Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture V

MULTILINEAR ALGEBRA

DUAL SPACES

Let (V, K) be a vector space (over a field K; $K = \mathbb{R}$ or $\mathbb{C}$ throughout). The **dual** vector space V^* is, by definition

$$V^* = \{ f : V \to K \ / \ f \text{ linear} \}$$

[terminology: the elements in V^* are called
- linear functions
- linear functionals
- linear forms
- (algebraic) 1-forms
- covectors
]

In more detail, $f \in V^*$ satisfies

$$f(\underset{K}{\alpha} \cdot \underset{V}{v} + \underset{K}{\beta} \cdot \underset{V}{w}) = \underbrace{\alpha \cdot f(v)}_{K} + \underbrace{\beta f(w)}_{K}$$

operations in V

operations in K

☆ V^* is actually a vector space, upon defining linear combinations in the following fashion:

this is a function, so one has to specify its action on elements in its domain

$$\underbrace{(\alpha \cdot f + \beta \cdot g)}_{V^*}(\underset{V}{v}) := \alpha \cdot f(v) + \beta g(v)$$

operations in K

operations in V^*, defined via

and checking vector space axioms.

★ Let $\dim_K V = n < \infty$ (finite dimensional vector space)

. Then $\dim_K V^* = n$ (hence $V \cong V^*$)

* isomorphic non canonically

Pf. Let $e = (e_1 \ldots e_n)$ be a basis of V. Consider the dual forms $\{e^*_i\}_{i=1\ldots n}$, defined via

Kronecker's delta

alternative notation: $\leadsto e^i(e_j) = \delta^i_j = \delta_{ij}$

$$e^*_i(e_j) = \delta_{ij} \qquad \delta_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$$

We want to show that $e^* = (e^*_1 \ldots e^*_n)$ is a basis for V^*, called the dual basis of $(e_1 \ldots e_n)$

[recall that it is enough to define $f \in V^*$ on a basis of V, and "extend by linearity", since $f(v) = f(\sum_i \underset{\substack{\uparrow \\ \text{components of } v \\ \text{w. resp. to } (e_1 \ldots e_n)}}{\alpha_i} e_i) = \sum \alpha_i f(e_i)$]. Observe that e^*_j picks up the j^{th} component of v = linear combination of the e_i's (components are uniquely defined) : $\{e^*_j(v) = \alpha_j\}$

One immediately finds:

$$\boxed{f = \sum_{i=1}^n f(e_i) e^*_i}$$

(that is: the e^*_i's generate V^*)

Indeed, if $v = \sum_{i=1}^n \alpha_i e_i$, then, on the one hand,

$$f(v) = \sum_{i=1}^n \alpha_i f(e_i)$$ and, on the other hand,

$$\left(\sum_{i=1}^n f(e_i) e^*_i\right)(v) = \sum_{i,j=1}^n \alpha_j f(e_i) \underbrace{e^*_i(e_j)}_{= \delta_{ij}} = \sum_{i=1}^n \alpha_i f(e_i).$$

Furthermore, the e^*_i's are linearly independent:

if $\sum \beta_i e^*_i = 0 \leadsto$ (the zero-functional)

then, $\forall v \in V$, $(\sum \beta_i e^*_i)(v) = 0$. Choosing $v = e_j$

yields $\quad 0 = \sum_i \beta_i \, e_i^*(e_j) = \sum_i \beta_i \, \delta_{ij} = \beta_j$,

i.e. $\beta_j = 0 \quad \forall\, j = 1 \dots n$, whence the conclusion. $\square$

☆ Notice that $V \cong V^*$, but non-canonically (i.e. the established isomorphism is basis dependent.

Define $V^{**} = (V^*)^* \equiv$ bidual of V

In finite dimensions, $V \cong V^{**}$ canonically (i.e. independently of the choice of a basis): this follows from setting, for any $v \in V$,

$$v^{**} \in V^{**}, \text{ defined via}$$

$$\boxed{v^{**}(\underset{\substack{\cap \\ V^*}}{f}) := f(v)} \qquad (f \in V^*)$$

The map $\boxed{V \ni v \longmapsto v^{**} \in V^{**}}$

is linear, injective, and $\dim V^{**} = n$, hence it is surjective as well (in view of the nullity + rank theorem), so it is an isomorphism.

Examples

1. $\mathbb{R}^n$ $(\mathbb{C}^n ..)$, $(e_1 \dots e_n)$ canonical basis

$$\mathbb{R}^n \ni x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \qquad x_i \in \mathbb{R}$$

compact notation: $x = (x^i)$ notice this

$$e_i = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \leftarrow i$$

dual basis $(e_1^* \dots e_n^*)$

$$e_i^* = (0, 0 \dots 1, 0 \dots 0)$$
$$\uparrow i$$

We may realize covectors as row vectors.

compact notation $f_a = (a_i)$ notice this

$$(\mathbb{R}^n)^* = \{ \underbrace{(a_1 \dots a_n)}_{a^T} \}_{a_i \in \mathbb{R}} \qquad a = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}$$

$$(a_1 \dots a_n) \leftrightarrow f_a \qquad \boxed{f_a(x) = a^T x = \sum_i a_i x_i}$$

components of f with respect to the dual basis $(e_1^* \dots e_n^*)$

$x \in \mathbb{R}^n$

a^T x (matrix product)

compact notation: $f_a(x) = a_i x^i$ Einstein's convention

2. Within the geometric vector space:

$\vec{F}$: force $\quad \vec{a}$: displacement

The work exerted by $\vec{F}$ along $\vec{a}$ is given by

$$\boxed{\vec{F} \cdot \vec{a} = \|\vec{F}\| \cdot \|\vec{a}\| \cos \vartheta}$$

(elementary scalar product)

$$l = l_{\vec{F}} \text{ , defined as } \boxed{l_{\vec{F}}(\vec{a}) = \vec{F} \cdot \vec{a}}$$

is a 1-form (work 1-form)

3. The differential of a function

Let $f : \mathcal{U} \subset \mathbb{R}^n \to \mathbb{R}$, $f \in \mathcal{C}^1(\mathcal{U})$

open set — to fix ideas

Let $x_0 \in \mathcal{U}$, $x+h \in \mathcal{U}$ Then

$$f(x_0+h) - f(x_0) = \underbrace{df\big|_{x_0}}_{\text{linear part of the increment}} \cdot h + o(h)$$

$$\left(\frac{\|o(h)\|}{\|h\|} \to 0 \right)$$

The linear operator

$$df\big|_{x_0} : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$h \longmapsto df\big|_{x_0} h$$

(defined for all $h \in \mathbb{R}^n$) now

★ differential of f at x_0

alternative notation: $f_*\big|_{x_0}$ push-forward — See prologue as well

is indeed a 1-form, and it is represented, concretely, by a $1 \times n$ - matrix (a row matrix)

$$df\big|_{x_0} = \left(\frac{\partial f}{\partial x_1}(x_0), \dots \frac{\partial f}{\partial x_n}(x_0) \right) = \nabla f(x_0)^t$$

"gradient" (abuse of language ⚠) see also below

Aside

Fréchet differential:

$f : \mathcal{U} \subset V \to W$ normed vector spaces

$$\boxed{f(a+h) - f(a) = T_a h + o(h)}$$

with $\dfrac{\|o(h)\|_W}{\|h\|_V} \to 0$ as $h \to 0$

T_a: Fréchet differential of f at a

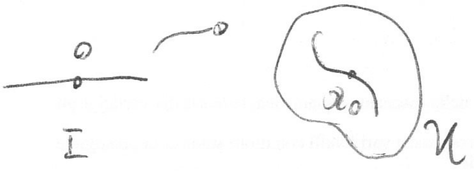
see prologue as well

Example: $V = \mathbb{R}^n$, $W = \mathbb{R}^m$, T_a is represented by the Jacobian matrix, (which is an $m \times n$ - matrix)

Remark. The following observation will be important in the sequel

[everything $\mathcal{C}^\infty$, in order to fix ideas]

Let $x = x(t) \in \mathcal{U}$, $t \in I$ I interval (containing 0)



$$x(0) = x_0 \in \mathcal{U}$$

Set $\dot{x}(0) = h$

velocity in 0

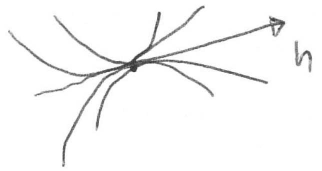
Set $F = F(t) = f(x(t))$ $\quad (= (f \circ x)(t))$

Then
$$(df|_{x_0})(h) = \frac{dF}{dt}(0)$$

$$\left(= \sum_{i=1}^{n} f^{0}_{x_i} h_i \right)$$

$$f^{0}_{x_i} = \frac{\partial f}{\partial x_i}(x_0)$$

<u>independently of</u> $x = x(t)$, provided $\dot{x}(0) = h$ (fixed)



4. Integral

Let $V = C_c^0(\mathbb{R})$ (compactly supported real continuous functions on $\mathbb{R}$)

(an infinite dimensional vector space)

[recall:
$$\operatorname{supp} f = \overline{\{ x \in \mathbb{R} \,/\, f(x) \neq 0 \}}$$
closure]

Set

$$\boxed{\int_{\mathbb{R}} : \quad V \ni f \longmapsto \int_{\mathbb{R}} f \in \mathbb{R}}$$

Riemann integral

[notice that Riemann integration does not require a measure the standard measure of parallelepipeds being sufficient]

$$\mu(P) := \prod_{i=1}^{n} (b_i - a_i) \qquad \text{obvious notation}$$

Then $\boxed{\int_{\mathbb{R}}}$ is a linear functional (continuous and positive, cf functional analysis course)

ASIDE:

The (measure theoretic) Riesz representation theorem tells us that $\int_{\mathbb{R}}$ is in fact integration with respect to the Lebesgue measure (that is $\int_{\mathbb{R}} f = \int_{\mathbb{R}} f(x)\, d\mu(x)$

Lebesgue measure