Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture VII

MULTILINEAR ALGEBRA
- Dual homomorphism
- Covariance & contravariance
- Example from analysis

★ Dual (or adjoint) homomorphism

Let $T \in \mathrm{Hom}(V,W)$ $\qquad \dim_K V = n \qquad \dim_K W = m$

↑ homomorphism
linear map
linear transformation
linear operator

operations in V

linearity: $T(\alpha \cdot v_1 + \beta \cdot v_2)$

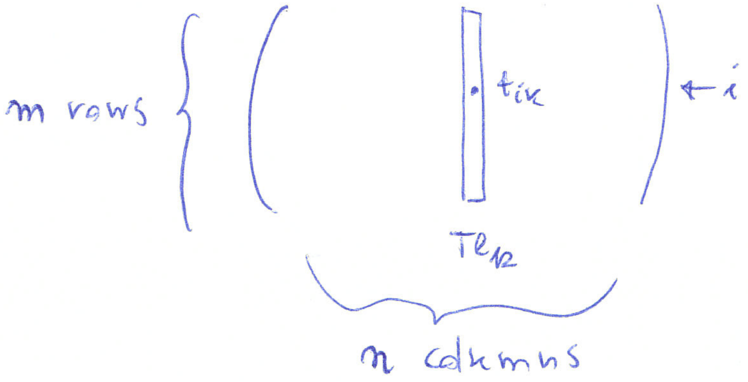
$= \alpha \cdot T v_1 + \beta \cdot T v_2$.

operations in W

Let $e=(e_1 \dots e_n)$, $f=(f_1 \dots f_m)$ bases in V and W, respectively

The $m \times n$ matrix $\underset{\uparrow \text{final}}{m}{}_{f\underset{\text{initial}}{e}}(T) = (t_{ik})_{\substack{i=1\dots m\\ k=1\dots n}}$ representing T

with respect to the given bases is



that is

$$Te_k = \sum_{i=1}^{m} t_{ik} f_i$$

and, clearly

$$\boxed{t_{ik} = f_i^*(Te_k)} \qquad (\lozenge)$$

↑ i-th vector of the dual basis $f^* = (f_1^* \dots f_m^*)$ of W^*

The **adjoint** (or **dual**) homomorphism T' (of T)

is an element in $\mathrm{Hom}(\underset{\nwarrow}{W^*}, \underset{\uparrow}{V^*})$

caveat!

$T: V \to W$

$T': W^* \to V^*$ defined as

$$\boxed{(\underbrace{T' \underset{\substack{\cap \\ W^*}}{\ell}}_{\substack{\cap \\ V^*}})(\underset{\substack{\cap \\ V}}{v}) := \underset{\substack{\cap \\ W^*}}{\ell}(\underset{\substack{\cap \\ W}}{Tv})} \qquad \begin{array}{l} \forall v \in V \\ \forall \ell \in W^* \end{array}$$

Any matrix representation of T' will be of type $n \times m$; in particular, if $e^* = (e_1^* \dots e_n^*)$, $f^* = (f_1^* \dots f_m^*)$ denote the dual bases of e, f, respectively, one finds

$$\boxed{m_{\underset{\substack{\nearrow \\ \text{final}}}{e^*}\underset{\substack{\nwarrow \\ \text{initial}}}{f^*}}(T') = m_{fe}(T)^t}$$

(This provides an intrinsic meaning to the notion of transpose of a matrix)

Pf. $$[m_{e^*f^*}(T')]_{ik} \overset{\text{recall}(\diamond)}{=} e_i^{**}(T'f_k^*) \underset{\substack{\nearrow \\ \text{recall } x^{**}(y^*) = y^*(x)}}{=} (T'f_k^*)(e_i)$$

$$\underset{\substack{\text{by definition} \\ \text{of } T'}}{=} f_k^*(Te_i) \underset{(\diamond) \text{ again}}{=} t_{ki}$$

and this concludes the proof □

Example   Let $T : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$\longleftarrow$ $\nwarrow$ columns

$$T \equiv \underset{\substack{\nearrow \\ \text{canonical} \\ \text{bases}}}{m_{fe}}(T) = \begin{pmatrix} 1 & 1 \\ 2 & 4 \\ 0 & 5 \end{pmatrix}$$

we show that $T' = T^t$ using the very definition

$T' : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$\longleftarrow$ $\nwarrow$ rows

$$\ell : \mathbb{R}^3 \longrightarrow \mathbb{R} \qquad \ell \leftrightarrow (x, y, z)$$

$$(T'\ell)\Big(\underbrace{\begin{pmatrix}1\\0\end{pmatrix}}_{e_1}\Big) := \ell(Te_1) = (x \;\; y \;\; z)\begin{pmatrix}1\\2\\0\end{pmatrix}$$

$$= x + 2y$$

$$(T'\ell)\Big(\underbrace{\begin{pmatrix}0\\1\end{pmatrix}}_{e_2}\Big) := \ell(Te_2) = (x \;\; y \;\; z)\begin{pmatrix}1\\4\\5\end{pmatrix}$$

$$= x + 4y + 5z$$

Therefore

$$(x \;\; y \;\; z) \xmapsto{T'} (x + 2y, \; x + 4y + 5z)$$

$$\|$$

$$\underbrace{\begin{pmatrix} 1 & 2 & 0 \\ 1 & 4 & 5 \end{pmatrix}}_{T^t}\begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

∗ Transformation laws (contravariance & covariance)

This is a key issue, and it will be analysed from different perspectives

Ⓘ Let $T \in GL(V)$ $T: V \to V$

invertible endomorphisms

notation: general linear group associated to V

The problem is the following : find $S': V^* \to V^*$, $S \in GL(V^*)$

(stemming from $S: V \to V$, $S \in GL(V)$)

such that, $\forall \ell \in V^*, \forall v \in V$,

$$\boxed{(S'\ell)(Tv) = \ell(v)}$$

that is, if vectors in V are transformed via T, how should vectors in V transform, in order that the corresponding evaluations do not change?
["Gattopardo" Principle]

One immediately finds, given $\ell \in V^*$, $v \in V$

$$(S'\ell)(Tv) \underset{\text{def.}}{=} \ell(STv) = \ell(v) \quad \forall v \in V, \ \forall \ell \in V^*$$

this being true if and only if $ST = I \Rightarrow S = T^{-1}$

$$\Rightarrow S' = (T^{-1})'$$ (which is easily seen to equal $(T')^{-1}$)

The upshot is that

to $T: V \to V$ "contravariance"

there corresponds $(T')^{-1} = (T^{-1})'$ "covariance"

vectors in V: contravariant vectors "vectors"
V^*: covariant vectors "covectors"

vectors from $\underset{\text{"vectors"}}{V}$ and $\underset{\text{"covectors"}}{V^*}$ can be distinguished by their behaviour under linear transformations.

Ⓘⓘ

(II) Let us deal with the same problem from a slightly different, more concrete standpoint.

Again consider the scalar $\underset{\substack{\uparrow\\ V^*}}{l}(\underset{\substack{\uparrow\\ V}}{v})$

fix bases $e = (e_1 \dots e_n)$ and $e' = (e_1' \dots e_n')$

together with the corresponding dual bases.

Then $v = \sum_{i=1}^{n} x_i e_i = \sum_{i=1}^{n} x_i' e_i'$

and one has

$$x' = \overbrace{\underbrace{m_{e'e}(I)}_{\substack{\parallel \\ A \\ \text{non singular...}}}}^{\text{change of basis matrix}} x \qquad A \in GL(n, K)$$

Similarly, one has

$$l = \sum_{i=1}^{n} y_i e_i^* = \sum_{i=1}^{n} y_i' e_i'^*$$

and, obviously $y' = B y$

for some $B \in GL(n, K)$

what is then B ?

We have, successively

$$l(v) = y^t x = y'^t x'$$

$$v \leftrightarrow \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = x$$

$$l \leftrightarrow y^t = (y_1 \dots y_n)$$

$$y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$$

But $y^t x = y^t A^{-1} A x$

$$= (A^{-t} y)^t x'$$

$$\Rightarrow (A^{-t} y)^t x' = y'^t x'$$

$$\forall x' \in \mathbb{R}^n$$

$$\Rightarrow \left\{ y' = A^{-t} y \right\} \quad \text{i.e.} \quad \boxed{B = A^{-t}}$$

Therefore $v \longmapsto \overset{A}{\overset{\parallel}{B^{-t}}} v$

$$l \longmapsto \underset{A^{-t}}{\underset{\parallel}{B}} \cdot l$$

This is of course consistent with the previous discussion.

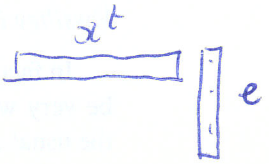
Ⅲ Still another point of view

“vintage” definition

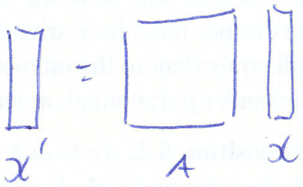
Let $e = \begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \leftarrow$ vectors

$$v = \sum_i x_i e_i \equiv x^t e \qquad x^t = (x_1 \dots x_n)$$

$$= \sum x'_i e'_i \equiv x'^t e'$$



Then, if $x' = \underset{\substack{\cap \\ GL(n,\mathbb{R})}}{A} x$



covariance, i.e. coordinate-like transformation, i.e. coordinates transform covariantly

Then $e' = A^{-t} e$

i.e bases transform contravariantly

Indeed:

$$x'^t e' = (Ax)^t e' = x^t A^t e' \qquad \forall x \in \mathbb{R}^3$$

$$\|$$

$$x^t e$$

$$\Rightarrow \quad e = A^t e' \quad , \quad e' = A^{-t} e$$

covariance: coordinate type transformation

contravariance: basis type transformation

★ Important example

Let $\underline{r} = \underline{r}(u,v)$ $(u,v) \in \mathcal{U} \subset \mathbb{R}^2$ • $\underline{r} \in C^\infty$
smooth parametric surface region • $\underline{r}$ injective
• $\underline{r}_u \times \underline{r}_v \neq \underline{0}$

(This concept is invariant under regular parameter changes)

$$\begin{cases} u' = u'(u,v) \\ v' = v'(u,v) \end{cases} \qquad (u,v) \in \mathcal{U}$$

the differential of $(u,v) \xmapsto{f} (u',v')$ reads at a generic point:

$$\begin{cases} du' = \frac{\partial u'}{\partial u} du + \frac{\partial u'}{\partial v} dv \\ dv' = \frac{\partial v'}{\partial u} du + \frac{\partial v'}{\partial v} dv \end{cases} \quad \text{i.e.} \quad \boxed{\begin{pmatrix} du' \\ dv' \end{pmatrix} = \begin{pmatrix} \frac{\partial u'}{\partial u} & \frac{\partial u'}{\partial v} \\ \frac{\partial v'}{\partial u} & \frac{\partial v'}{\partial v} \end{pmatrix} \begin{pmatrix} du \\ dv \end{pmatrix}}$$

J : Jacobian matrix

One has a corresponding transformation (basis change) on the tangent plane (at any point)

$$\begin{cases} \frac{\partial \underline{r}}{\partial u'} = \frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial u'} + \frac{\partial \underline{r}}{\partial v}\frac{\partial v}{\partial u'} \\ \frac{\partial \underline{r}}{\partial v'} = \frac{\partial \underline{r}}{\partial u}\frac{\partial u}{\partial v'} + \frac{\partial \underline{r}}{\partial v}\frac{\partial v}{\partial v'} \end{cases} \qquad \begin{pmatrix} \frac{\partial \underline{r}}{\partial u'} \\ \frac{\partial \underline{r}}{\partial v'} \end{pmatrix} = \begin{pmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{pmatrix} \begin{pmatrix} \frac{\partial \underline{r}}{\partial u} \\ \frac{\partial \underline{r}}{\partial v} \end{pmatrix}$$

abstractly:
(remove $\underline{r}$)

$$\boxed{\begin{pmatrix} \frac{\partial}{\partial u'} \\ \frac{\partial}{\partial v'} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{\partial u}{\partial u'} & \frac{\partial v}{\partial u'} \\ \frac{\partial u}{\partial v'} & \frac{\partial v}{\partial v'} \end{pmatrix}}_{J^{-t}} \begin{pmatrix} \frac{\partial}{\partial u} \\ \frac{\partial}{\partial v} \end{pmatrix}}$$

directional derivatives along coordinates

remember $J_{f^{-1}} = J_f^{-1}$

From $$\frac{\partial u}{\partial u} = \frac{\partial v}{\partial v} = 1 \quad , \quad \frac{\partial u}{\partial v} = \frac{\partial v}{\partial u} = 0$$

we conclude that, setting

$$e = \left(\frac{\partial}{\partial u} , \frac{\partial}{\partial v} \right)$$

basis for tangent space (at a point)

then $$e^* = (du , dv)$$

basis for cotangent space (at a point)

We shall resume this discussion later on.