquad t \in I}$

$(F = f \circ x)$

Then $\dot{F}(t) \equiv \dfrac{dF}{dt}(t) = \dfrac{\partial f}{\partial x^j} \dfrac{dx^j}{dt}$ — (Chain rule) Einstein convention

$$\boxed{\dot{F}(0) = \underbrace{\overset{\circ}{f_{x^j}}}_{} \dot{x}_j(0) \equiv \underbrace{f_*|_p}_{\text{differential of } f \text{ at } p}(\dot{x})}$$

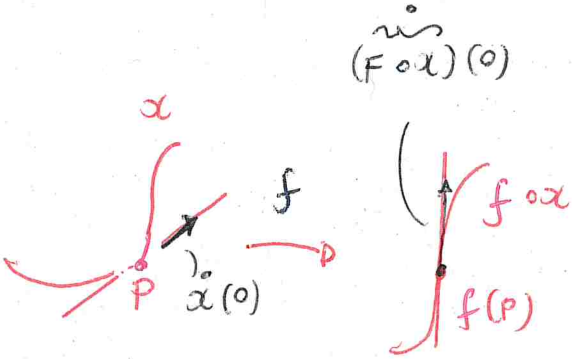
$\bullet = \dfrac{d}{dt}$

velocity field of $x = x(t)$

$\dot{x} = \begin{pmatrix} \dot{x}^1 \\ \vdots \\ \dot{x}^n \end{pmatrix}$ } the previous h

differential of f at p! in this case an $n \times 1$ - matrix i.e. a dual vector

in general one works in components



In general, a smooth map $f: \mathbb{R}^n \to \mathbb{R}^m$ transforms curves in $\mathbb{R}^n$ through p into curves in $\mathbb{R}^m$ through $F(p)$, and $F_*|_p$ relates their corresponding velocity vectors (a rephrasing of the chain rule)

Dini's theory

☆ Inverse function theorem

$f : \underset{\text{open set}}{\mathcal{U} \subset \mathbb{R}^n} \longrightarrow \mathbb{R}^n$

$p \in \mathcal{U}$ If $f_*|_p$ is an isomorphism,

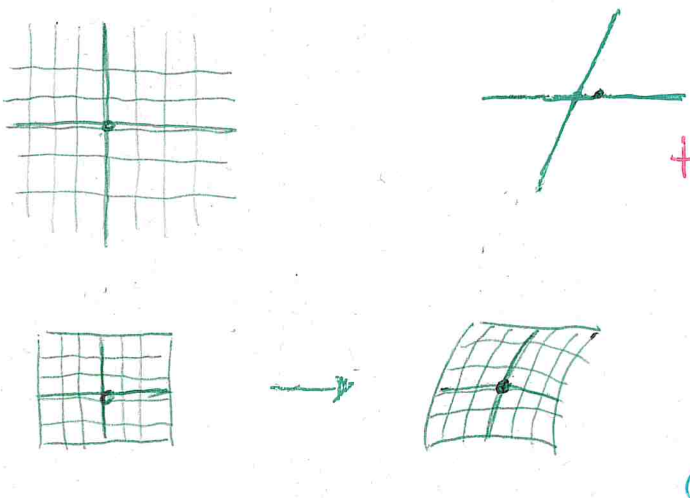
then f is locally a diffeomorphism (invertible, with smooth inverse)

locally: in a suitable open neighborhood $\mathcal{V}$, $\mathcal{U} \supset \mathcal{V} \ni p$

($f|_{\mathcal{V}} : \underset{\mathbb{R}^n \,\cup\!\!\!\!\!\supset}{\mathcal{V}} \longmapsto f(\mathcal{V}) \subset \mathbb{R}^n$ invertible, with smooth inverse)

(proven by the contraction lemma (Banach - Caccioppoli theorem)

example in $\mathbb{R}^2$



transversal intersection

images of coordinate lines: genuine curvilinear coordinates (intersecting transversally)

From the inverse function theorem one gets the rank theorem & the implicit function theorem

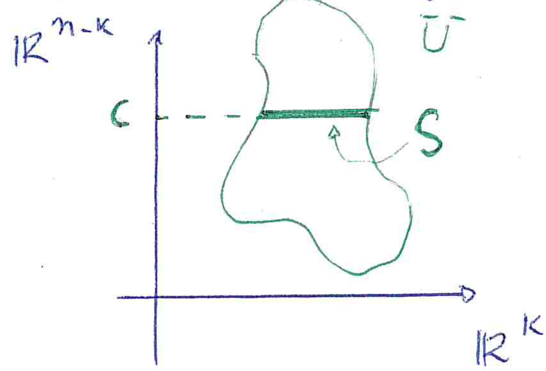
∗ k-slices in $\mathbb{R}^n$

$$\boxed{S = \{(x^1, \dots x^k, x^{k+1}, \dots x^n) \in \bar{U} \;/\; x^{k+1} = c^{k+1}, \dots x^n = c^n\}}$$

open set in $\mathbb{R}^n$

c^i constants

S is called a k-slice of $\bar{U}$



∗ The rank theorem

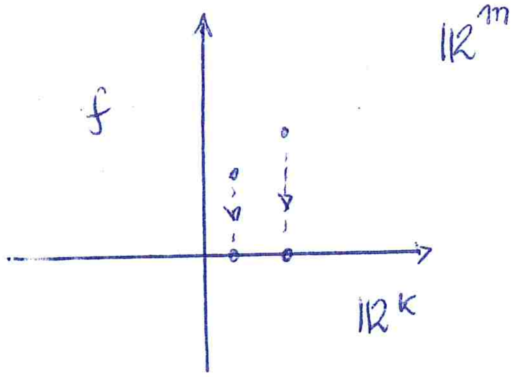
Let $f : \mathbb{R}^m \to \mathbb{R}^n$ smooth, with constant rank $k \le m$. Then, $\forall p \in \mathbb{R}^m$, there exists coordinates $(x^1 \dots x^m)$ centred at p (i.e. $x^i(p) = 0$, $i = 1 \dots m$) and coordinates $(y^1, \dots, y^n)$ centred at $f(p)$ such that

i.e., one can perform a change of coordinates via suitable diffeomorphisms

$$f(x^1, \dots x^k, x^{k+1} \dots x^m) = (x^1, \dots x^k, 0, 0 \dots 0)$$

a slice in $\mathbb{R}^n$

$k \le \min(m, n)$



* Elementary Dini's theory revisited via the inverse function theorem

1. Curves in $\mathbb{R}^2$ (given implicitly)

$$f : \mathbb{R}^2 \to \mathbb{R} \qquad f(x,y) = 0$$

0-level set of f

(without loss of generality (wlog))

$P_0 : (x_0, y_0) \qquad f(P_0) = 0$

Assume $\frac{\partial f}{\partial y}(P_0) \neq 0$. Then locally $\exists!$ $y = y(x)$, $y_0 = y(x_0)$ (smooth..)

such that $f(x, y(x)) \equiv 0 \quad \forall x \in I$ a suitable interval

This is the standard Dini result (actually, part of it...)

Define

$$F : \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \longmapsto \begin{pmatrix} X = x \\ Y = f(x,y) \end{pmatrix} \in \mathbb{R}^2$$

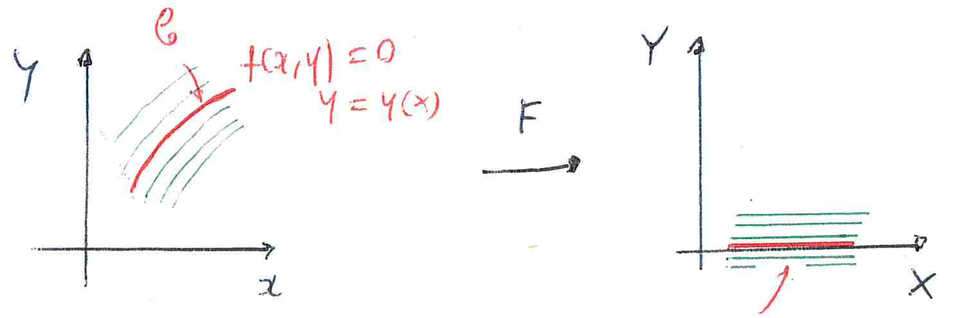
$$dX = dx$$
$$dY = f_x\, dx + f_y\, dy \quad \leadsto \quad F_* : \begin{pmatrix} 1 & 0 \\ f_x & \boxed{f_y} \end{pmatrix}$$

compute formally, for the time being

$$f_y^0 \neq 0 \implies F_x^0 \text{ isomorphism}$$

$\Rightarrow$ locally F is a diffeomorphism (inverse function theorem)

level sets are "rectified"



$F(\mathcal{C})$: a 1-slice in $\mathbb{R}^2$

2. surfaces in $\mathbb{R}^3$ (implicit description)

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad f(x,y,z) = 0 \qquad f(P_0) = 0$$

$P_0 : (x_0, y_0, z_0)$

$$\to \quad f_z^0 \neq 0$$

Then locally around $P_0' : (x_0, y_0)$, $\exists\, \varphi$ s.t. $z = \varphi(x,y)$, $z_0 = z(x_0, y_0)$

with $\quad f(x, y, \varphi(x,y)) \equiv 0 \qquad$ part of (Dini's theorem)

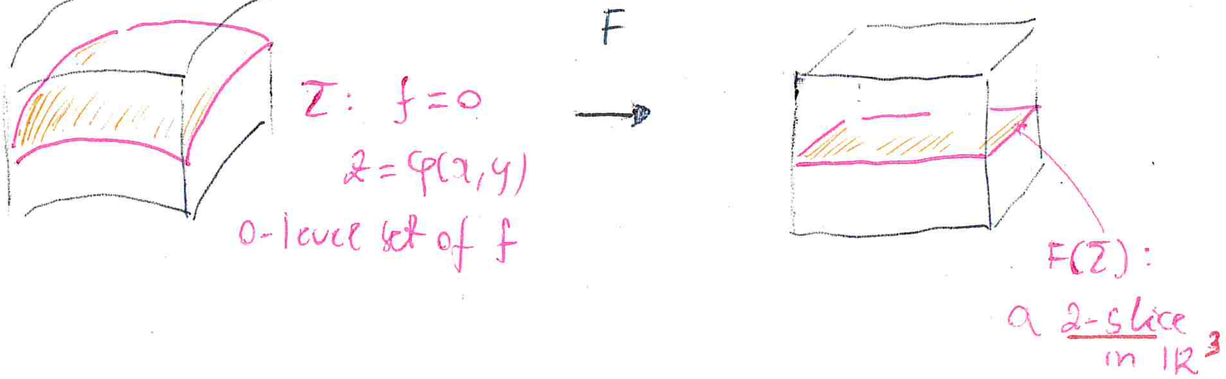
Reinterpretation in terms of the inverse function theorem

$$F : \mathbb{R}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} X = x \\ Y = y \\ Z = f(x,y,z) \end{pmatrix}$$

$$F_* = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ f_x & f_y & \boxed{f_z} \end{pmatrix} \qquad f_z^0 \neq 0 \Rightarrow F_*^0 \text{ isomorphism}$$

$\Rightarrow$ locally F is a diffeomorphism

local appearance of F



3. Curves in $\mathbb{R}^3$ (implicit description)

Given $\mathcal{C}: \begin{cases} f=0 \\ g=0 \end{cases}$ $\quad \mathbb{R}^3 \to \mathbb{R}^2, \ \begin{pmatrix} x\\y\\z\end{pmatrix} \overset{(f,g)}{\longmapsto} \begin{pmatrix}0\\0\end{pmatrix}$ $\quad f=f(x,y,z)$, $g=g(x,y,z)$ smooth

0-level set of (f,g)

$P_0 : (x_0, y_0, z_0) \in \mathcal{C} \ : \ f(P_0) = g(P_0) = 0$

Jacobian

$$\frac{\partial(f,g)}{\partial(y,z)}(P_0) \neq 0 \implies \mathcal{C} \text{ locally given by } \begin{cases} x = x \\ y = y(x) \\ z = z(x) \end{cases} \text{ smooth}$$

$$\frac{\partial(f,g)}{\partial(y,z)} \equiv \begin{vmatrix} \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}$$

with $y_0 = y(x_0)$, $z_0 = z(x_0)$

$$\begin{cases} f(x, y(x), z(x)) \equiv 0 \\ g(x, y(x), z(x)) \equiv 0 \end{cases}$$

Define

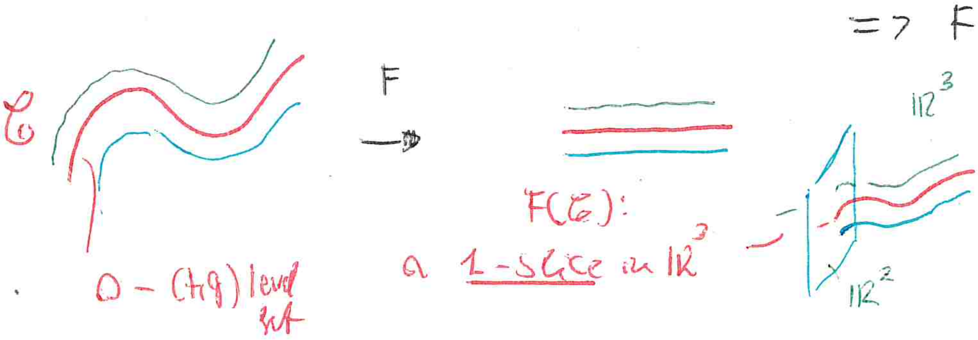
$$F : \mathbb{R}^3 \to \mathbb{R}^3$$

$$\begin{pmatrix} x\\y\\z\end{pmatrix} \longmapsto \begin{pmatrix} X = x \\ Y = f(x,y,z) \\ Z = g(x,y,z) \end{pmatrix}$$

$$F_* = \begin{pmatrix} 1 & 0 & 0 \\ f_x & f_y & f_z \\ g_x & g_y & g_z \end{pmatrix}$$

$\frac{\partial(f,g)}{\partial(y,z)}(P_0) \neq 0 \implies F_*^0$ isomorphism

$\implies F$ local diffeomorphism



★ Immersions, embeddings, submersions

$$F: \mathbb{R}^n \to \mathbb{R}^m$$

$$F_*\big|_p : T_p\mathbb{R}^n \to T_{F(p)}\mathbb{R}^m$$

$p \in \mathbb{R}^n$

- F immersion : $F_*\big|_p$ injective $\forall p$ (therefore $n \leq m$)

- F embedding : F immersion + $F: \mathcal{U} \to F(\mathcal{U}) \subset \mathbb{R}^m$ homeomorphism with respect to the relative topology (induced from the ambient space)

  $\mathcal{U}$: domain of $\mathcal{U}$ (open)

  equivalently:

  F injective
  F_* injective
  F homeomorphism onto image (relative topology)

- F submersion : $F_*\big|_p$ surjective $\forall p$ (hence $n \geq m$)

From Dini's theory:

$\underbrace{F \text{ injective } \& \ F_* \text{ injective}}_{F \text{ injective immersion}}$

$\Rightarrow$ F is locally an embedding

For $n=m$, F_* injective $\Leftrightarrow$ F_* surjective (N+R theorem)

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Lecture II

REVIEW OF CALCULUS IN SEVERAL VARIABLES II

Examples

① $F : \mathbb{R} \to \mathbb{R}$
$\quad x \mapsto e^x$ $\qquad \operatorname{Im}(F) = \mathbb{R}^+ = \{ y \in \mathbb{R} / y > 0 \}$

F is a <u>submersion</u>! $\quad F_*|_x = e^x \qquad (d e^x = e^x dx)$

explicitly:

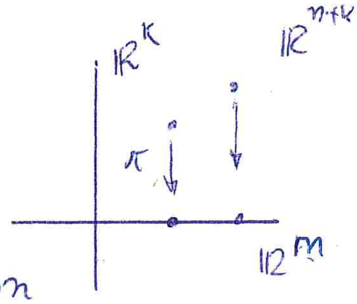
$$e^x : \underset{\cong \mathbb{R}}{T_x \mathbb{R}} \to \underset{\cong \mathbb{R}}{T_{e^x} \mathbb{R}}$$
$$h \mapsto e^x . h$$

surjective $\forall x \in \mathbb{R}$
(actually, an isomorphism)

F is not surjective: however, it becomes surjective as a map $F : \mathbb{R} \to F(\mathbb{R}) = \mathbb{R}^+$ (and it is obviously an embedding)

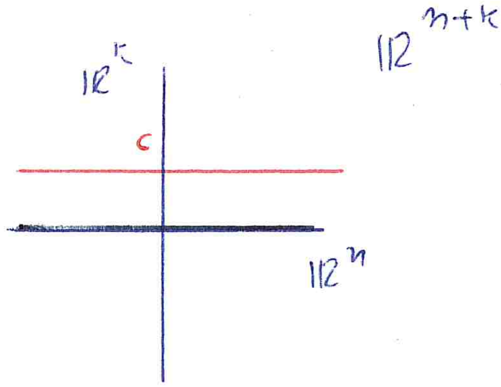
② $\pi : \mathbb{R}^{n+k} \to \mathbb{R}^n$
$\quad (x^1 .. x^n, x^{n+1}, .. x^{n+k}) \mapsto (x^1, x^2, x^n)$

(π projection) is a surjective submersion



③ $i : \mathbb{R}^n \to \mathbb{R}^{n+k}$
$\quad (x^1 .. x^n) \mapsto (x^1 ... x^n, 0, 0, 0)$
$\qquad$ or $(x^1 ... x^n, c^1 ... c^k)$

embedding



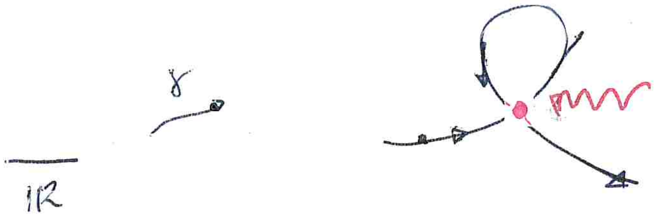
4 $\quad \gamma : \underset{\text{open interval}}{I} \longrightarrow \mathbb{R}^n \qquad n \geq 1$

smooth curve

γ immersion : $\quad \dot{\gamma}(t) \neq 0 \quad \forall t \in I$

γ embedding : $\quad \gamma$ injective , $\dot{\gamma} \neq 0$, $\gamma : I \to \gamma(I)$ homeomorphism

4'



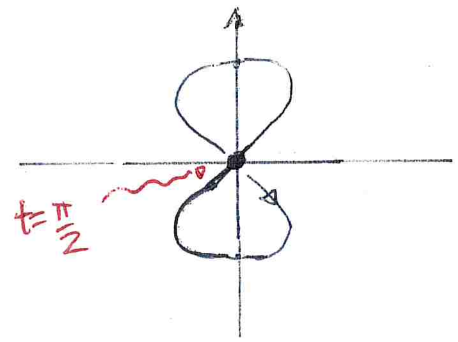
an immersion (when $\dot{\gamma} \neq 0$) but not an embedding γ is not injective, so it cannot be a homeomorphism

4''

$$\gamma : (\overbrace{-\tfrac{\pi}{2}, \tfrac{3\pi}{2}}^{I}) \longmapsto \mathbb{R}^2$$

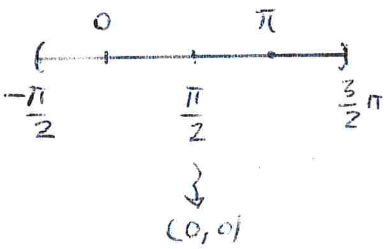
$$\gamma(t) = (\sin 2t, \cos t) \qquad x^2 = 4y^2(1-y^2)$$

(lemniscate)



$\dot{\gamma} = (2\cos 2t, -\sin t)$

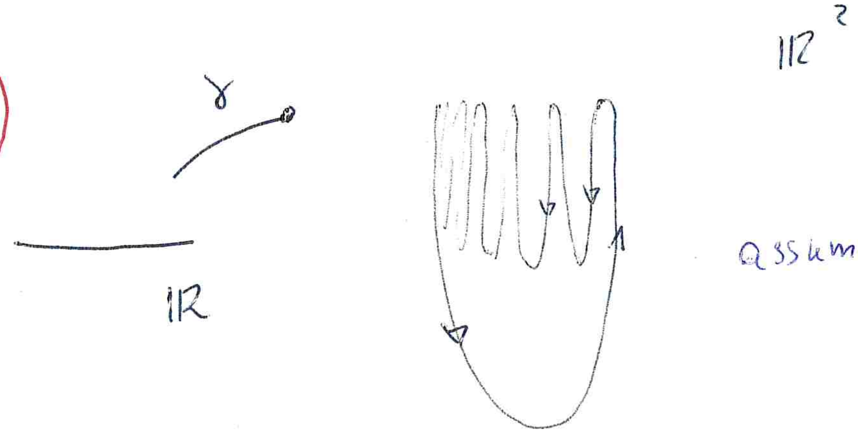
$\dot{\gamma}(\frac{\pi}{2}) = (2, -1)$



I is not compact but

$\gamma(I)$ is compact ($\Rightarrow \gamma$ cannot be an isomorphism: compactness is a topological property

4'''



assume $\mathring{\gamma} \neq \emptyset$

γ is not an embedding: $\mathbb{R}$ is locally arcwise connected but $\gamma(\mathbb{R})$ is not!

X locally arcwise connected: $\forall p \in X$, $\forall\, \mathcal{U} \ni p$, $\exists\, p \in V \subset \mathcal{U}$ arcwise connected

points like this do not admit any arcwise connected neighbourhood

"Kronecker foliation"

5 A quite important example!

$$\gamma : \mathbb{R} \longrightarrow \mathbb{T}^2 = S^1 \times S^1$$

$$t \longmapsto \gamma(t) = \left(e^{2\pi i t}, e^{2\pi i c t}\right) \quad c \in \mathbb{R} \setminus \mathbb{Q}$$

$\operatorname{Im}\gamma$ is dense in $\mathbb{T}^2$ and it not locally arcwise connected (again γ cannot be a homeomorphism)

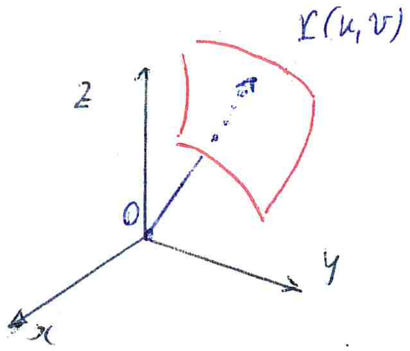
Alternatively, $\mathbb{Z} \subset \mathbb{R}$ is a discrete set (i.e. it has no limit points) whilst $\gamma(\mathbb{Z})$ is not

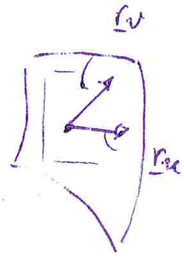
⑥ Parametric surfaces in $\mathbb{R}^3$

Σ described via

$$\underline{r}: \quad \mathcal{U} \ni (u,v) \longmapsto \underline{r}(u,v) \in \mathbb{R}^3$$

$$F(u,v) \equiv \underline{r}(u,v) = \begin{pmatrix} x(u,v) \\ y(u,v) \\ z(u,v) \end{pmatrix}$$





regular surface — recall $\mathcal{C}^\infty$

$$\begin{cases} \bullet\ \underline{r} \in \mathcal{C}^\infty \\ \bullet\ \underline{r} \text{ injective} \\ \bullet\ \underline{r}_u \times \underline{r}_v \equiv \dfrac{\partial \underline{r}}{\partial u} \times \dfrac{\partial \underline{r}}{\partial v} \neq \underline{0} \end{cases}$$

$\Longleftrightarrow \begin{pmatrix} | & | \\ \underline{r}_u & \underline{r}_v \\ | & | \end{pmatrix}$ has maximal rank $= 2$

$\Longleftrightarrow \underline{r}_u, \underline{r}_v$ l.i.

this is an example of injective immersion, since

$$F_* = \begin{pmatrix} | & | \\ \underline{r}_u & \underline{r}_v \\ | & | \end{pmatrix}$$

and the above conditions amount to injectivity

$\rightsquigarrow$ locally $\underline{r}$ is an embedding as well

In view of the preceding examples, it is easy to conceive examples of immersions which are not embeddings.