4. Spheres (as hypersurfaces)

$$S^n = \{ x \in \mathbb{R}^{n+1} : f(x) = x_0^2 + x_1^2 + \ldots x_n^2 - 1 = 0 \}$$

$$x = \begin{pmatrix} x_0 \\ \vdots \\ x_n \end{pmatrix}$$

n-dimensional unit sphere in $\mathbb{R}^{n+1}$

$$f_*\big|_x = (2x_0, 2x_1, \ldots, 2x_n) \in (\mathbb{R}^{n+1})^*$$

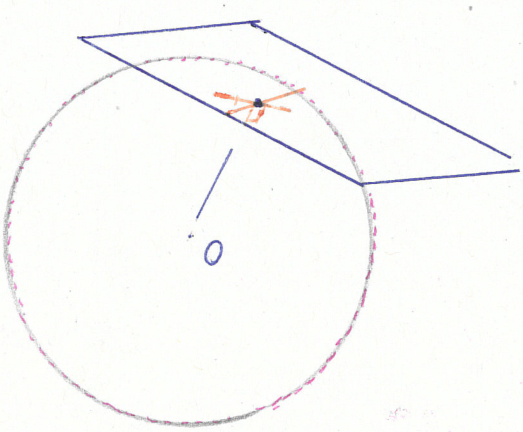
$$f_*\big|_x \cdot h = 2 \sum_{i=0}^{n} x_i h_i \qquad h = \begin{pmatrix} h_0 \\ \vdots \\ h_n \end{pmatrix}$$

$\rightsquigarrow$ f_* is surjective $\forall\, x \in S^n$

$\left(\sum_{i=0}^{n} x_i^2 = 1 \right)$

Tangent space at x_0 : $\sum_{i=0}^{n} x_i^0 (x_i - x_i^0) = 0$

equipping $\mathbb{R}^{n+1}$ with the standard metric, this can be read as an orthogonality condition



DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

Prof. Mauro Spera
UCSC - Brescia

Lecture IV

REVIEW OF CALCULUS IN SEVERAL VARIABLES IV

Further examples

5. Hyperboloids

$$H^n_c = \{ x \in \mathbb{R}^{n+1} \;/\; g_c(x) := x_0^2 - x_1^2 - \dots - x_n^2 - c = 0 \}$$

$c \in \mathbb{R}$

For $c \neq 0$ H^n_c is a hypersurface of $\mathbb{R}^{n+1}$ (a generalised hyperboloid)

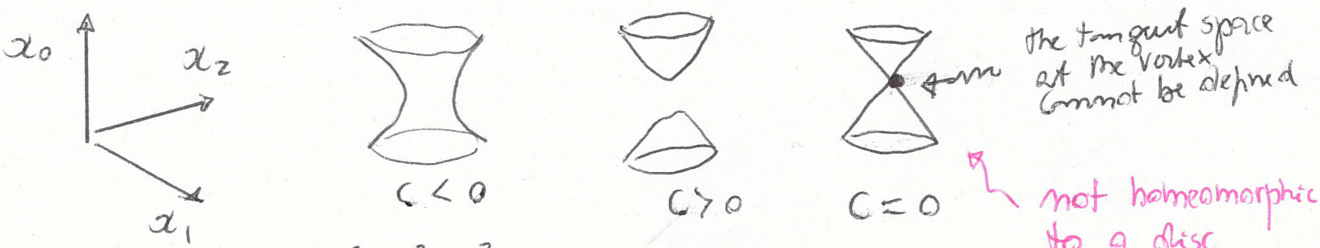
$$(g_c)_*\big|_x = (2x_0, -2x_1, \dots, -2x_n)$$

which is surjective $\forall c \neq 0$, $\forall x \in H^n_c$

Let $c = 0$. Then $(g_0)_*\big|_0 = 0$ (not surjective)

$H^n_0 \setminus \{0\}$ is a submanifold (indeed, a cone without its vertex)

Example, in $\mathbb{R}^3$



$x_0^2 - x_1^2 - x_2^2 - c = 0$

hyperbolic hyperboloid (one-sheeted; ruled)

elliptic hyperboloid (two-sheeted)

cone

6. Tori

$$\boxed{\mathbb{T}^n = \left\{ \underset{\cap\,\mathbb{C}^n}{z = (z_1 \ldots z_n)} \;/\; |z_i|^2 = 1 \right\}}$$

n-dimensional torus

$$\underbrace{S^1 \times S^1 \times \ldots S^1}_{n \text{ copies}}$$

$$= \left\{ \underset{\cap\,\mathbb{R}^{2n}}{x = (x_1 \ldots x_{2n})} \;\Big|\; f(x) := (x_1^2 + x_2^2 - 1, \ldots, x_{2n-1}^2 + x_{2n}^2 - 1) = \underbrace{(0, \ldots 0)}_{n} \right\}$$

f_* is readily computed:

$$\boxed{f_*\Big|_x = \left(2x_1, 2x_2, \ldots\ldots 2x_{2n-1}, 2x_{2n} \right)}$$

and it is everywhere surjective (i.e., in this case, it never vanishes) (f submersive)

$\mathbb{T}^n$ becomes an n-dimensional submanifold of $\mathbb{R}^{2n}$

7. The special orthogonal group $SO(n)$

A quite important and instructive example $\quad n \geq 2$

$$SO(n) = \{ A \in \underbrace{M_n(\mathbb{R})}_{\text{real } n\times n \text{ matrices}} \mid \underbrace{A^T A = A A^T = I_n}_{\text{this condition defines the orthogonal group } O(n)}, \underbrace{\det A = 1}_{\text{this condition further identifies } SO(n)} \}$$

special orthogonal group

(isometries of the Euclidean vector space $(\mathbb{R}^n, \langle , \rangle_{\text{standard}})$ having determinant equal to $+1$

In general, $A \in O(n) \Rightarrow \det A = \pm 1$ (clear); the converse is obviously false.

We want to check that

$SO(n)$ is a submanifold of $M_n(\mathbb{R}) \cong \mathbb{R}^{n^2}$ having dimension $\dfrac{n(n-1)}{2}$

The idea is to realise it as a level set $f^{-1}(0)$

Let

$$f : \underbrace{GL_n^+(\mathbb{R})}_{\text{non singular matrices with positive determinant}} \longrightarrow \underbrace{\mathrm{Sym}_n}_{\text{symmetric matrices : } B^T = B}$$

transpose

$$A \longmapsto f(A) := \underbrace{A^T A - I_n}_{\text{this is a symmetric matrix}}$$

Clearly $\boxed{SO(n) = f^{-1}(0)}$ $\subset \mathrm{Sym}_n$

Let us check that f is submersive everywhere.
Its differential reads

transpose

(♦) $$\boxed{\left(f_*\big|_{\substack{A \\ \cap \\ SO(n)}}\right) \underset{\substack{\cap \\ M_n(\mathbb{R})}}{H} = A^T H + H^T A}$$

Indeed, let $A = A(t)$, $t \in I$ (an interval containing 0) be a smooth curve in $Gl_n^+(\mathbb{R})$ such that

$$A(0) = A \ , \quad \mathring{A}(0) = H$$ ← velocity at $t=0$

(for instance, $A(t) = A + tH$ for sufficiently small t).

We have to compute

$$\boxed{\frac{d}{dt}\left[A(t)^T A(t) - I\right]\Big|_{t=0}}$$

getting, successively,

$$\underbrace{(\mathring{A^T})(0)}_{= (\mathring{A}(0))^T = H^T} \underbrace{A(0)}_{=A} + \underbrace{A(0)^T}_{=A^T} \underbrace{\mathring{A}(0)}_{=H} = H^T A + A^T H$$

transposition commutes with derivation

this yields (♦)

Now, let $S \in \mathrm{Sym}_n$ any real symmetric matrix. Set $H := \frac{AS}{2}$

Then $\left(\frac{AS}{2}\right)^T A + A^T \frac{AS}{2} = \frac{1}{2} S^T A^T A + \frac{1}{2} A^T A S$

$= \frac{1}{2} S \underbrace{A^T A}_{= I} + \frac{1}{2} \underbrace{A^T A}_{= I} S = \frac{1}{2} S + \frac{1}{2} S = S$

and this proves that $\left\{ f_*\big|_A \text{ is surjective } \forall A \in SO(n) \right\}$, and we are done.

The tangent space $T_A\, SO(n)$ is then

$$\boxed{\ker f_*\big|_A = \left\{ H \in M_n(\mathbb{R}) \mid H^T A + A^T H = 0 \right\}}$$

In particular, tangent space to $SO(n)$ at the identity I_n

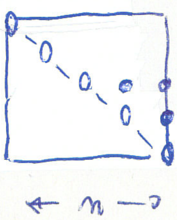
$$\boxed{T_{I_n}\, SO(n) = \left\{ H \in M_n(\mathbb{R}) \mid H^T + H = 0 \right\}}$$

That is, the antisymmetric matrices

skew symmetric

[This is the Lie algebra $\mathfrak{so}(n)$ of the Lie group $SO(n)$ as we shall see later on]

Their dimension is clearly $\frac{n(n-1)}{2} = 1 + 2 + \dots (n-1)$ (Gauss)

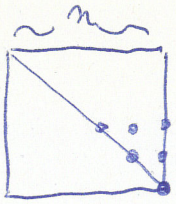


★ Further remarks

An independent check

codimension

$$\dim \mathrm{Sym}_n = \frac{n(n+1)}{2} = \operatorname{codim} SO(n)$$



$1+2+\dots n$ (Gauss)

$$\dim SO(n) = n^2 - \frac{n(n+1)}{2} = \frac{2n^2-n^2-n}{2} = \frac{n^2-n}{2} = \frac{n(n-1)}{2} \quad \checkmark$$

Let $A \in M_n(\mathbb{R})$. Then $A = \underbrace{\frac{A+A^T}{2}}_{\in\, \mathrm{Sym}_n} + \underbrace{\frac{A-A^T}{2}}_{\in\, \mathfrak{so}(n)}$

and this expression is <u>unique</u>. Therefore

direct sum

$$M_n(\mathbb{R}) = \mathrm{Sym}_n \oplus \mathfrak{so}(n)$$

Actually, the above is an <u>orthogonal</u> direct sum upon setting

trace

$$\langle A, B \rangle := \mathrm{Tr}(A^T B)$$

Frobenius, or <u>Hilbert-Schmidt</u> inner product

Check that this is indeed an inner product and that $\langle A, B\rangle = 0$ if $A \in \mathrm{Sym}_n$, $B \in \mathfrak{so}(n)$

Recall that for any $n\times n$ matrix $A = (a_{ij})$,

$\mathrm{Tr}\, A = \sum_{i=1}^{n} a_{ii}$ (diagonal elements) and one has ★ $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$; from this one proves the <u>cyclical property</u> $\mathrm{Tr}(ABC) = \mathrm{Tr}(BCA) = \mathrm{Tr}(CAB)$

and $\mathrm{Tr}(SAS^{-1}) = \mathrm{Tr}(A)$ (similarity invariance)

invertible matrix