Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

Prof. Mauro Spera, UCSC, Brescia

MULTILINEAR ALGEBRA

Lecture X

- k-forms (continued)
- products of k 1-forms
- Further examples
- Linear algebraic digression
- Grassmann algebra

✱ k-forms (continued)

If $\omega_1, \omega_2 \ldots \omega_k \in \Lambda^1(V^*)$, one finds

again $\dim V = n \ldots$

$$\underbrace{(\omega_1 \wedge \omega_2 \wedge \ldots \wedge \omega_k)}_{\in \Lambda^k(V^*)}(v_1, v_2 \ldots, v_k) =$$

$\leftarrow v$ fixed $\rightarrow$

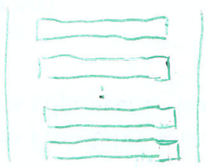
determinant

$$\begin{vmatrix} \omega_1(v_1) & \cdots & \omega_k(v_1) \\ \vdots & & \\ \omega_1(v_k) & \cdots & \omega_k(v_k) \end{vmatrix} \quad \omega \text{ fixed}$$

which represents the volume of a suitable hyperparallelopode in $\mathbb{R}^k$

(set $\mathbb{R}^n \ni \xi \longmapsto (\omega_1(\xi) \ldots \omega_k(\xi)) \in \mathbb{R}^k$

(row notation))



Consider, in particular, the k-forms

$$e^*_{i_1} \wedge e^*_{i_2} \wedge \dots \wedge e^*_{i_k} \qquad i_1 < i_2 < \dots < i_k$$

They yield a basis of $\Lambda^k(V^*)$:

recall $e^*_i(e_j) = \delta_{ij}$

One easily checks that

$$(e^*_{i_1} \wedge \dots e^*_{i_k})(e_{j_1}, e_{j_2} \dots e_{j_k}) =$$

$$= \begin{cases} \pm 1 & \text{if } \{i_1 \dots i_k\} = \{j_1 \dots j_k\} \\ & \text{(sign according to parity of the corresponding permutation} \\ 0 & \text{otherwise} \end{cases}$$

and this implies

$$e^*_I = e^*_{i_1} \wedge \dots e^*_{i_k}$$

defined previously

+ Pull-back of k-forms

Given $T \in \mathrm{Hom}(V,W) \cong W \otimes V^*$

and $\omega \in \Lambda^k(W^*)$, the "pulled-back"

form $T^*\omega \in \Lambda^k(V^*)$ (pull-back of ω)

is defined as follows:

$$\boxed{(T^*\omega)(\underset{\overset{\cap}{V}}{v_1}\dots \underset{\overset{\cap}{V}}{v_k}) := \omega(\overset{W}{\overset{\cup}{Tv_1}} \dots \overset{W}{\overset{\cup}{Tv_k}})}$$

$$\underset{T^*\omega}{\overset{\Lambda^k(V)}{\cup}} \longleftarrow \underset{\omega}{\overset{\Lambda^k(W^*)}{\cup}}$$

$$T: V \longrightarrow W$$

It is immediately checked that, given homomorphisms

$$U \xrightarrow{S} V \xrightarrow{T} W \quad , \quad \boxed{(T\circ S)^*\underset{\overset{\cap}{\Lambda^k(W^*)}}{\omega} = S^* T^* \omega}$$

notice the reverse order

$$\left\{ ((T\circ S)^*\omega)(\overset{U}{\overset{\cup}{x_1}} \dots \overset{U}{\overset{\cup}{x_k}}) = \omega(TSx_1, \dots, TSx_k) \right.$$

$$= \omega(T(Sx_1) \dots T(Sx_k)) = (T^*\omega)(Sx_1 \dots Sx_k)$$

$$\left. = S^*(T^*\omega)(x_1 \dots x_k) = ((S^*T^*)\omega)(x_1 \dots x_k) \right\}$$

* Further examples

* In $\mathbb{R}^3$ let

$$\omega^1_a = \sum_{i=1}^{3} a_i e^*_i \qquad \underline{a} = (a_1, a_2, a_3)$$

$$\omega^1_b = \sum_{i=1}^{3} b_i e^*_i \qquad \underline{b} = (b_1, b_2, b_3)$$

Compute $\omega^1_a \wedge \omega^1_b$. One gets

$$\omega^1_a \wedge \omega^1_b = \sum_{i,j} a_i b_j \, e^*_i \wedge e^*_j = \ldots =$$

$$= \begin{matrix} z & (a_1 b_2 - a_2 b_1)\, e^*_1 \wedge e^*_2 & + \\ x & (a_2 b_3 - a_3 b_2)\, e^*_2 \wedge e^*_3 & + \\ y & (a_3 b_1 - a_1 b_3)\, e^*_3 \wedge e^*_1 & \end{matrix}$$

components of $\underline{a} \times \underline{b}$

flux 2-form (different notation)

$(\underline{a} \times \underline{b}) \times \underline{c} \neq \underline{a} \times (\underline{b} \times \underline{c})$

indeed $\underline{a} = \underline{i} = \underline{b}$, $\underline{c} = \underline{j}$

$\underline{0} \neq \underline{i} \times (\underline{i} \times \underline{j}) = \underline{i} \times \underline{k} = -\underline{j}$

$\times$ fulfils the Jacobi identity

$\underline{a} \times (\underline{b} \times \underline{c})$ + cyclic $= \ldots = \underline{0}$

Now set $\omega^2_c(v_1, v_2) := \langle c, v_1 \times v_2 \rangle$

$$= \det(\underline{c}, \underline{v_1}, \underline{v_2})$$

One easily checks that

$$\omega^2_c = c_1 e^*_2 \wedge e^*_3 + c_2 e^*_3 \wedge e^*_1 + c_3 e^*_1 \wedge e^*_2$$

Indeed r.h.s. $(e_1, e_2) = \ldots \; c_3$

and $\omega^2_c(e_1, e_2) = \det(c, e_1, e_2) = \begin{vmatrix} c_1 & 1 & 0 \\ c_2 & 0 & 1 \\ c_3 & 0 & 0 \end{vmatrix} \underset{\text{Laplace}}{=} c_3$

etc.

Therefore $\omega^1_a \wedge \omega^1_b = \omega^2_c$, $\underline{c} = \underline{a} \times \underline{b}$

$\Rightarrow$ $\wedge$, with a grain of salt (a metric is involved!) can be viewed as a generalization of $\times$. But beware: $\wedge$ is associative, $\times$ is not (see box)

* A linear-algebraic intermezzo

* **Direct sums** Given vector spaces U and W over the same field K, their direct sum $U \oplus W$ is defined as follows:

$$U \oplus W = \{ (u, w) \,/\, u \in U, \; w \in W \}$$

with operations

$$(u_1, w_1) + (u_2, w_2) := (\underbrace{u_1 + u_2}_{\text{in } U}, \underbrace{w_1 + w_2}_{\text{in } W})$$

$+$ to be defined

$$\alpha \cdot (u, w) := (\underbrace{\alpha u}_{\text{in } U}, \underbrace{\alpha \cdot w}_{\text{in } W})$$

$\cdot$ to be defined

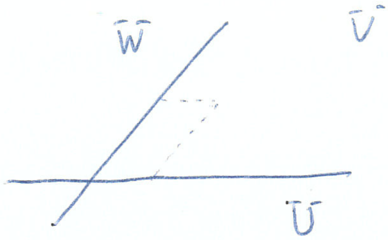
Notice that
$$U \cong \{(u, 0)\}_{u \in U} \leq U \oplus W$$
$$W \cong \{(0, w)\}_{w \in W} \leq U \oplus W$$

(i.e. U and W are naturally realized as vector subspaces of $U \oplus W$)

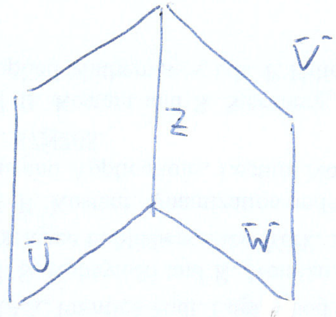
Now, given V and $U \leq V$, $W \leq V$, V is the <u>sum</u> of U and W, notation $V = U + W$, if any $v \in V$ can be written as $v = u + w$ for some $u \in U$, $w \in W$.

If u and v are unique (and this is equivalent to $U \cap W = \{0\}$), then U and W are said to be in direct sum, and one writes $V = U \oplus W$

In general, recall Grassmann's formula $\dim(U + W) + \dim(U \cap W) = \dim U + \dim W$



here $V = U \oplus W$



$Z = U \cap W \quad (\leq V)$

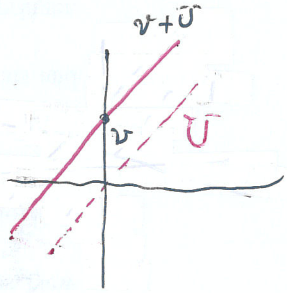
one has $V = U + W$

but the sum is not direct

★ Quotient spaces (fundamental for the sequel: homology & cohomology theories)

Let $U \leq V$. Define the quotient vector space

$$V/U = \{ [v] := v + U \}$$



linear varieties, i.e. affine subspaces of V of "direction", or "position" U

quotient of abelian groups. (U is normal in V)

structured as follows: $[v_1] + [v_2] := [v_1 + v_2]$

abelian group structure with respect to +

$\alpha \cdot [v] = [\alpha \cdot v]$ in V

to be defined

one checks that + and · are well-defined, i.e. independent of the choice of a vector in its class. One also has

$$V/U \cong W \quad , W \text{ any direct complement of } U \text{ in } V$$

(i.e. W such that $U \oplus W = V$)

★ Exterior (or Grassmann) algebra

$\dim V = n$

Let

$$\underset{\substack{|||\\ \Lambda}}{\Lambda(V^*)} = \underbrace{\bigoplus_{k=0}^{n}}_{\text{direct sum}} \underbrace{\Lambda^k(V^*)}_{k\text{-forms}}$$

$\sim\!\!/\!/\ \Lambda^k$

(it is a vector space)

Extend $\wedge$ by distributivity, one has a quadruple

$$(\Lambda, \underbrace{+, \cdot}_{\text{vector space operations}}, \underbrace{\wedge}_{\text{exterior product}})$$

[Λ is structured via $+, \cdot, \wedge$]

called exterior, or Grassmann algebra (over V^*); its elements are called forms on V, elements in Λ^k are k-forms (forms of degree k)

One easily finds that

$$\dim \Lambda(V^*) = 2^n$$

($\sum_{k=0}^{n} \binom{n}{k} = 2^n$, this immediately following from

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

upon setting $a = b = 1$).