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Lecture VI
MULTILINEAR ALGEBRA
MUSICAL ISOMORPHISMS

Let $K = \mathbb{R}$, and let $(V, \langle \cdot, \cdot \rangle)$ be a Euclidean vector space, i.e. $\langle \cdot, \cdot \rangle$ is an inner product:
(scalar product)

[work in finite dimensions]

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow \mathbb{R}$$

is a function fulfilling the following properties

1. bilinearity

(linearity in both arguments)

$$\langle \alpha v_1 + \beta v_2, w \rangle = \alpha \langle v_1, w \rangle + \beta \langle v_2, w \rangle \quad \text{etc}$$

2. symmetry

$$\langle v, w \rangle = \langle w, v \rangle \qquad \forall\, v, w \in V$$

3. positive definiteness

$$\langle v, v \rangle \geq 0 \quad \text{and equality holds if and only if } v = 0$$

$\langle \cdot, \cdot \rangle$ induces specific isomorphisms (musical isomorphisms) between V and V^*

$\flat$: flat
$\sharp$: sharp
(inverse to each other)

A = LA
$A^\sharp = B^\flat$ ($LA^\sharp = SI^\flat$)
B = SI ...

$$\begin{array}{ccc} V & & V^* \\ \Psi & & \end{array}$$

$$\flat : v \longmapsto v^\flat = \langle v, \cdot \rangle$$

inner product against a fixed vector

$C^\sharp = D^\flat$

$\flat$ is clearly injective

(indeed $v^\flat = 0$ iff $\langle v, w\rangle = 0 \quad \forall\, w \in V$

$\Rightarrow$ in particular $\langle v, v\rangle = 0 \Rightarrow v = 0$

(positive definiteness))

hence surjective ($N+R$), and its inverse is called

$\sharp$. This is also expressed by means of the

Riesz representation theorem, in the following guise:

$\star$ Let $(V, \langle\, ,\rangle)$ be a Euclidean vector space,

$\dim_{\mathbb{R}} V = n$. Let $\ell \in V^*$. Then, $\exists !\, u \in V$

such that, $\forall\, v \in V$, one has

$$\left\{\ell(v) = \langle u, v\rangle\right\}$$

[conversely, as we have already observed, $\forall\, u \in V$, the position $\ell_u(v) := \langle u, v\rangle$ defines a linear functional]

Proof. Let $(e_1 \dots e_n)$ be an orthonormal basis of V,

i.e. $\langle e_i, e_j\rangle = \delta_{ij}$ $i = 1..n$. [Such a basis can

be manufactured from any basis via the Gram-Schmidt

procedure] . Then $v = \sum \alpha_i e_i$ (α_i uniquely determined)

Thus $\ell(v) = \sum_{i=1}^{n} \alpha_i\, \ell(e_i)$. Set $u_i = \ell(e_i)$

and $u = \sum_{i=1}^{n} u_i e_i$. Then u is the sought-for vector

Indeed: $\langle u, v \rangle = \langle \sum_i u_i e_i , \sum \alpha_j e_j \rangle$

$$= \sum_{i,j} u_i \alpha_j \underbrace{\langle e_i, e_j \rangle}_{g_{ij}} = \sum_{i=1}^{n} u_i \underset{\substack{\| \\ f(e_i)}}{\alpha_i} = l(v) \qquad \square$$

Notice that we employed bases...

We wish to be more explicit

Concretely, $\langle \, , \, \rangle$ can be represented, given any basis $(e_1, \dots e_n)$, via a matrix $G = (g_{ij} = \langle e_i, e_j \rangle)$

In fact if $v = \sum v^i e_i$ $\quad w = \sum_j w^j e_j$

notice this

(compactly: $v = v^i e_i$, $w = w^j e_j$ Einstein's convention)

$$\boxed{\langle v, w \rangle = \sum_{i,j} v^i w^j \langle e_i, e_j \rangle = \sum_{i,j} g_{ij} v^i w^j}$$

compactly: $g_{ij} v^i w^j$

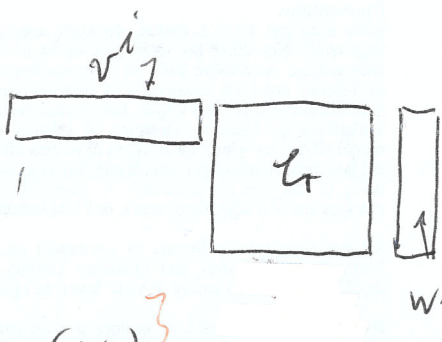
sum over i & j

with G symmetric and positive definite.

Conversely, given a basis in V and a symmetric, positive definite G, one defines an inner product via the above formula. The latter, in turn, can be rewritten as follows:

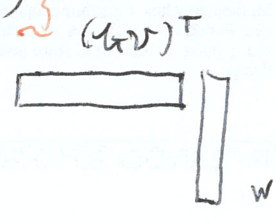
$$\langle v, w \rangle = v^T \cdot G \cdot w$$

(matrix products...!)



But $v^T G w =$

$$v^T G^T w = (Gv)^T w = l_{Gv}(w)$$



where l_{Gv} is the <u>linear functional</u>

corresponding to the <u>row vector</u> $(Gv)^T$

One sets $\qquad v_i := g_{ij} v^j \quad \sim$ Einstein's summation convention

So $\qquad \langle v, w \rangle = v_i w^i$

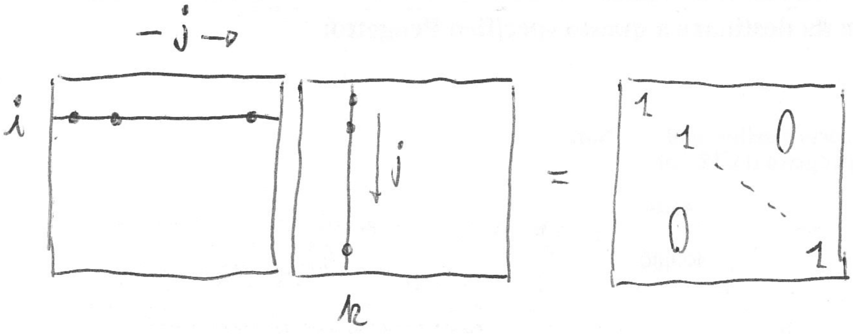
Therefore $\qquad \boxed{\flat} : (v^i) \longmapsto (v_i)$ $\qquad$ This explains the musical terminology

i.e. it <u>lowers</u> indices.

Let us visualise $\sharp = \flat^{-1}$: this will involve the <u>inverse</u> G^{-1} of G. $\quad$ Set $\quad G^{-1} = (g^{ij})$

Then by definition $\qquad g^{ij} g_{jk} = \delta^i_k = \delta_{ik}$

again sum over j



$$G^{-1} G = G G^{-1} = I$$

Start from l, i.e. from a row vector $(v_i) = v^T$.

One finds, successively:

components of l with respect to the dual basis $(e_1^* \dots e_n^*)$ $= (e^1 \dots e^n)$

$\left((A^{-1})' = (A^T)^{-1} \equiv A^{-T} \right)$

$$l(w) = v^T w = v^T \ell_t^{-1} \ell_t w = v^T \ell_t^{-T} \ell_t w$$

$$= (\ell_t^{-1} v)^T \ell_t w \equiv \langle v^{\#}, w \rangle$$

that is: $v^{\#} = \ell_t^{-1} v$

in components: $\boxed{v^i = g^{ij} v_j}$

namely, indices are <u>raised</u>

Summarizing:

$$\boxed{\begin{aligned} v_i &= g_{ij} v^j \quad & \flat \\ v^i &= g^{ij} v_j \quad & \# \end{aligned}}$$

$$v_i w^i = g_{ij} v^j w^i = g_{ij} w^i v^j = \dots = v^i w_i$$

observe

Example

Let $(\mathbb{R}^2, \langle , \rangle)$, with $\langle , \rangle$ represented, w.r. to the canonical basis, by

$$G = (g_{ij}) = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

> recall:
>
> $v_i = g_{ij} v^j$
>
> $v^i = g^{ij} v_j$

Let $v = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Find $v^\flat$.

One has

$$v^\flat = (Gv)^T$$

$$= \left[\begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right]^T = \begin{pmatrix} 2 \\ 4 \end{pmatrix}^T = (2,4)$$

Now, given $\omega = (2,4)$, find $\omega^\#$.

We have

$$\omega^\# = \overset{\text{caveat}}{G^{-1}\omega^T} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{4} \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

as expected, since $\flat = \#^{-1}$, $\# = \flat^{-1}$.