

Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

Prof. Mauro Spina, UCSC, Brescia

Lecture VIII

MULTILINEAR ALGEBRA

- Tensor products
- Spaces of tensors
- Examples

* Tensor products

Let V, W (finite dimensional) vector spaces over a field K ($\dim_K V = n, \dim_K W = m$);

their tensor product $V \otimes_K W$ (or $V \otimes W$, if no confusion arises) is defined as follows

$$V \otimes_K W := \left\{ \pi : \begin{array}{c} \downarrow \quad \downarrow \\ V^* \times W^* \end{array} \longrightarrow K \mid \pi \text{ bilinear} \right\}$$

It is naturally a vector space, generated by π 's of the form

$$\boxed{v \otimes w \equiv \pi : \begin{array}{ccc} \alpha^* & , & \beta^* \\ \uparrow & & \uparrow \\ V^* & & W^* \end{array} \longmapsto \underbrace{\alpha^*(v)}_K \cdot \underbrace{\beta^*(w)}_K}_{\text{product in } K}} \quad \boxed{\quad}$$

decomposable vectors in $V \otimes W$

Notice that $\alpha(v \otimes w) = \alpha v \otimes w = v \otimes \alpha w$

$\forall \alpha \in K, v \in V, w \in W$, and

$$(\alpha v_1 + \beta v_2) \otimes w = \alpha(v_1 \otimes w) + \beta(v_2 \otimes w) \text{ etc.}$$

(as maps).

A more abstract algebraic definition is possible, but we decided to be more concrete

Given, as usual, bases $e = (e_1, \dots, e_n)$, $f = (f_1, \dots, f_m)$ in V and W , resp., it is easily checked that

$e \otimes f \equiv \left(e_i \otimes f_j \right)_{\substack{i=1 \dots n \\ j=1 \dots m}}$ yields a basis for

$V \otimes W$, i.e., any $\pi \in V \otimes W$ can be written as

$$\boxed{\pi = \sum_{i,j} c_{ij} e_i \otimes f_j}, \text{ for uniquely determined } c_{ij} \in K.$$

Hence $\dim V \otimes W = \dim V \cdot \dim W = n \cdot m$

Notice, in particular,
$$\left\{ (e_i \otimes f_j) (e_k^*, f_l^*) = e_k^*(e_i) \cdot f_l^*(f_j) = \delta_{ik} \cdot \delta_{lj} \right\}$$

and that, if $v = \sum_i \alpha_i e_i$, $w = \sum_j \beta_j f_j$,

then
$$\boxed{v \otimes w = \sum_{i,j} \underbrace{\alpha_i \beta_j}_{c_{ij}} e_i \otimes f_j}$$

Important remark

$$\boxed{\text{Hom}(V, W) \cong W \otimes V^*}$$

canonically

This comes from
$$\begin{array}{c} \text{leave it as it stands} \end{array} \quad \begin{array}{c} \uparrow \\ (w \otimes v^*) \end{array} \quad \begin{array}{c} \uparrow \\ \text{leave it as it stands} \end{array} \quad \begin{array}{c} \text{leave it as it stands} \end{array}$$

$\begin{array}{c} \downarrow K \\ W \\ \downarrow \\ W \end{array}$

Concretely, choosing bases

$$w = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}_{m \times 1} \quad v^* = \begin{bmatrix} \end{bmatrix}_{1 \times n}$$

$$w \otimes v^* = \begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix}_{m \times 1} \cdot \begin{bmatrix} \end{bmatrix}_{1 \times n} = \begin{bmatrix} \end{bmatrix}_{m \times n} \text{ matrix}$$

matrix product

$$\boxed{w \otimes v^*} = \left\{ \sum_{i,j} a_{ij} (f_i \otimes e_j^*) \right\}$$

$$\sum_{i,j} a_{ij} (f_i \otimes e_j^*)(\cdot, e_k) = \sum_{i,j} a_{ij} \underbrace{e_j^*(e_k)}_{\delta_{jk}} f_i$$

$$= \sum_i a_{ik} f_i \quad , \quad \text{i.e.} \quad e_k \mapsto \sum_i a_{ik} f_i$$

$$\mapsto \left(\begin{bmatrix} \vdots \\ \vdots \\ \vdots \end{bmatrix} \right)_{k\text{-column}}$$

of the matrix representation
of a homomorphism T

One can naturally define $V_1 \otimes V_2 \otimes V_3$ etc.

either directly or via $(V_1 \otimes V_2) \otimes V_3$

(canonically isomorphic to $V_1 \otimes (V_2 \otimes V_3)$) and concluding by induction.

We shall stick to the case in which $V_i = V$ or V^*

Define the **space of (p, q) -tensors**, $\mathcal{T}_{p,q} = \underbrace{V^* \otimes \dots \otimes V^*}_q \otimes \underbrace{V \otimes \dots \otimes V}_p$

contravariance index

covariance index

notice

notice

$$\mathcal{T}_{p,q} = \left\{ \pi: \underbrace{V \times V \times \dots \times V}_q \times \underbrace{V^* \times \dots \times V^*}_p \rightarrow K \mid \pi \text{ multilinear, i.e. linear in each argument} \right\}$$

$$V \otimes V^* \cong V^* \otimes V$$

canonically

$$V \otimes V \otimes V^* \cong \underbrace{V \otimes V \otimes V^*}_{\cong \underbrace{V \otimes V \otimes V^*}_{\cong \dots}}$$

$$V^* \otimes V \otimes V \cong \underbrace{V^* \otimes V \otimes V}_{\cong \dots}$$

$$V \otimes V^* \otimes V$$

etc.

$T \in \mathcal{T}_{p,q}$ is p -times contravariant
 q -times covariant

Fixing a basis $e = (e_1, \dots, e_n)$ in V , together with the dual basis

$$e^* = (e_1^*, \dots, e_n^*) \quad , \quad \text{we have}$$

$$T = \sum_{I, J} T_{J, I}^{i_1 \dots i_p} \underbrace{e_{j_1}^* \otimes e_{j_2}^* \otimes \dots \otimes e_{j_q}^*}_{\text{components of } T \text{ with respect to the basis } (e^{j_1} \otimes e^{j_2} \otimes \dots \otimes e^{j_q} \otimes e_{i_1} \otimes \dots \otimes e_{i_p})} \otimes \underbrace{e_{i_1} \otimes \dots \otimes e_{i_p}}_{\text{components of } T \text{ with respect to the basis } (e^{j_1} \otimes e^{j_2} \otimes \dots \otimes e^{j_q} \otimes e_{i_1} \otimes \dots \otimes e_{i_p})}$$

$I = (i_1, \dots, i_p)$
 p -multiindex
 $J = (j_1, \dots, j_q)$
 q -multiindex

Schematically: $T = T_J^I e^J \otimes e_I$

notice that $e_j^* \mapsto e^j$!

Einstein's convention

this notation would be more appropriate

★ Examples

$$V, V^*, \text{End}(V) \cong V \otimes V^* \cong V^* \otimes V$$

endomorphisms

are easily recovered.

$$\boxed{V = \mathcal{L}_{1,0}} \quad , \quad \boxed{V^* = \mathcal{L}_{0,1}}$$

$$\boxed{\text{End}(V) \cong \mathcal{L}_{1,1}}$$

notice this i : row index

$$T = (a_j^i)$$

↑ notice this

$$v = x^i e_i$$

↑
Einstein's conv.

a_j^i ↑ row index
↓ column index

$$T = a_j^i e^j \otimes e_i$$

★ Compute TV using tensor notation (Einstein's convention being employed throughout)

$$(a_j^i e^j \otimes e_i) (x^k e_k) = a_j^i x^k e^j (e_k) e_i = a_j^i x^k \underset{\substack{\parallel \\ e_j^*(e_k) \\ \parallel \\ \delta_{jk}}}{e^j(e_k)} e_i = a_j^i x^k \delta_{jk} e_i$$

$$= a_j^i x^j e_i \Rightarrow (x^i) \mapsto (a_j^i x^j)$$

$$\boxed{(x^i)} \mapsto \boxed{(a_j^i)} \boxed{(x^j)}$$

* Another example (crucial in Riemannian geometry)

$\langle , \rangle$ inner product on a Euclidean vector space V

$\langle , \rangle$ is a $(0,2)$ -tensor

$\langle , \rangle: V \times V \rightarrow \mathbb{R}$

bilinear

upon choosing a basis $e = (e_1, \dots, e_n)$

(+ symmetric
+ positive definite)

$$v = x^i e_i$$

$$w = y^j e_j$$

$$\langle e_i, e_j \rangle = g_{ij}$$

$$\langle v, w \rangle = g_{ij} x^i y^j$$

* Tensor product of tensors

$$T \in \mathcal{T}_{p,q} \quad S \in \mathcal{T}_{p',q'} \quad \leadsto$$

$$T \otimes S \in \mathcal{T}_{p+p', q+q'} \quad \text{is concretely given}$$

as follows:

$$\begin{aligned} \text{if } T &= T^I_J e^J \otimes e_I & |I| &= p \quad |J| = q \\ S &= S^K_H e^H \otimes e_K & |K| &= p' \quad |H| = q' \end{aligned}$$

$$\begin{aligned} T \otimes S &= T^I_J S^K_H e^J \otimes e^H \otimes e_I \otimes e_K \\ &= W^Z_\gamma e^Z \otimes e_\gamma \end{aligned}$$

with

$$W^Z_\gamma = T^I_J S^K_H$$

examples: ① $T = (v^i) \in V$

that is: $T = v^i e_i$

$$S = (w_j) \quad (S = w_j e^j) \in V^*$$

$$T \otimes S = v^i w_j e^i \otimes e_j \sim (v^i w_j) \in V \otimes V^* \quad (\cong \text{End}(V))$$

$$\begin{array}{c} \downarrow \\ \boxed{v} \xrightarrow{w^T} \end{array} = \boxed{\begin{array}{c} v^i w_j \\ \downarrow \\ i \quad j \end{array}}$$

$$a_{ij} = v^i w_j$$

④

Special case: Tensor product of covariant tensors:

$$S \in \mathcal{T}_{0,q} \quad T \in \mathcal{T}_{0,q'} \quad S \otimes T \in \mathcal{T}_{0,q+q'}$$

$$\boxed{(S \otimes T) (\overset{\downarrow}{v_1} \dots \overset{\downarrow}{v_{q+q'}}) := S(v_1 \dots v_q) T(v_{q+1} \dots v_{q+q'})}$$

($\neq T \otimes S$ in general) intrinsic definition

In the sequel we shall discuss covariant tensors (tensorfields) with prescribed symmetry properties

(e.g. Riemannian metrics (symmetric)
differential forms (antisymmetric) and so on)

⑤

Contraction

Fixing one or more arguments produces tensors as well, with different indices! This is referred to as contraction. For the time being let us briefly discuss couple of examples:

① $g = (g_{ij}) \in \mathcal{T}_{0,2}$ $v = (v^i) \in \mathcal{T}_{1,0}$ $g \otimes v = (g_{ij} v^k) \in \mathcal{T}_{1,2}$

$\left\{ \begin{matrix} \sim \\ \sim \end{matrix} \right\}$ $\mathcal{T}_{0,2}$ $\mathcal{T}_{1,0}$ $\left\{ \begin{matrix} \sim \\ \sim \end{matrix} \right\}$ get $j=k$ (and sum (Einstein))

get $\boxed{g_{ij} v^j =: v_i}$ $(v_i) = b((v^j))$

The covariant index j is lowered $\langle v, \cdot \rangle$

★ g has been contracted with v

② $T = (a^i_j) \in \mathcal{T}_{1,1}$ $v = (v^k) \in \mathcal{T}_{1,0}$

contract: $\boxed{a^i_j v^j =: w^i}$ $(w^i) \in \mathcal{T}_{1,0}$

again a decrease of covariance index

⑥ Another basic example of contraction: The trace of an endomorphism

$$T = (a^i_j) \quad T \in \text{End}(V) \cong V \otimes V^*$$

take $i = j$ (and sum, as usual), get

$$a^i_i = \text{Tr } T \quad (\text{a number, i.e. a } \mathbb{Z}_{0,0} \text{ tensor})$$

trace of T

More complicated examples emerge, for instance, in Riemannian geometry, giving rise to

partial traces, and will be discussed later on.