# Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

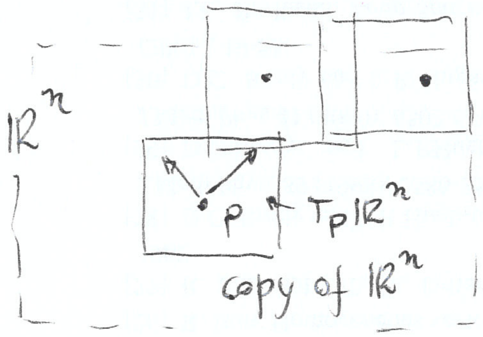
Prof. Mauro Spera, UCSC, Brescia

## Lecture XI — VECTOR FIELDS & DIFFERENTIAL 1-FORMS ON $\mathbb{R}^n$

( Tangent vectors and vector fields
{ cotangent vectors and differential forms
( analytic interpretations

* **Tangent vectors and vector fields**

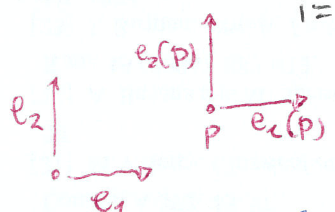
Let $p \in \mathbb{R}^n$. Let $T_p\mathbb{R}^n$ denote a copy of $\mathbb{R}^n$, thinking of its elements as "applied" vectors at $p$, and call them tangent vectors at $p$. The vector space $T_p\mathbb{R}^n$ itself is called tangent space to $\mathbb{R}^n$ at $p$.




In the sequel, a more formal definition will be given.

Given real, smooth functions $X_i = X_i(p)$, the map $X : \mathbb{R}^n \to \mathbb{R}^n$ given by

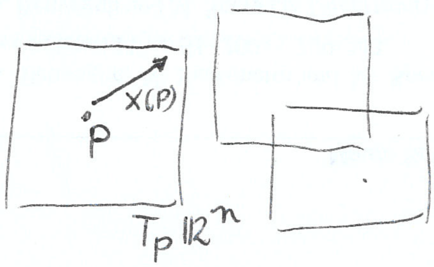
$$X(p) := \sum_{i=1}^{n} X_i(p)\, e_i(p) \in T_p\mathbb{R}^n$$

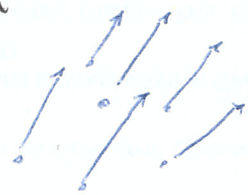



$e_i(p) \equiv e_i$ applied at $p$ ↑ canonical basis

actually: $X : p \longmapsto X(p)$, $p \in \mathbb{R}^n$, $X(p) \in T_p\mathbb{R}^n = \mathbb{R}^n$

is called a (smooth) vector field on $\mathbb{R}^n$



The union

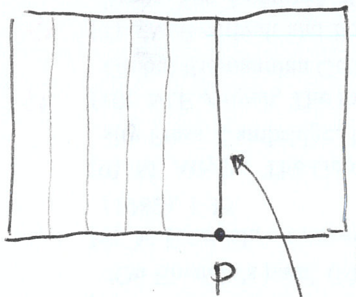
$$T\mathbb{R}^n := \bigcup_{p\in\mathbb{R}^n} T_p\mathbb{R}^n$$

actually, the disjoint union of copies of $\mathbb{R}^n$ labelled by $p \in \mathbb{R}^n$

is called tangent bundle of $\mathbb{R}^n$ (or associated to $\mathbb{R}^n$)

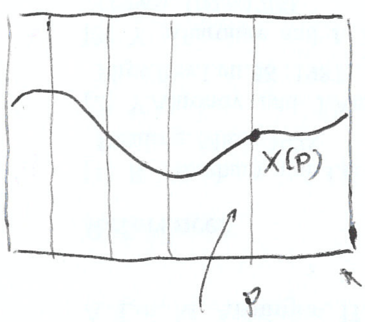
the various $T_p\mathbb{R}^n$ constitute the fibres of the tangent bunde

A vector field is a section of the tangent bundle




$T_p\mathbb{R}^n$ = fibre of $T\mathbb{R}^n$ at $p$

the "vertical" $\mathbb{R}^n$ : typical fibre




depiction of a vector field

$$\mathbb{R}^n \ni p \longmapsto X(p) \in T_p\mathbb{R}^n = \mathbb{R}^n$$

fibre at $p$

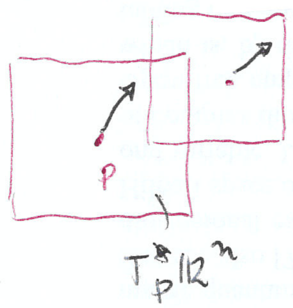
* Cotangent vectors and differential forms

Let $p \in \mathbb{R}^n$. Let us denote by $T_p^* \mathbb{R}^n$ a replica of $(\mathbb{R}^n)^*$ (dual of $\mathbb{R}^n$), thinking of its elements (dual vectors, or covectors) as being "applied" at $p$. The vector space $T_p^* \mathbb{R}^n$ is called cotangent space of $\mathbb{R}^n$ at $p$

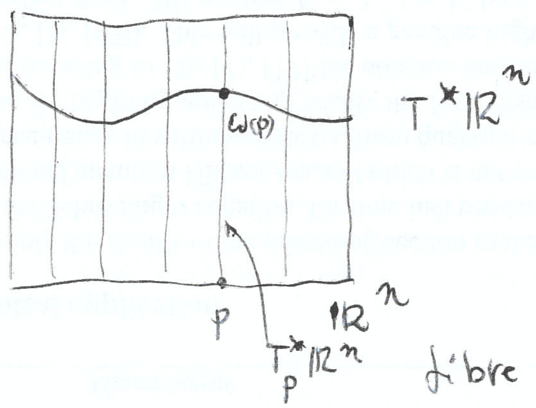
A differential 1-form $\omega$ is a map (smooth) $\omega : \mathbb{R}^n \to (\mathbb{R}^n)^*$

$$\omega = \omega(p) := \sum_{i=1}^{n} \underbrace{\omega_i(p)}_{\in\, \mathscr{C}^\infty(\mathbb{R}^n)} \underbrace{e_i^*(p)}_{=\, e_i^* \text{ applied at } p} \in T_p^* \mathbb{R}^n$$

The set $T^* \mathbb{R}^n = \bigcup_{p \in \mathbb{R}^n} T_p^* \mathbb{R}^n$ is called cotangent bundle of $\mathbb{R}^n$.




In a similar vein, differential 1-forms are the sections of the cotangent bundle




portrait of a differential 1-form

$\omega : p \to \omega(p) \in T_p^* \mathbb{R}^n$

Now – and this is a crucial point –

let us interpret tangent vectors as directional derivatives:

$$X(p) \longleftrightarrow \sum_{i=1}^{n} X_i(p) \underbrace{\frac{\partial}{\partial x^i}\Big|_p}_{\equiv\ \partial_i|_p}$$

directional derivative, applied to a generic smooth function $f$, the vector $X(p) = \begin{pmatrix} X_1(p) \\ \vdots \\ X_n(p) \end{pmatrix}$

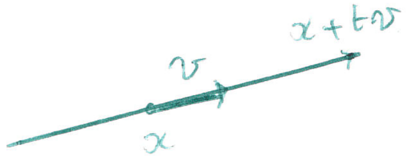
$i^{th}$ partial derivative

Recall, from analysis (see prologue as well)

$$(\diamond) \quad \frac{\partial f}{\partial v}(x) := \frac{d f(x+tv)}{dt}\Big|_{t=0} \qquad v = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$$

$$\underset{\text{chain rule}}{=} \sum_{i=1}^{n} v_i \frac{\partial f}{\partial x^i}(x) = df\big|_x (v)$$

differential of $f$ at $x$




Shortly: $v \longleftrightarrow v \cdot \nabla = \sum v_i \frac{\partial}{\partial x^i} \equiv v^i \partial_i$ $\qquad \nabla = \begin{pmatrix} \frac{\partial}{\partial x^1} \\ \vdots \\ \frac{\partial}{\partial x^n} \end{pmatrix}$

☆☆ Then we have, in view of $(\diamond)$, the following identifications:

$$\boxed{e_i \longleftrightarrow \partial_i \qquad \underset{\equiv\ e^j}{e^*_j} \longleftrightarrow dx^j}$$

notice $e_i = (\delta_i^k)_{k=1..n}$

Indeed: $\quad dx^j(e_i) = \delta_i^k \frac{\partial x^j}{\partial x^k} = \delta_i^k \cdot \delta_k^j = \delta_i^j \quad (= \delta_{ij})$

Therefore, from now on, we shall replace

$e_i$ by $\frac{\partial}{\partial x^i}$ $\left(\frac{\partial}{\partial x^i}\Big|_p\right)$

$\parallel$

$e_i(p)$

constant vector field

and $e^i \equiv e_i^* = e^i(p)$ by $dx^i$

constant 1-form

We have the following fundamental formula

$X \in \mathcal{X}(\mathbb{R}^n)$, $f \in \Lambda^0(\mathbb{R}^n) \equiv C^\infty(\mathbb{R}^n)$

vector fields    0-form

$$\boxed{X(f) = df(X) \overset{\text{duality}}{\equiv} (df, X)}$$

That is, at every point $p$, $X(f)(p) = (df)_p(X(p))$

directional derivative of $f$ along $X$ evaluated at $p$

evaluation of the differential at $p$ on $X(p)$

Pf $X = b^j \partial_j$ $\quad df = \frac{\partial f}{\partial x^i} dx^i$

$\boxed{X(f)} = \boxed{b^j \partial_j f}$ $\quad \boxed{df(X)} = \left(\frac{\partial f}{\partial x^i} dx^i\right)\left(b^j \frac{\partial}{\partial x^j}\right) =$

$= \frac{\partial f}{\partial x^i} b^j \, dx^i\left(\frac{\partial}{\partial x^j}\right) = \frac{\partial f}{\partial x^i} b^j \delta^i_j = \boxed{b^j \frac{\partial f}{\partial x^j}}$ $\square$

More generally, given $X \in \mathfrak{X}(\mathbb{R}^n)$, $X = b^j \partial_j$

$\omega \in \Lambda^1(\mathbb{R}^n)$, $\omega = a_i dx^i$,

$$\left[\begin{aligned} \underset{\substack{\cap \\ \Lambda^0(\mathbb{R}^n) \\ \| \\ C^\infty(\mathbb{R}^n)}}{\omega(X)} &= a_i dx^i (b^j \partial_j) = a_i b^j dx^i(\partial_j) \\ &= a_i b^j \delta^i_j = a_i b^i \end{aligned}\right]$$