Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture XII

DIFFERENTIAL FORMS ON $\mathbb{R}^n$

As usual $n \geq 1$

Differential k-forms
properties
functoriality
examples

* Differential k-forms

Let $\Lambda^k_p(\mathbb{R}^n)$ denote the $\binom{n}{k}$ dimensional space of k-forms on $T_p\mathbb{R}^n$ [notice the slight change of notation]

A basis thereof is given by $\left(dx^{i_1}\big|_p \wedge \dots dx^{i_k}\big|_p \right)$

($\spadesuit$) $i_1 < i_2 < \dots < i_k$, $i_j \in \{1, 2, \dots, n\}$

notice the upper indices appended to coordinates: this is in view of applying tensor notation

Obviously

$$dx^{i_1}\big|_p \wedge \dots dx^{i_k}\big|_p = dx^{i_1} \wedge \dots dx^{i_k}\big|_p$$

Also set $dx^{i_1} \wedge \dots dx^{i_k} =: dx^I$ $\quad I = (i_1 \dots i_k)$
(abridged notation)

The differential k-forms (on $\mathbb{R}^n$) are then the (smooth) functions

$$\omega : \mathbb{R}^n \ni p \longmapsto \omega_I(p)\, dx^I \qquad \text{use } (\spadesuit)$$

Einstein

$$= \omega_{i_1 \dots i_k}(p)\, dx^{i_1} \wedge \dots dx^{i_k}\big|_p$$

sometimes, for clarity, Σ will be added

$\omega_I \equiv \omega_{i_1 \dots i_k} \in \Lambda^0(\mathbb{R}^n)$
smooth function

Notation: $\Lambda^k(\mathbb{R}^n)$

Again a bundle interpretation can be given, but for the time being we do not further delve into it.

* Properties of differential forms
(we shall often omit the adjective "differential")

Given k-forms $\omega_1 = a_I \, dx^I$, $\omega_2 = b_I \, dx^I$,

their linear combination (with $\alpha, \beta \in \Lambda^0(\mathbb{R}^n)$) is

$$\alpha \omega^1 + \beta \omega^2 = (\alpha a_I + \beta b_I)\, dx^I$$

(as in the algebraic case: everything is carried out pointwise)

Notice that $\mathcal{X}(\mathbb{R}^n)$ (vector fields) and $\Lambda^k(\mathbb{R}^n)$ are in fact modules over $\Lambda^0(\mathbb{R}^n)$
$X \in \mathcal{X}(\mathbb{R}^n) \Rightarrow fX \in \mathcal{X}(\mathbb{R}^n)$
$\omega \in \Lambda^k(\mathbb{R}^n) \Rightarrow f\omega \in \Lambda^k(\mathbb{R}^n)$
etc.

The wedge product between ω_1 (k-form) and ω_2 (l-form), $\omega_1 = a_I \, dx^I$, $\omega_2 = b_J \, dx^J$

(again defined pointwise) reads:

$$\omega_1 \wedge \omega_2 = a_I b_I \, dx^I \wedge dx^J$$

Let us check that

(a) $\overset{k}{(\omega} \wedge \overset{s}{\varphi}) \wedge \overset{r}{\nu} = \omega \wedge (\varphi \wedge \nu)$

$\omega = a_I dx^I$, $\varphi = b_J dx^J$, $\nu = c_K dx^K$ (associativity, therefore one can safely write $\omega \wedge \varphi \wedge \nu$ without ambiguity)

(b) $\omega \wedge (\varphi + \nu) = \omega \wedge \varphi + \omega \wedge \nu$ (easy)
if $r = s$

(c) $\overset{k}{\omega} \wedge \overset{s}{\varphi} = (-1)^{ks} \, \overset{s}{\varphi} \wedge \overset{k}{\omega}$ graded commutativity

Proof of (a): $(\omega \wedge \varphi) \wedge \vartheta = (a_I dx^I \wedge b_J dx^J) \wedge c_K dx^K =$

$= (a_I b_J \, dx^I \wedge dx^J) \wedge (c_K \, dx^K) = a_I b_J c_K \, dx^I \wedge dx^J \wedge dx^K$

$= \text{r.h.s.}$

Proof of (c): $\omega \wedge \varphi = a_I b_J \, dx^{i_1} \wedge \dots dx^{i_k} \wedge dx^{j_1} \wedge \dots dx^{j_s})$

$= - a_I b_J \, dx^{i_1} \wedge \dots dx^{j_1} \wedge dx^{i_k} \wedge \dots dx^{j_s})$

$\Rightarrow$ there are $\underbrace{k + k + \dots k}_{s \text{ times}} = k \cdot s$ sign changes before abutting at $\varphi \wedge \omega$, yielding the $(-1)^{ks}$ factor in the r.h.s.

Notice that in general $\omega \wedge \omega \neq 0$

(If $\omega \in \Lambda^k(\mathbb{R}^n)$, k odd, then $\omega \wedge \omega = (-1)^{k^2} \omega \wedge \omega$ odd

$= -\omega \wedge \omega \Rightarrow \omega \wedge \omega = 0$)

If k is even, then one has a tautology: $\omega \wedge \omega = \omega \wedge \omega$

Example. In $\mathbb{R}^4$, take $\omega = dx^1 \wedge dx^2 + dx^3 \wedge dx^4 \in \Lambda^2(\mathbb{R}^4)$

This is an example of symplectic form

Then $\omega \wedge \omega = \underbrace{dx^1 \wedge dx^2 \wedge dx^1 \wedge dx^2}_{= 0} + dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

$+ dx^3 \wedge dx^4 \wedge dx^1 \wedge dx^2 + \underbrace{dx^3 \wedge dx^4 \wedge dx^3 \wedge dx^4}_{0}$

$= dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4$

$= 2 \, \underbrace{dx^1 \wedge dx^2 \wedge dx^3 \wedge dx^4}_{\text{volume form on } \mathbb{R}^4}$

* Pull-back of differential forms

Given a smooth function $f: \mathbb{R}^n \to \mathbb{R}^m$,

and a k-form $\omega \in \Lambda^k(\mathbb{R}^m)$, one can construct

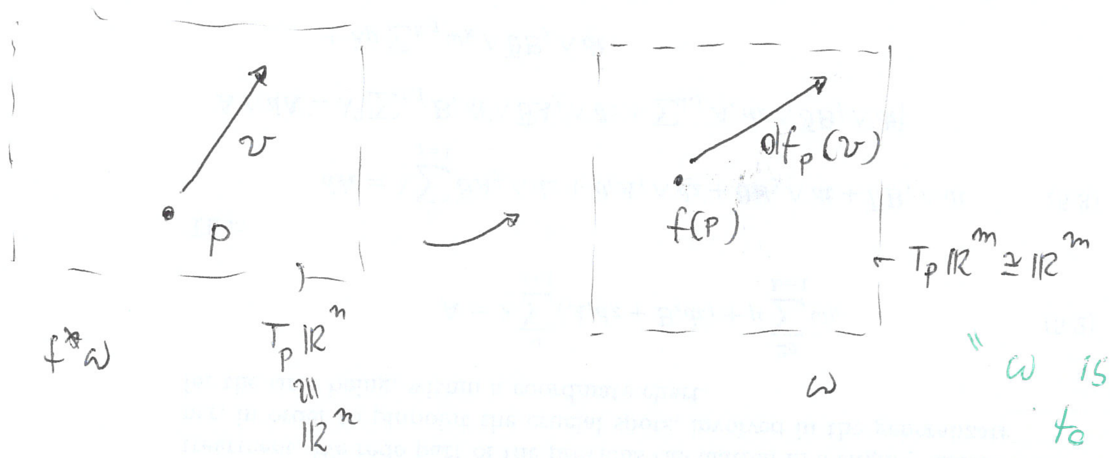
a k-form $f^*\omega \in \Lambda^k(\mathbb{R}^n)$ (pull-back of ω via f)

in the following guise:

pointwise, the k-form is to be evaluated on k-tuple of vectors from $T_p\mathbb{R}^n$

$$\boxed{(f^*\omega)(\underset{\in \mathbb{R}^n}{p})(\underbrace{v_1 \dots v_k}_{\in T_p\mathbb{R}^n}) := \omega(f(p))(\underbrace{df_p(v_1), \dots df_p(v_k)}_{\in T_{f(p)}\mathbb{R}^m})}$$

differential of f



" ω is pulled back to $f^*\omega$ "

If $g \in \Lambda^0(\mathbb{R}^m)$ (a smooth function $g: \mathbb{R}^m \to \mathbb{R}$)

set $f^*g := g \circ f$

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

* Properties of pull-back ("functoriality")

compatibility with the various operations

(a) $f^*(\omega + \varphi) = f^*\omega + f^*\varphi$

$\uparrow$ k-forms

(easy)

$\mathbb{R}^n \xrightarrow{f} \mathbb{R}^m \xrightarrow{g} \mathbb{R}$

(b) $f^*(g \cdot \omega) = \underbrace{f^*(g)}_{= g \circ f} f^*\omega$, $\quad g \in \Lambda^0$

(easy)

(c) $f^*(\varphi_1 \wedge \varphi_2 \wedge \dots \varphi_k) = f^*\varphi_1 \wedge f^*\varphi_2 \wedge \dots f^*\varphi_k$

1 forms

this will hold in general

Let us prove it: $f^*(\varphi_1 \wedge \dots \varphi_k)(v_1 \dots v_k) =$

evaluated at p

$= (\varphi_1 \wedge \dots \varphi_k)(df(v_1), \dots df(v_k))$

$=$ (recall!) $\det(\varphi_i(df(v_j))) = \det(f^*\varphi_i(v_j))$

$= (f^*\varphi_1 \wedge \dots f^*\varphi_k)(v_1, \dots v_k)$ $\square$

(d) in general: $f^*(\omega \wedge \varphi) = f^*\omega \wedge f^*\varphi$

By virtue of associativity, one has, in general

$$f^*(\omega_1 \wedge \dots \omega_k) = f^*\omega_1 \wedge \dots f^*\omega_k$$

any form

Also (e) $(f \circ g)^*\omega = g^*(f^*\omega)$

Let us prove (d) and (e)

$$\boxed{f^*(\omega \wedge \varphi) = f^*\omega \wedge f^*\varphi} \qquad f_* \equiv df$$

k-form l-form

differential (push-forward

$$f^*(\omega\wedge\varphi)(v_1 - v_{k+l}) = (\omega\wedge\varphi)(f_* v_1, f_* v_2 —— f_* v_{k+l})$$

$$= \frac{1}{k!\,l!} \sum_\nu (-1)^\nu \, \omega(f_* v_{i_1} - f_* v_{i_k})\, \varphi(f_* v_{i_{k+1}}, — f_* v_{i_{k+l}})$$

$$= \frac{1}{k!\,l!} \sum_\nu (-1)^\nu (f^*\omega)(v_{i_1} - v_{i_k})(f^*\varphi)(v_{i_{k+1}} - v_{i_{k+l}})$$

$$= (f^*\omega \wedge f^*\varphi)(v_1 —— v_{k+l}) \qquad \square$$

$$\boxed{(f\circ g)^*\omega = g^*(f^*\omega)}$$

$$\left[(f\circ g)^*\omega\right](v_1 - v_k) = \omega\left((f\circ g)_*(v_1) —— (f\circ g)_*(v_k)\right)$$

k-form

$$= \omega\left(f_*\cdot g_*(v_1) —— f_*\cdot g_*(v_k)\right)$$

(chain rule)

$$= (f^*\omega)(g_*(v_1) —— g_*(v_k))$$

$$= g^*(f^*\omega)(v_1 —— v_k)$$

Let us interpret pull-back operationally

Pick $f : \mathbb{R}^n \to \mathbb{R}^m$ (one can in fact work on $\mathcal{U} \subset \mathbb{R}^n$ open)

concretely

$$\begin{cases} y^1 = f^1(x^1 \dots x^n) \\ y^2 = f^2(x^1 \dots x^n) \\ \vdots \\ y^m = f^m(x^1 \dots x^n) \end{cases} \qquad y = f(x)$$

Now:

$$(f^* dy^i)(v) \underset{\text{def}}{=} dy^i(df(v)) \underset{\text{chain rule}}{=} d(y^i \circ f)(v)$$

def of pull-back

$$= d(f^* y^i)(v) = df^i(v)$$

$y^i \circ f = f^i$ as a function of x

Therefore, operationally, if $\omega = a_I(y) dy^I$,

then $f^* \omega = a_I(f(x)) df^I$ (functoriality of pull-back: it "respects" wedge products)

$$dy^I = \frac{\partial y^I}{\partial x^J} dx^J$$

"partial Jacobians"

If $I = i$, $J = j$ (single indices) $\left(\frac{\partial y^i}{\partial x^j}\right)$ is the Jacobian matrix of df

$$\omega = a_I dy^I \longmapsto f^* \omega = a_I \frac{\partial y^I}{\partial x^J} dx^J$$

$$a_I \frac{\partial y^I}{\partial x^J} \equiv a'_J$$

See how practical tensor notation is!

See also below

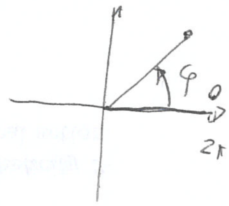
Example (extremely important)

1. $\omega = \dfrac{-y}{x^2+y^2}\,dx + \dfrac{x}{x^2+y^2}\,dy$ (in $U = \{ r>0,\ 0<\varphi<2\pi \}$

polar coordinates

defined for $(x,y) \neq (0,0)$

Let $f : \begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$ $x^2+y^2=r^2$



$\varphi = \arctan \dfrac{y}{x}$ $x \neq 0$

Then $f^*\omega = \ldots = d\varphi$ angular form

It is instructive to compute this directly

$$dx = dr\cos\varphi + r\,d\cos\varphi$$
$$= \cos\varphi\,dr - r\sin\varphi\,d\varphi$$
$$dy = \sin\varphi\,dr + r\cos\varphi\,d\varphi$$

These two terms cancel out

$$-\frac{r\sin\varphi}{r^2}\left(\cos\varphi\,dr - r\sin\varphi\,d\varphi\right) + \frac{r\cos\varphi}{r^2}\left(\sin\varphi\,dr + r\cos\varphi\,d\varphi\right)$$

$$= (\underbrace{\sin^2\varphi + \cos^2\varphi}_{1})\,d\varphi = d\varphi$$

other examples

Sums over $l_1 \ldots l_k$ are omitted

2. $$dy^{i_1} \wedge \ldots dy^{i_k} = \frac{\partial y^{i_1}}{\partial x^{l_1}}\,dx^{l_1} \wedge \ldots \frac{\partial y^{i_k}}{\partial x^{l_k}}\,dx^{l_k}$$

$$= \underbrace{\frac{\partial(y^{i_1} \ldots y^{i_k})}{\partial(x^{l_1} \ldots x^{l_k})}}_{\text{"partial Jacobians"}}\,dx^{l_1} \wedge \ldots dx^{l_k}$$

This is clear from the very definition of determinant (involving the appropriate sums over permutations, weighted with $(-1)^\sigma$..

Whenever two equal dx appear, one gets zero by skew-symmetry

Take, for instance

$2'.\quad f : \mathbb{R}^2 \to \mathbb{R}^2 \qquad \begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$

$\omega = dx \wedge dy \qquad f^*\omega = \dfrac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv$

"area 2-form"

(oriented)

without | |: one has an oriented area element

Check this directly:

$$\left[\quad \begin{aligned} dx &= x_u\, du + x_v\, dv \\ dy &= y_u\, du + y_v\, dv \end{aligned} \qquad dx \wedge dy = \underbrace{(x_u y_v - x_v y_u)}_{\frac{\partial(x,y)}{\partial(u,v)}}\, du \wedge dv \quad \right]$$

we have omitted the symbol f^*

$2''.\quad f : \mathbb{R}^3 \to \mathbb{R}^2 \qquad dy^1 \wedge dy^2 = \dfrac{\partial y^1}{\partial x^{\ell_1}} \dfrac{\partial y^2}{\partial x^{\ell_2}}\, dx^{\ell_1} \wedge dx^{\ell_2}$

$$= \underbrace{\left(\frac{\partial y^1}{\partial x^1}\frac{\partial y^2}{\partial x^2} - \frac{\partial y^1}{\partial x^2}\frac{\partial y^2}{\partial x^1} \right)}_{\frac{\partial(y^1,y^2)}{\partial(x^1,x^2)}} dx^1 \wedge dx^2 + \text{similar terms}$$

$$\frac{\partial(y^1,y^2)}{\partial(x^1,x^2)} = \begin{vmatrix} \frac{\partial y^1}{\partial x^1} & \frac{\partial y^1}{\partial x^2} \\ \frac{\partial y^2}{\partial x^1} & \frac{\partial y^2}{\partial x^2} \end{vmatrix}$$

$3.\quad \varphi : \mathbb{R}^2 \to \mathbb{R}^3 \qquad \varphi \equiv \underline{r}$

$(u,v) \mapsto (x,y,z)$

F : flux 2-form $\left(\in \Lambda^2(\mathbb{R}^3)\right)$

$F = F_1\, dy \wedge dz + F_2\, dz \wedge dx + F_3\, dx \wedge dy$

$$\varphi^* F = \Big[F_1(\overbrace{x(u,v), y(u,v), z(u,v)}^{\varphi(u,v)}) \frac{\partial(y,z)}{\partial(u,v)} + F_2 \frac{\partial(z,x)}{\partial(u,v)} + F_3 \frac{\partial(x,y)}{\partial(u,v)} \Big] du \wedge dv = \underline{F} \cdot \underbrace{d\underline{\sigma}}_{\text{area element}}$$

$d\underline{\sigma} = \underline{n}\, d\sigma$

$$= \langle \underline{F}, \underline{r}_u \times \underline{r}_v \rangle\, du \wedge dv$$

$\langle \underline{F}, \underline{r}_u \times \underline{r}_v \rangle = \det(\underline{F}, \underline{r}_u, \underline{r}_v)$