Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

Prof. Mauro Spera, UCSC, Brescia

Lecture XIV — TOPOLOGICAL AND DIFFERENTIABLE MANIFOLDS

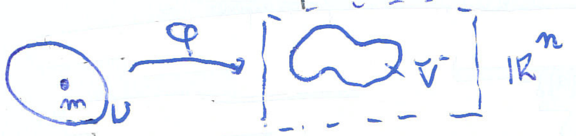
- Topological manifolds
- Differentiable manifolds
- Another definition
- Examples

✱ Topological manifolds

A topological space M is called **topological manifold of dimension n** (or, **topological n-manifold**) if

1. M is Hausdorff
2. M has a countable basis (also: second countable)
3. M is **locally euclidean**: $\forall\, m \in M$, $\exists\, U \ni m$ neighbourhood of m i.e. an open set containing m and a **homeomorphism** $\varphi : U \to V$ (V hom. to an open ball in $\mathbb{R}^n$, the latter being equipped with the standard topology)

with n independent of m



In words: every point in M admits a neighbourhood homeomorphic to an open ball in $\mathbb{R}^n$ (with n fixed)

$\varphi : U \to V$ is called **local chart**

(also, local patch, coordinate system)

Notes: ◆ Hausdorff: any two points admit disjoint neighbourhoods

◆ A basis in a topological space $(X, \mathcal{T})$ is a subset $\mathcal{B} \subset \mathcal{T}$ $(\subset \mathcal{P}(X))$ such that $\forall\, A \in \mathcal{T}$, $A = \bigcup_{\lambda \in \Lambda} B_\lambda$

Λ an index set

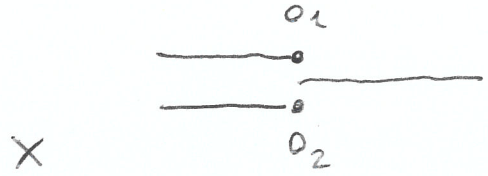
In $\mathbb{R}^n$, open balls with rational radii and rational centres (i.e. with rational coordinates) give rise to a countable basis thereof.

Observe that, if $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $B_1 \cap B_2 \in \mathcal{T}$ and there exists $B \in \mathcal{B}$ such that $B \subset B_1 \cap B_2$, since $B_1 \cap B_2 = \bigcup_{\lambda \in \Lambda} B_\lambda$ for suitable $B_\lambda \in \mathcal{B}$.

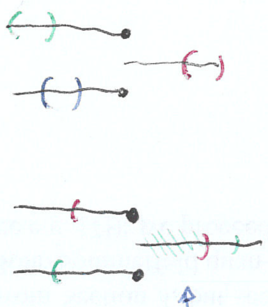
One can prove that, given on a set X a family $\mathcal{B} \subset \mathcal{P}(X)$ containing $\emptyset$ and X, and such that $\bigcup_{B \in \mathcal{B}} B = X$, and $\forall\, B_1, B_2 \in \mathcal{B}$ $\exists\, B \subset B_1 \cap B_2$, $B \in \mathcal{B}$, then

$\exists!$ topology $\mathcal{T}$ admitting $\mathcal{B}$ as a basis: the open sets in $\mathcal{T}$ are unions of sets in $\mathcal{B}$...

Notice that $3 \not\Rightarrow 1$

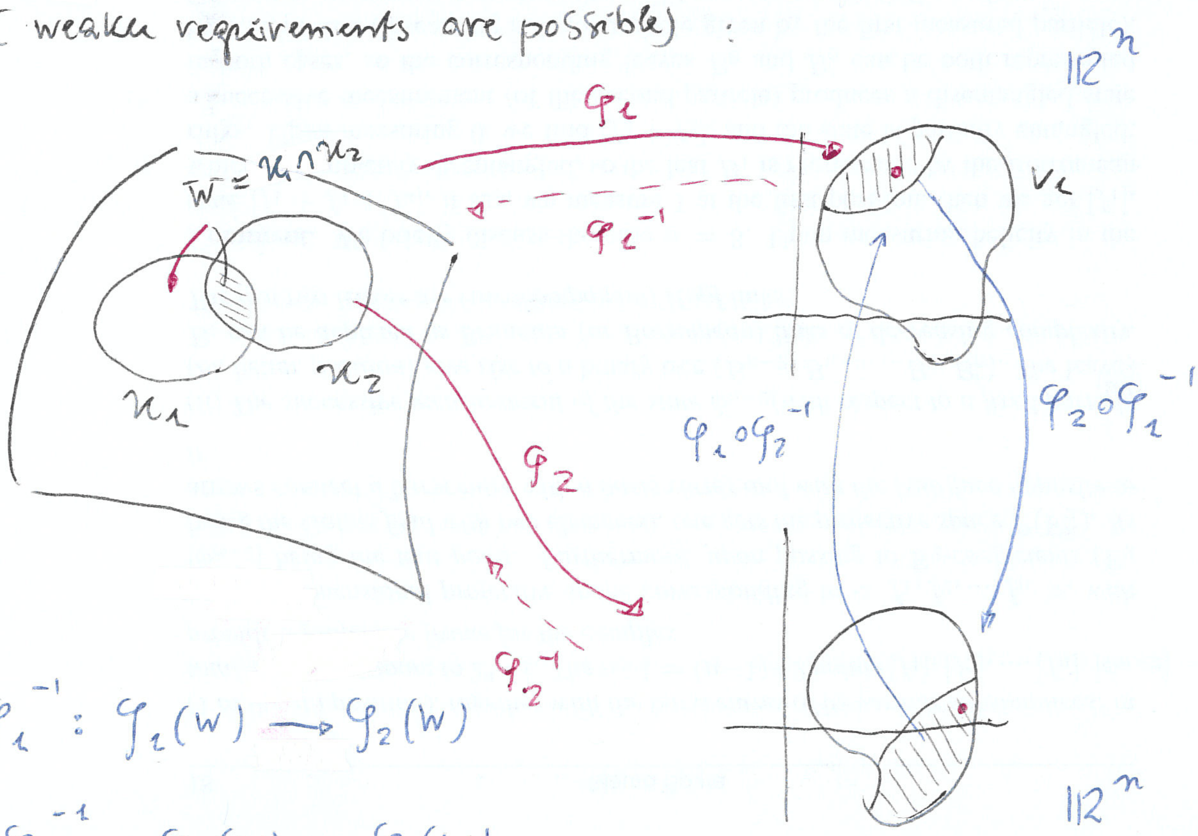


basis:



one obtains a topology that obviously makes it locally euclidean. X is not Hausdorff since o_1 and o_2 cannot be separated by disjoint neighbourhoods.

In order to get a differentiable manifold, we require the overlap maps (also: transition maps, coordinate change maps ...) to be smooth (weaker requirements are possible):



$\varphi_2 \circ \varphi_1^{-1} : \varphi_1(W) \to \varphi_2(W)$

$\varphi_1 \circ \varphi_2^{-1} : \varphi_2(W) \to \varphi_1(W)$

Therefore, a differentiable manifold (of dimension n) M is a topological space which is Hausdorff, has countable basis, equipped with an atlas

$$\mathcal{A} := \{ (\mathcal{U}_\alpha, \varphi_\alpha) \}_{\alpha \in \mathfrak{A}}$$

i.e. a collection of local charts fulfilling the following properties

differentiable structure

index set

(i) $\bigcup_{\alpha \in \mathfrak{A}} \mathcal{U}_\alpha = M$

(namely, $\{\mathcal{U}_\alpha\}_{\alpha \in \mathfrak{A}}$ is an open covering (or cover) of M,

(ii) $\varphi_\alpha : \mathcal{U}_\alpha \to V_\alpha$ is a homeomorphism

local chart

V_α ⊂ ball in $\mathbb{R}^n$

(iii) and, if $\mathcal{U}_\alpha \cap \mathcal{U}_\beta =: W_{\alpha\beta} \neq \emptyset$

the overlap maps

transition maps

$\varphi_\beta \circ \varphi_\alpha^{-1}$ are diffeomorphisms: smooth maps with smooth inverse

$$\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(W_{\alpha\beta}) \longrightarrow \varphi_\beta(W_{\alpha\beta})$$

open in $\mathbb{R}^n$ open in $\mathbb{R}^n$

$$\varphi_\alpha \circ \varphi_\beta^{-1} : \varphi_\beta(W_{\alpha\beta}) \longrightarrow \varphi_\beta(W_{\alpha\beta})$$

are smooth

they are maps between open sets in $\mathbb{R}^n$, so the concept of smoothness is meaningful for them...

One could be more sophisticated.

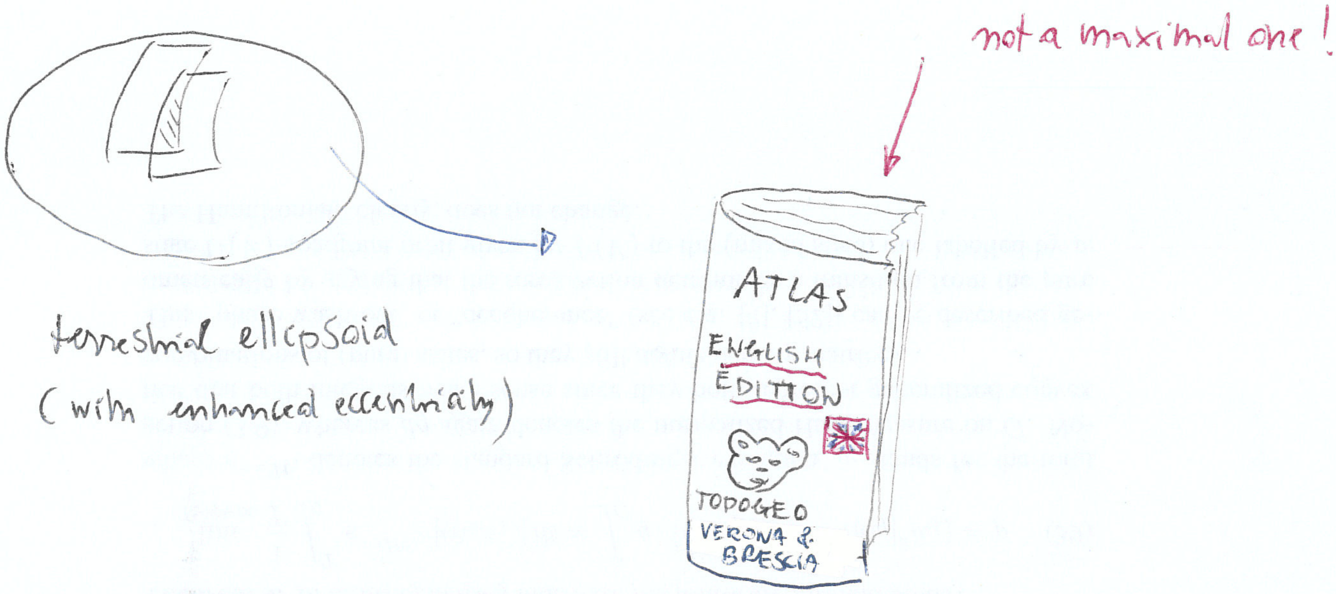
Two atlases are said to be compatible (or equivalent) if their union is still an atlas.

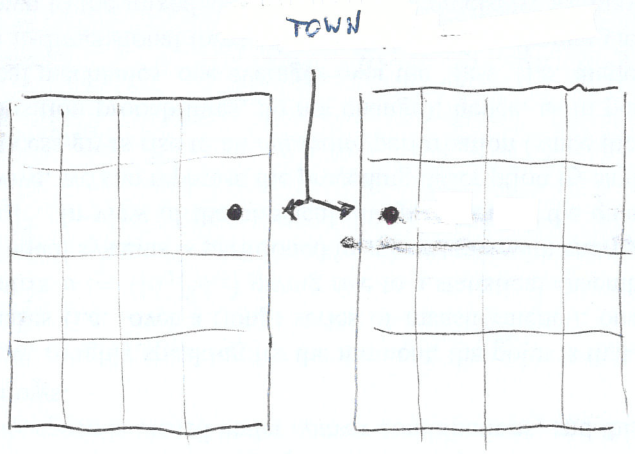
A maximal atlas is the union of all atlases compatible with a fixed atlas (existence follows from Zorn's lemma). In theoretical investigations it turns out to be convenient to work with a maximal atlas: it gives us a sort of universal receptacle of charts wherefrom we can take those suiting our needs.

More formally, a differentiable manifold of dimension n (n-dimensional differentiable manifold) is a pair $(M, [\mathcal{A}])$, with M a topological n-manifold and $[\mathcal{A}]$ the equivalence class determined by a maximal atlas: this is also called a differentiable structure.

Note. One can speak of C^k-manifolds or C^ω-manifolds (transition charts being real-analytic)). Upon replacing $\mathbb{R}^n$ with $\mathbb{C}^n$, and requiring (bi-)holomorphy (holomorphicity) (complex analiticity) one abuts at the notion of complex manifold of dimension n. If $n=1$, one obtains a Riemann surface (historically, the latter concept is due to H. Weyl (1913))

* Basic motivation : cartography

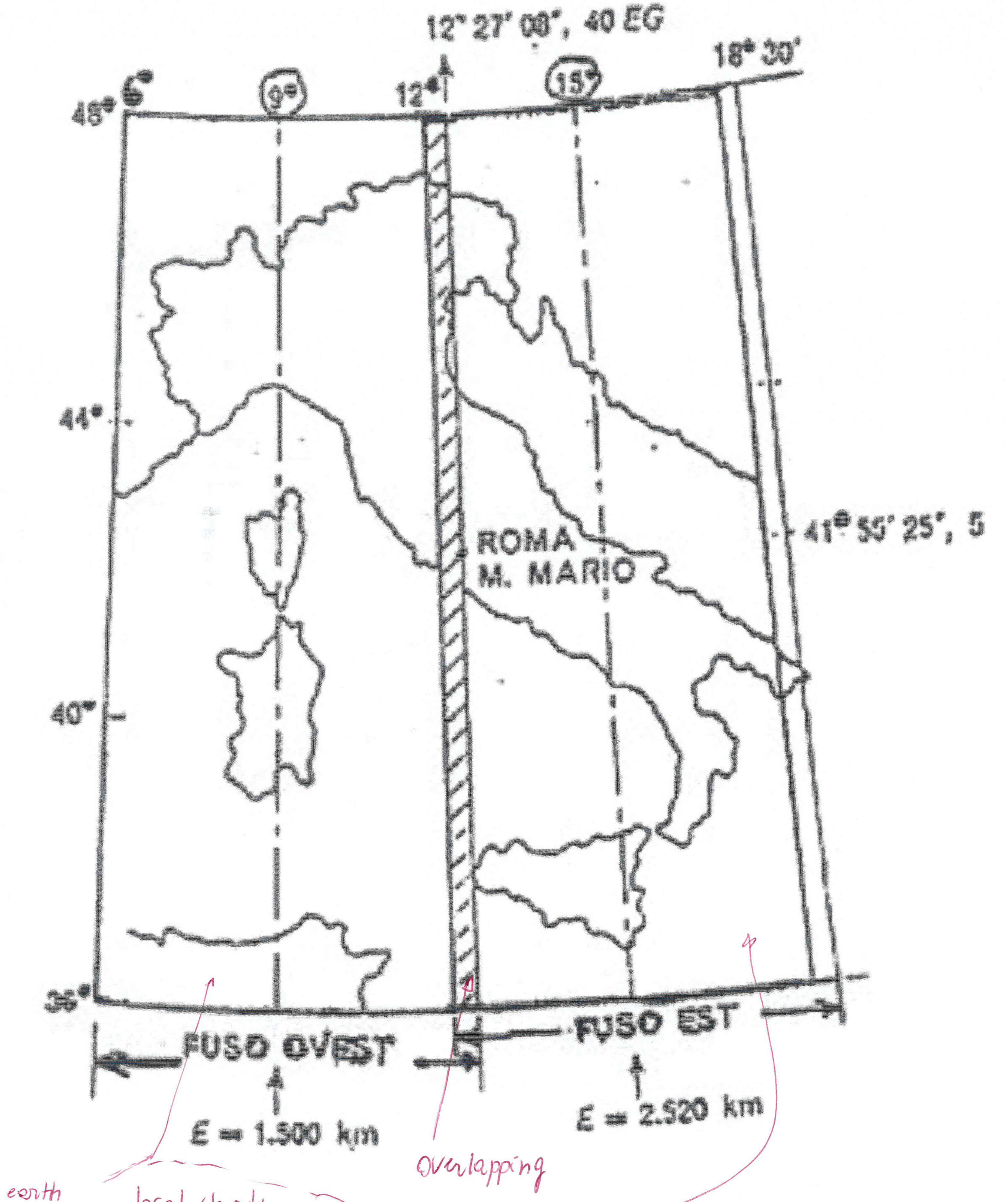




A transition map is involved, invisible to the ... final user

★ Gauss-Boaga projection

italian version of the UTM projection



★ Another (equivalent) definition of smooth manifold without starting from a topological space.

Let M be a set, such that $\exists\ f_\alpha : \mathcal{U}_\alpha \to M$, $\alpha \in \mathcal{O}$, $\mathcal{U}_\alpha$ open $\subset \mathbb{R}^n$

no topology on it, a priori

f_α injective

[observe that charts go in the opposite direction, but this is not important]

such that

(1). $\displaystyle\bigcup_{\alpha \in \mathcal{O}} f_\alpha(\mathcal{U}_\alpha) = M$

(2). $\forall\ \alpha, \beta \in \mathcal{O}$ such that $f_\alpha(U_\alpha) \cap f_\beta(U_\beta) \equiv W_{\alpha\beta} \neq \phi$,

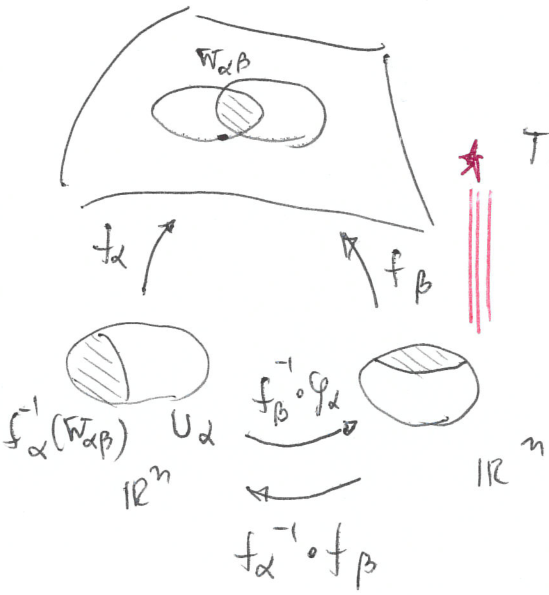
$f_\alpha^{-1}(W_{\alpha\beta})$ and $f_\beta^{-1}(W_{\alpha\beta})$ are open in $\mathbb{R}^n$ and such that

$f_\alpha^{-1} \circ f_\beta$ and $f_\beta^{-1} \circ f_\alpha$ are smooth

well defined in view of injectivity

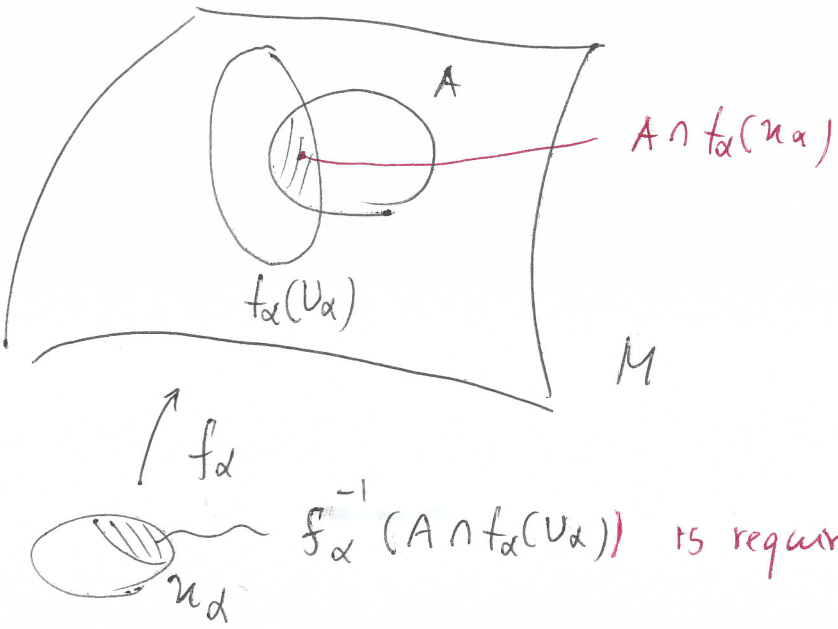
(3). The above family is maximal with respect to the properties 1 and 2

$\mathcal{A} = \{(\mathcal{U}_\alpha, f_\alpha)\}_{\alpha \in \mathcal{O}}$ atlas (diff. structure)



★ This gives us a natural topology $\mathcal{T}$ on M:

$A \subset M$ is open if $f_\alpha^{-1}(A \cap f_\alpha(U_\alpha))$ is open in $\mathbb{R}^n$



$f_\alpha^{-1}(A \cap f_\alpha(U_\alpha))$ is required to be open in $\mathbb{R}^n$

★ One checks that $\mathcal{T}$ fulfils the axioms of a topology.
($\mathcal{T}$ contains ϕ, M and is closed under arbitrary unions and finite intersections)

The extra requirements : Hausdorff + countable basis are then postulated.

uniqueness of limits

existence of partitions of unity, see below

This approach is useful in applications, in cases there is no a priori topology to be imposed on set.