Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture IX

★ k-forms

MULTILINEAR ALGEBRA
- k-forms
- Examples
- wedge product
- Examples

Let (V, K) be a vector space $(K = \mathbb{R}$ or $\mathbb{C})$, $\dim_K V = n$

A k-form (more precisely, algebraic k-form) on V is a function $\omega : \underbrace{V \times \ldots \times V}_{k} \longrightarrow K$

A k-form is actually a skew-symmetric covariant tensor (on V_K)

which is

1. k-linear (i.e. linear in each argument)
2. skew-symmetric (alternating), that is
   antisymmetric

$$\omega(v_1, v_2 \ldots, \alpha v_j^{(1)} + \beta v_j^{(2)}, \ldots v_k) =$$

$$= \alpha \cdot \omega(v_1, v_2, \ldots v_j^{(1)}, \ldots v_k) + \beta \cdot \omega(v_1, v_2, v_j^{(2)}, \ldots v_k) \qquad j = 1, 2, \ldots, k$$

and

$$\omega(v_1, \ldots v_i \ldots v_j \ldots v_k) = -\omega(v_1 \ldots v_j \ldots v_i \ldots v_k) \qquad i \neq j$$

Notice that $\omega(v_1 \ldots \overset{i}{v} \ldots \overset{j}{v} \ldots v_k) = 0$

and that, in general

$$\omega(v_{i_1} \ldots v_{i_k}) = \underbrace{(-1)^\sigma}_{\text{sign of the permutation } \sigma} \omega(v_1 \ldots v_k)$$

S_n symmetric group

sign of the permutation σ $\equiv$ parity of σ

$\sigma : \begin{pmatrix} 1 & 2 & \ldots & k \\ i_1 & i_2 & \ldots & i_k \end{pmatrix}$

$(-1)^\sigma = \pm 1$

$+$: even permutation
$-$: odd permutation

Recall : $\sigma : \begin{pmatrix} 1 & \dots & k \\ i_1 & \dots & i_k \end{pmatrix}$ is $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$ if you go from

$(1, 2 \dots k)$ to $(i_1 \dots i_k)$ performing an $\begin{matrix}\text{even}\\\text{odd}\end{matrix}$

number of transpositions, i.e. switches $(\cdot \;\; \overset{\curvearrowright}{\cdot \;\; \cdot} \;\; \cdot)$

[you may accomplish this in many ways, for a fixed permutation σ, however, the parity remains unaltered :

$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$ $\quad (1\ 2\ 3\ 4) \to (2\ 1\ 3\ 4)$

$\to (2\ 1\ 4\ 3) \to (2\ 4\ 1\ 3)$

3 switches : parity : -1 odd number of transpositions

Alternatively :

$(1\ 2\ 3\ 4) \to (1\ 2\ 4\ 3) \to (2\ 1\ 4\ 3)$

$\to (2\ 4\ 1\ 3)$

3 switches : parity : -1

(notice that one may use in general just simple transpositions, i.e. exchanges of adjacent elements)

or, for instance

$(1\ 2\ 3\ 4) \to (1\ 2\ 4\ 3) \to (1\ 4\ 2\ 3)$

$\to (4\ 1\ 2\ 3) \to (4\ 2\ 1\ 3) \to (2\ 4\ 1\ 3)$

5 switches : parity : -1

More technically, one has a group homomorphism

$p : S_n \to \mathbb{Z}_2$ (parity)

$\sigma \mapsto (-1)^\sigma$

with kernel A_n (even permutations)

$S_n / A_n \cong \mathbb{Z}_2$

Set $\Lambda^k(V^*)$ = vector space of k-forms

subspace $\to \ \rotatebox{90}{$\subseteq$}$

$\mathcal{T}_{0,k}$

* Examples

(1). The determinant (of a square matrix) as an n-form

$$\det : M_n(K) \ni A \longrightarrow K$$

$$A = \left(\begin{array}{cccc} [\;] & [\;] & \dots & [\;] \\ C_1 & C_2 & & C_n \end{array} \right)$$

i.e A is (looked upon as a matrix of columns)

1 & 2 hold and, moreover

3. $\det(I_n) = 1$

I_n: identity $n \times n$ matrix

* Geometric interpretation : volume of a "hyperparallelepiped" formed with the columns of A (or rows..)

(2) Flux of a (constant, for the time being) field $\underline{F}$ through a surface : to fix ideas, a space parallelogram formed with two l. i. vectors $\underline{a}$, $\underline{b}$.

"mixed product"

$$\Phi_F : (\underline{a}, \underline{b}) \longmapsto \langle \underline{F}, \underline{a} \times \underline{b} \rangle$$

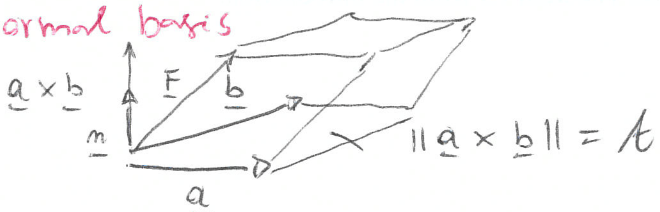
$\langle \;,\; \rangle$: standard scalar product in geometric vector space

$\times$: vector product

flux 2-form

$$= \det(\underline{F}, \underline{a}, \underline{b})$$

build-up columns with the components of $\underline{F}, \underline{a}, \underline{b}$ w.r. to an orthonormal basis



$$\Phi_F = \langle \underline{F}, \underline{n} \rangle A = \langle \underline{F}, \underline{A} \rangle$$

$\underline{A} = A\,\underline{n}$: area vector

★ **Theorem** $\dim \Lambda^k(V^*) = \begin{cases} \binom{n}{k} & \text{if } 0 \le k \le n \\ 0 & \text{if } k > n \end{cases}$

$(\Lambda^k(V^*) = \{0\}$ for $k > n)$

Note that $\Lambda^1(V^*) = V^*$; $\Lambda^0(V^*) = K$
and $\dim \Lambda^k(V^*) = \dim \Lambda^{n-k}(V^*)$

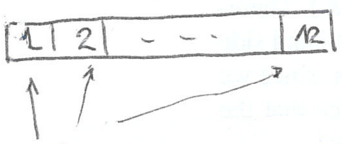
Proof (sketch). Let $e = (e_1, \dots e_n)$ be a basis of V.

a k-form w is completely determined, in view of k-linearity and skew-symmetry, by the values

$$w(e_{i_1}, e_{i_2}, \dots e_{i_k}), \quad i_1 < i_2 < \dots i_k$$

[notice that, permuting any two entries in, one gets
$\pm\, w(e_{i_1} \dots e_{i_k})$]

Therefore, the number of "free parameters" is given by combinations of n objects in k places (the arguments of the k-form), that is, by the binomial coefficient $\binom{n}{k}$, if $0 \le k \le n$ $(\Lambda^0(V^*) = K)$



$e_1 \dots e_n$ are to be allocated in the various slots

Notice that if $k > n$, in the allocation, at least a basis vector is being inserted twice, so by skew-symmetry one gets 0.

A basis for $\Lambda^k(V^*)$ is given as follows $(k \le n)$

Let $I = (i_1, \dots i_k)$, $i_1 < i_2 < \dots i_k$ a multi-index, set

$$e^*_I(e_{j_1}, \dots e_{j_k}) = \begin{cases} \pm 1 & \text{if } J \text{ is a permutation of } I \\ 0 & \text{otherwise} \end{cases}$$

$\pm$ according to parity

Then $w = \sum_I w(e_{i_1}, \dots e_{i_k})\, e^*_I$, and moreover the e^*_I are linearly independent.

We shall obtain a more explicit description of e^*_I later on

★ Exterior (or wedge) product of forms

use the preceding notation

Let ω^k be a k-form

ω^l be an l-form

The wedge product of ω^k and ω^l, denoted by
exterior product

in this order

$\omega^k \wedge \omega^l$, is a $(k+l)$-form defined as follows:

Actually, it is an antisymmetrized tensor product

$$\boxed{(\omega^k \wedge \omega^l)(v_1, \dots v_{k+l}) := \frac{1}{k!\,l!} \sum_{\nu} (-1)^{\nu}\, \omega^k(v_{i_1} \dots v_{i_k})\, \omega^l(v_{i_{k+1}} \dots v_{i_{k+l}})}$$

$$\nu = \begin{pmatrix} 1 & 2 & \dots & k+l \\ i_1 & i_2 & \dots & i_{k+l} \end{pmatrix}$$

(sum over all permutations)

$(-1)^{\nu}$ parity of ν

take any permutation of $v_1 \dots v_{k+l}$, distribute the arguments in the manner indicated

other conventions are possible

Properties: 1. Graded commutativity

$$\omega^k \wedge \omega^l = (-1)^{kl}\, \omega^l \wedge \omega^k$$

2. distributivity $(\alpha\, \omega_1^k + \beta\, \omega_2^k) \wedge \omega^l =$

$$= \alpha\, \omega_1^k \wedge \omega^l + \beta\, \omega_2^k \wedge \omega^l \quad \text{etc}$$

3. associativity: $(\omega^k \wedge \omega^l) \wedge \omega^p = \omega^k \wedge (\omega^l \wedge \omega^p)$

$$\equiv \omega^k \wedge \omega^l \wedge \omega^p$$ (no ambiguity arising)

Let us check 1.

Notice that in order to go from

$$(i_1 \dots i_k, i_{k+1} \dots i_{k+l}) \text{ to } (i_{k+1} \dots i_{k+l}, i_1, \dots i_k),$$

one needs kl (simple) transpositions. Also,

$$(i_1 \dots \underbrace{i_k, i_{k+1}, \dots i_{k+l}}_{})$$

each element is moved to the final position by means of k transpositions, and l elements intervene...

see box

the parity of

$$\nu' = \begin{pmatrix} 1 & 2 & \dots & & & k+l \\ i_{k+1} & i_{k+2} & i_{k+l}, & i_1, & i_2 \dots & i_k \end{pmatrix}$$

is $(-1)^{\nu}(-1)^{kl}$

and $(-1)^{\nu} = (-1)^{\nu}(-1)^{2kl}$

$$\left((1\ 2\ \dots\ k+l) \xrightarrow[(-1)^{\nu}]{} (i_1, i_2 \dots i_{k+l}) \xrightarrow[(-1)^{kl}]{} (i_{k+1}, i_{k+2} \dots i_1 \dots i_k) \right)$$

We are now prepared for the actual computation:

$$(\omega^k \wedge \omega^l)(v_1, \dots, v_{k+l}) = \frac{1}{k!\,l!} \sum_{\nu} (-1)^{\nu} \omega^k(v_{i_1} \dots v_{i_k}) \omega^l(v_{i_{k+1}} \dots v_{i_{k+l}})$$

$$= \frac{1}{k!\,l!} \sum_{\nu} (-1)^{\nu} \omega^l(v_{i_{k+1}} \dots v_{i_{k+l}}) \omega^k(v_{i_1} \dots v_{i_k})$$

$$= (-1)^{kl} \frac{1}{k!\,l!} \sum_{\nu'} \underbrace{(-1)^{\nu}(-1)^{kl}}_{\text{parity of } \nu'} \omega^l(v_{i_{k+1}} \dots v_{i_{k+l}}) \omega^k(v_{i_1} \dots v_{i_k})$$

again, this is a sum over all permutations

$$= (-1)^{kl} (\omega^l \wedge \omega^k)(v_1 \dots v_{k+l})$$

$\square$

In order to grasp the geometrical meaning of $\wedge$ (crucial for the sequel), let us compute (exercise), in $\mathbb{R}^2$

$$(e_1^* \wedge e_2^*)(v_1, v_2) = \qquad v_1 = \alpha_1 e_1 + \alpha_2 e_2, \quad v_2 = \beta_1 e_1 + \beta_2 e_2$$

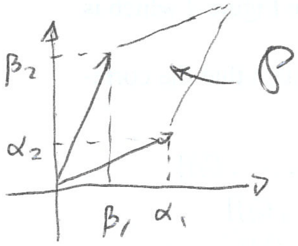
bilinearity

$$\alpha_1 \beta_1 \underbrace{(e_1^* \wedge e_2^*)(e_1, e_1)}_{= 0} + \alpha_1 \beta_2 \underbrace{(e_1^* \wedge e_2^*)(e_1, e_2)}_{= +1}$$

equal arguments

$$+ \alpha_2 \beta_1 \underbrace{(e_1^* \wedge e_2^*)(e_2, e_1)}_{= -1} + \alpha_2 \beta_2 \underbrace{(e_1^* \wedge e_2^*)(e_2, e_2)}_{= 0}$$

$$= \alpha_1 \beta_2 - \alpha_2 \beta_1 = \begin{array}{c} \begin{vmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{vmatrix} \\ \;v_1 \quad\; v_2 \end{array}$$



= oriented area of P

i.e, with sign

★ Check that, in $\mathbb{R}^3$, $(e_1^* \wedge e_2^* \wedge e_3^*)(v_1, v_2, v_3)$, where $v_1 = \sum \alpha_i e_i$, $v_2 = \sum \beta_i e_i$, $v_3 = \sum \gamma_i e_i$

$$= \dots = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}$$

<u>oriented</u> volume of the obvious parallelepiped

as anticipated

Let us check <u>associativity</u>. (This would follow from general arguments, nevertheless we sketch an explicit computation)

It is enough to establish it for monomials of the form $e^*_{i_1} \wedge e^*_{i_2} \ldots e^*_{i_k}$, therefore, by skew-symmetry and induction, it suffices to check, to fix ideas,

$$(e^*_1 \wedge e^*_2) \wedge e^*_3 = e^*_1 \wedge (e^*_2 \wedge e^*_3),$$

and, by multilinearity & skew-symmetry again, it is enough to verify that, if both sides are evaluated on (e_1, e_2, e_3), we get the same result (we shall find 1)

So compute:

$$((e^*_1 \wedge e^*_2) \wedge e^*_3)(e_1, e_2, e_3) =$$

$$\underset{\text{note}}{\frac{1}{2}} \Big[\overbrace{(e^*_1 \wedge e^*_2)(e_1, e_2)}^{1} \overbrace{e^*_3(e_3)}^{1} - \overbrace{(e^*_1 \wedge e^*_2)(e_2, e_1)}^{-1} \overbrace{e^*_3(e_3)}^{1} \Big]$$

+ other 4 summands equal to 0: e^*_3 must in fact act on e_1 or e_2, yielding 0

Let us evaluate $(e^*_1 \wedge e^*_2)(e_1, e_2)$ directly:

$$(e^*_1 \wedge e^*_2)(e_1, e_2) = \underbrace{e^*_1(e_1)}_{=1} \underbrace{e^*_2(e_2)}_{=1} - \underbrace{e^*_1(e_2)}_{=0} \underbrace{e^*_2(e_1)}_{=0}$$

$= +1$.

Therefore $((e^*_1 \wedge e^*_2) \wedge e^*_3)(e_1, e_2, e_3) = +1$

The r.h.s. is also easily seen to be $+1$ as well.

<u>Remark</u>: everything is also clear in view of the geometric interpretation