Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture XIII

EXTERIOR DIFFERENTIAL
- Exterior differential
- d commutes with f^*
- The de Rham complex (hint)

★ <u>Exterior differential</u> $\qquad \Lambda^k = \Lambda^k(\mathbb{R}^n)$ (or $\Lambda^k(\mathcal{U})$, $\mathcal{U} \subset \mathbb{R}^n$ open)

$\Lambda^k(\mathcal{U})$: k-forms on $\mathcal{U}$

Let us define $\quad d : \Lambda^k \longrightarrow \Lambda^{k+1}$

(exterior differential) via the position — if $\omega = a_I \, dx^I$

$$d(\underset{\Lambda^0}{a_I}\, dx^I) := da_I \wedge dx^I$$

in particular $\quad d(dx^I) = d(1 \cdot dx^I) = \underbrace{d1}_{=0} \wedge dx^I = 0$

example: $\quad \omega = dx \wedge dz + \sin z \, dx \wedge dy$

$$d\omega = d(dx \wedge dz) + d(\sin z) \wedge dx \wedge dy$$
$$= 0 + \cos z \, dz \wedge dx \wedge dy = \cos z \cdot dx \wedge dy \wedge dz$$

<u>Properties of d</u>

(a) $\qquad d(\omega_1 + \omega_2) = d\omega_1 + d\omega_2$

(valid if $\omega_1, \omega_2 \in \Lambda^k$, postulated in general)

(b) $\qquad d(\underset{\Lambda^k}{\omega} \wedge \underset{\Lambda}{\varphi}) = d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$

Λ: Grassmann

$k = \deg \omega$ — notice this

(antiderivation property)

generalizes Leibniz rule $\quad d(fg) = df \cdot g + f\,dg$

Let us prove it: let $\quad \omega = a_I \, dx^I \quad$ k-form

$\varphi = b_J \, dx^J \quad$ l-form

Then

↓ sum over I, J

$$d(\omega \wedge \varphi) = d(a_I b_J \, dx^I \wedge dx^J) =$$

$$= d(a_I b_J) \wedge dx^I \wedge dx^J = (\underbrace{da_I \, b_J}_{\substack{\text{Leibniz} \\ \| \\ b_J da_I}} + a_I \, db_J) dx^I \wedge dx^J$$

can be brought here

$$= b_J \underbrace{da_I \wedge dx^I}_{d\omega} \wedge dx^J + a_I \, db_J \wedge dx^I \wedge dx^J$$

can be brought here by means of k switches yielding a $(-1)^k$ factor

$$= d\omega \wedge \varphi + (-1)^k \omega \wedge \underbrace{db_J \wedge dx^J}_{d\varphi}$$

$$= d\omega \wedge \varphi + (-1)^k \omega \wedge d\varphi$$

Also notice, as a consistency check

$$db_J \wedge dx^I = (-1)^{k \cdot 1} dx^I \wedge db_J = (-1)^k dx^I \wedge db_J$$

and $\underset{\Lambda^0}{f} \wedge \underset{\Lambda^k}{\omega} = (-1)^{0 \cdot k} \omega \wedge f = \omega \wedge f$

and actually $f \wedge \omega = \omega \wedge f = f\omega$

(c) ⚠ $\quad d(\underset{\Lambda}{d\omega}) = 0$

Most important

namely $\boxed{d^2 = 0}$

Proof. if $f \in \Lambda^0$, then

$$d(df) = d\left(\frac{\partial f}{\partial x^i} dx^i\right) = d\left(\frac{\partial f}{\partial x^i}\right) \wedge dx^i =$$

$$= \frac{\partial^2 f}{\partial x^j \partial x^i} dx^j \wedge dx^i. \quad \text{But, in view of the Schwarz lemma}$$

$$\frac{\partial^2 f}{\partial x^i \partial x^j} = \frac{\partial^2 f}{\partial x^j \partial x^i}, \quad \text{so, by virtue of skew-symmetry,}$$

$$d(df) = d^2 f = 0$$

Now take $\omega = \sum_I a_I dx^I \in \Lambda^k$

we use summation symbol

In order to check that $d^2\omega = 0$, it is enough to verify it for a monomial $a_I dx^I$ No summation

Then

$$d(\underbrace{d(a_I dx^I)}_{\text{monomial}}) = d(da_I \wedge dx^I) = \underbrace{d^2 a_I}_{\overset{\parallel}{0}} \wedge dx^I$$

$$- da_I \wedge \underbrace{d(dx^I)}_{\overset{\parallel}{0}} = 0 \quad \square$$

notice this

(d) functoriality: $d(f^*\omega) = f^* d\omega$

⚠ pull-back commutes with d

Pf. First check that $f^* dg = d(f^* g)$

$$f^* dg = f^*\left(\frac{\partial g}{\partial y^i} dy^i\right) = \frac{\partial g}{\partial y^i}\frac{\partial y^i}{\partial x^j} dx^j$$

Einstein

$f: y = y(x)$

$$\underset{\text{(Chain rule)}}{=} \frac{\partial (g\circ f)}{\partial x^j} dx^j = d(\underbrace{g\circ f}_{f^* g}) = d(f^* g)$$

$f: \; y = y(x)$

Now let $\varphi = a_I \, dy^I$

↖ Einstein again

Then

$$d(f^*\varphi) = d\left(f^*(a_I)\, f^*(dy^I)\right)$$

$$= df^*(a_I) \wedge f^*(dy^I) + \underbrace{f^*(a_I) \wedge d(f^* dy^I)}_{\text{but this is zero!}}$$

$$df^*(dy^{i_1} \wedge \dots dy^{i_k}) = d(f^* dy^{i_1} \wedge \dots f^* dy^{i_k}) = d(d f^* y^{i_1} \wedge \dots d f^* y^{i_k}) = 0 \quad \text{since } d^2 = 0, \; dy^{i_1}(x)$$

$= $ (by the previous property)

$$f^*(da_I) \wedge f^*(dy^I)$$

$$= f^*(\underbrace{da_I \wedge dy^I}_{d\varphi}) = f^*(d\varphi) \qquad \square$$

★ Important examples. In $\mathbb{R}^3$ $d^2 = 0$ encapsulates the properties

Caveat!

$$\boxed{\begin{aligned} \text{curl grad } \varphi &= 0 \\ \text{div curl } \underline{F} &= 0 \end{aligned}}$$

$$\varphi \longmapsto d\varphi = \frac{\partial \varphi}{\partial x} dx + \frac{\partial \varphi}{\partial y} dy + \frac{\partial \varphi}{\partial z} dz$$

z-component of curl $\underline{A}$

$$\underline{A} \leftrightarrow A_1 dx + A_2 dy + A_3 dz \xmapsto{d} \left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right) dx \wedge dy + \text{similar terms}$$

$$\underline{F} \leftrightarrow F_1 \, dy \wedge dz + F_2 \, dz \wedge dx + F_3 \, dx \wedge dy$$

$$\xmapsto{d} \frac{\partial F_1}{\partial x} dx \wedge dy \wedge dz + \dots \text{ similar terms}$$

$$= \dots = \underbrace{\left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right)}_{\text{div } \underline{F}} dx \wedge dy \wedge dz$$

$$\Lambda^0 \xrightarrow{d} \Lambda^1 \xrightarrow{d} \Lambda^2 \xrightarrow{d} \Lambda^3 \xrightarrow{d} \dots$$

★ de Rham complex

Z^k : closed k-forms: $d\omega = 0$

B^k : exact k-forms : $\omega = da$, $a \in \Lambda^{k-1}$

$B^k \leq Z^k$ in view of $d^2 = 0$

$H^k = Z^k / B^k$ k-de Rham cohomology group

The Poincaré lemma tells us that

$$H^k(\mathbb{R}^n) = \begin{cases} \mathbb{R} & k=0 \\ \{0\} & k=1,\dots n \end{cases}$$ ← easy to prove!

that is, any closed k-form, for $k \geq 1$ is exact.

★ ★ ★ de Rham cohomology groups store topological information about a manifold M

(see second part of the course and M.S. notes "Topologia e geometria differenziale" (Topogeo))