☆ n-dimensional (smooth) submanifolds of $\mathbb{R}^{n+k}$

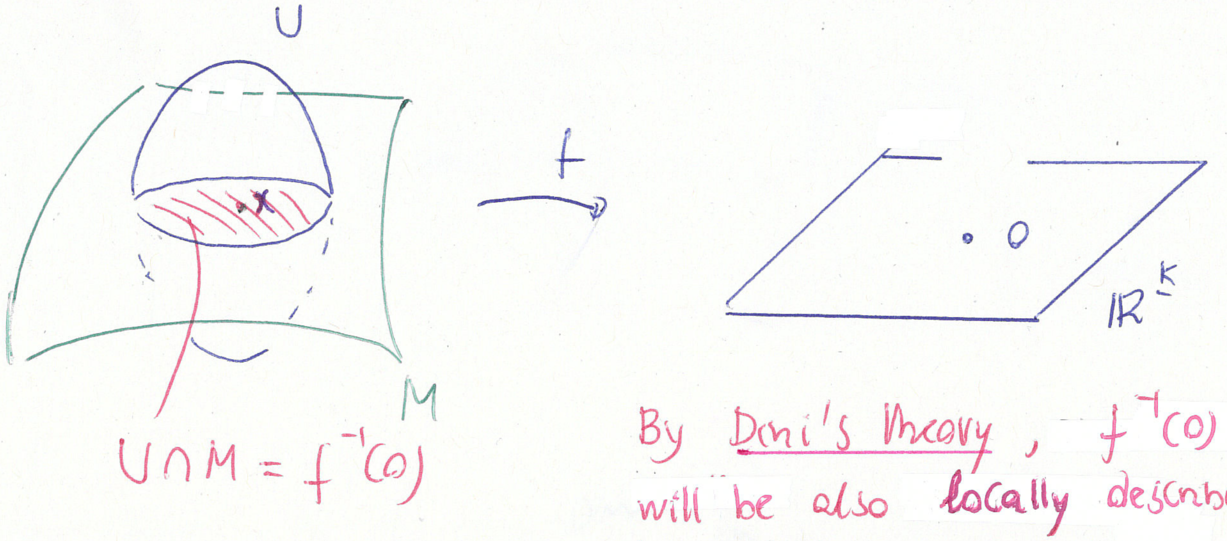
Definition: $M \subset \mathbb{R}^{n+k}$ is said to be a smooth n-dimensional submanifold of $\mathbb{R}^{n+k}$

if $\forall x \in M$, $\exists\ \mathcal{U} \ni x$ in $\mathbb{R}^{n+k}$
(neighbourhood)

and a submersion (smooth) $f : \mathcal{U} \to \mathbb{R}^k$

such that $$U \cap M = f^{-1}(0)$$

level set of f pertaining to $0 \in \mathbb{R}^k$
(0-level set)



By Dini's theory, $f^{-1}(0)$ will be also locally described by injective immersions (actually embeddings)

Given $x_0 \in f^{-1}(0)$,

$$\ker f_*|_x \leq \mathbb{R}^{n+k}$$

n-dimensional

subspace

is called the tangent space to M at x_0

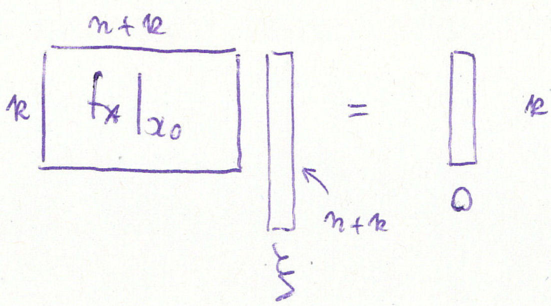
In fact, a moment's reflection shows that $\ker f_*|_{x_0}$ (an n-dimensional subspace of $\mathbb{R}^{n+k}$) consists precisely of the velocity vectors of curves issuing from x_0 and entirely lying on $U \cap M = f^{-1}(0)$ for sufficiently small t.

Indeed, for $x_0 \in f^{-1}(0)$ ($f(x_0) = 0$), let $x = x(t)$, $x(0) = x_0$, $t \in I$ (an interval containing 0) $\dot{x}(0) \equiv \xi \in \mathbb{R}^{n+k}$ be a smooth curve lying in $f^{-1}(0)$: the function $F = F(t) := f(x(t)) \equiv 0 \quad \forall t \in I$.

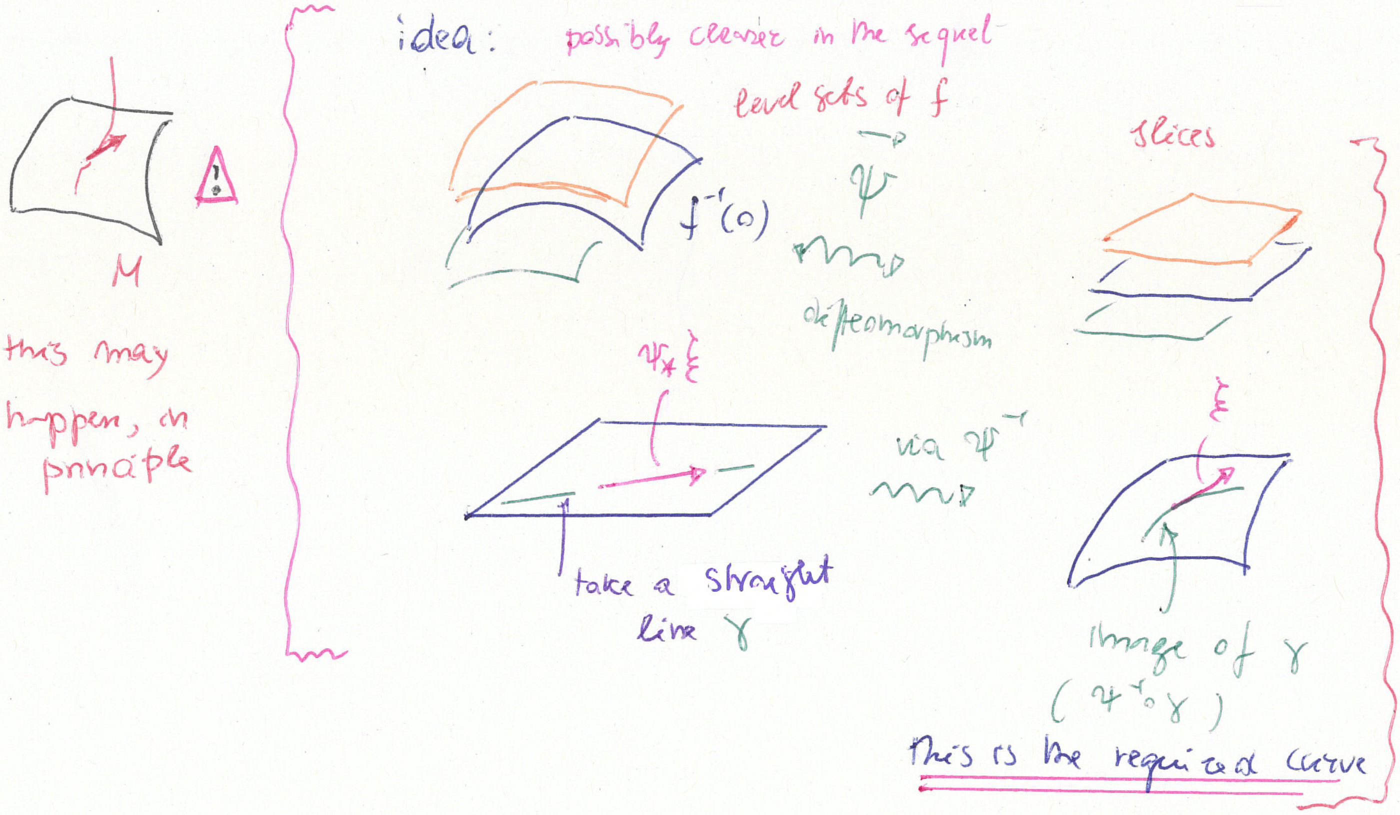
Differentiation with respect to t yields

$$\frac{\partial f}{\partial x^i} \dot{x}^i \equiv 0$$ and, in particular, at $t = 0$

$$\boxed{f_*|_{x_0} \xi = 0}$$



Conversely, as a consequence of Dini's theory, any vector in $\ker f_*|_{x_0}$ can be viewed as the velocity vector of a curve stemming from x_0 and <u>entirely</u> <u>contained</u> in $f^{-1}(0)$



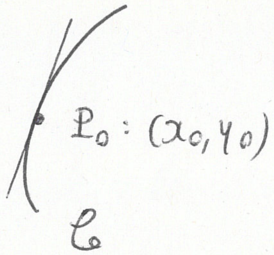
Examples

1. Curves in $\mathbb{R}^2$ (defined implicitly)

$f \in C^\infty(\mathbb{R}^2)$

$$\mathcal{C}: \quad f(x,y) = 0$$

0-level set of f



Let:

$$(\diamond) \qquad \frac{\partial f}{\partial y}(P_0) \neq 0$$

Then (Dini), there exists, locally $y = y(x)$ (C^∞)

$y(x_0) = y_0$, $\quad f(x, y(x)) \equiv 0 \quad x \in I$ interval containing x_0

Condition $(\diamond)$ ensures that f is submersive at P_0: (and locally as well)

$$f_*\big|_{P_0} = (f_x^0, f_y^0) \qquad (\Leftrightarrow \operatorname{rk}(f_*|_{P_0}) = 1 = \max)$$

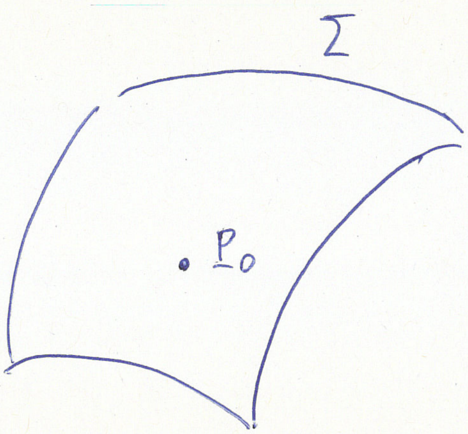
$f_y^0 \neq 0$

The tangent space to $\mathcal{C}$ at P_0 is precisely the (direction of the) tangent line to $\mathcal{C}$ at P_0: the equation of the latter is, indeed

$$(\diamond\diamond) \qquad f_x^0 (\underbrace{x - x_0}_{\xi_1}) + f_y^0 (\underbrace{y - y_0}_{\xi_2}) = 0 \qquad \underbrace{(f_x^0 \;\; f_y^0)}_{f_*|_{P_0}} \underbrace{\begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}}_{\xi} = 0$$

$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$ satisfies $(\diamond\diamond) \Leftrightarrow \xi \in \operatorname{Ker} f_*\big|_{P_0}$

2. Surfaces in $\mathbb{R}^3$ (defined implicitly)



$\Sigma: f(x,y,z)=0 \quad f\in C^\infty(\mathbb{R}^3)$

$P_0: (x_0, y_0, z_0)$

$(\diamond) \quad \frac{\partial f}{\partial z}(P_0) \neq 0 \quad \underset{\text{Dini}}{\Rightarrow}$

$z = z(x,y) \qquad z_0 = z(x_0,y_0)$

$f(x,y,z(x,y)) \equiv 0 \quad$ for $(x,y)\in \mathcal{U} \ni (x_0,y_0)$

Tangent plane to Σ at P_0:

$$(\diamond\diamond) \qquad \boxed{f_x^0 \underbrace{(x-x_0)}_{\xi_1} + f_y^0 \underbrace{(y-y_0)}_{\xi_2} + \underset{\neq 0}{f_z^0} \underbrace{(z-z_0)}_{\xi_3} = 0}$$

Condition $(\diamond)$ tells us that f is submersive at P_0 (and locally as well)

$$f_*|_{P_0} = (f_x^0, f_y^0, \underset{\neq 0}{f_z^0}) \qquad (\mathrm{rk}(f_*|_{P_0}) = 1 = \max)$$

$(\diamond\diamond)$ is tantamount to the condition

$$\boxed{\xi \in \ker(f_*|_{P_0})}$$

$$(f_x^0, f_y^0, f_z^0)\begin{pmatrix}\xi_1\\ \xi_2\\ \xi_2\end{pmatrix} = 0$$

3. Curves in $\mathbb{R}^3$ defined implicitly

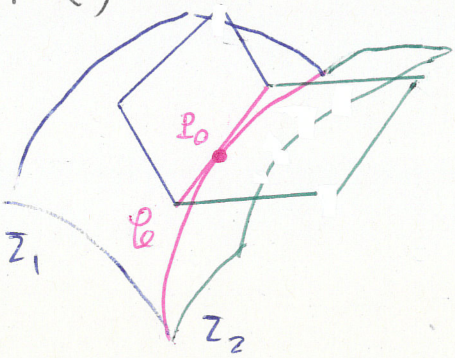
$$\mathcal{C}: \begin{cases} f(x,y,z) = 0 & \Sigma_1 \\ g(x,y,z) = 0 & \Sigma_2 \end{cases}$$

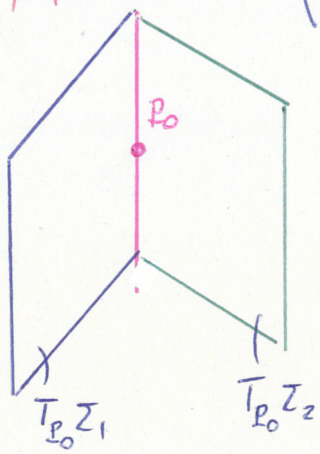
$\mathcal{C} = F^{-1}(0)$

$F: \mathbb{R}^3 \to \mathbb{R}^2$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} f(x,y,z) \\ g(x,y,z) \end{pmatrix}$$

$P_0: (x_0, y_0, z_0)$





surjectivity of $F_*|_{P_0}$ can be imposed by requiring e.g.

$$\frac{\partial(f,g)}{\partial(y,z)}(P_0) \neq 0$$

$\Downarrow$ Dini

locally around x_0:

$$\mathcal{C}: \begin{cases} x = x \\ y = y(x) \\ z = z(x) \end{cases}$$

$y(x_0) = y_0$, $z(x_0) = z_0$

$$\begin{cases} f(x, y(x), z(x)) \equiv 0 \\ g(x, y(x), z(x)) \equiv 0 \end{cases}$$

Tangent line to $\mathcal{C}$ at P_0

$$\begin{cases} f_x^0 \underbrace{(x - x_0)}_{\xi_1} + f_y^0 \underbrace{(y - y_0)}_{\xi_2} + f_z^0 \underbrace{(z - z_0)}_{\xi_3} = 0 & \leftarrow T_{P_0}\Sigma_1 \\ g_x^0 \underbrace{(x - x_0)}_{\xi_1} + g_y^0 \underbrace{(y - y_0)}_{\xi_2} + g_z^0 \underbrace{(z - z_0)}_{\xi_3} = 0 & \leftarrow T_{P_0}\Sigma_2 \end{cases}$$

intersecting transversally

rewrite this as follows

$$\underbrace{\begin{pmatrix} f_x^0 & f_y^0 & f_z^0 \\ g_x^0 & g_y^0 & g_z^0 \end{pmatrix}}_{\equiv\, F_*|_{P_0}} \underbrace{\begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix}}_{\equiv\, \xi} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \iff \boxed{\xi \in \operatorname{Ker} F_*|_{P_0}}$$