Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY

Prof. Mauro Spera, UCSC, Brescia

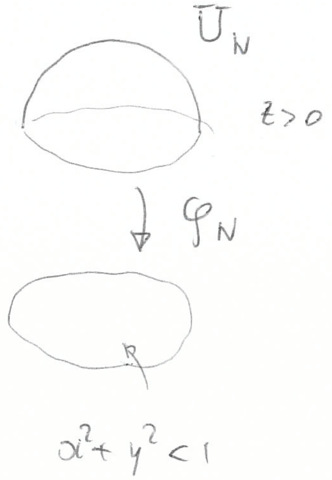
Lecture XV

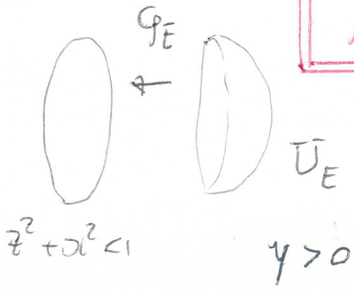
EXAMPLES OF DIFFERENTIABLE MANIFOLDS

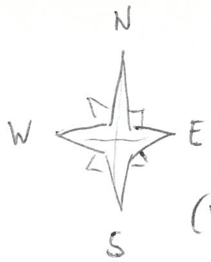
★ Examples

1. The sphere $S^2 = \{(x,y,z) \in \mathbb{R}^3 \,/\, x^2+y^2+z^2 = 1\}$

equipped with the relative topology (inherited from the standard one in $\mathbb{R}^3$).



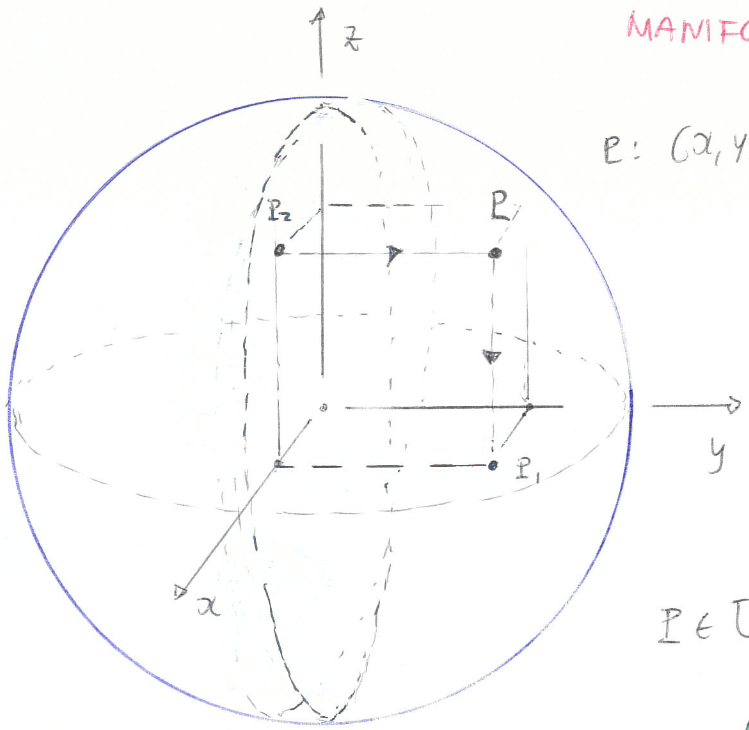




(rosa dei venti)

compass-card

↑ bussola



$P: (x,y,z)$

$P \in U_N \cap U_E$

$$\bullet \xrightarrow{\varphi_E^{-1}} \bullet \xrightarrow{\varphi_N} \bullet$$

compasso: compasses

a punte fisse: dividers

$$(z,x) \stackrel{\varphi_E^{-1}}{\longmapsto} (x,y,z)$$

$$(x,y,z) \stackrel{\varphi_N}{\longmapsto} (x,y)$$

$$y = \sqrt{1-x^2-z^2}$$

$$(z,x) \stackrel{\varphi_N \circ \varphi_E^{-1}}{\longmapsto} (x, \sqrt{1-x^2-z^2})$$

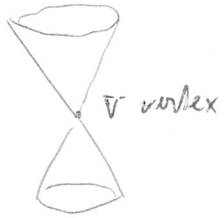
$(z,x) \in \varphi_E(U_E) = \{z^2+x^2<1\}$

$(x, \sqrt{1-x^2-z^2}) \in \varphi_N(U_N) = \{x^2+y^2<1\}$

★ $\boxed{\varphi_N \circ \varphi_E^{-1} \text{ is smooth, with smooth inverse}}$

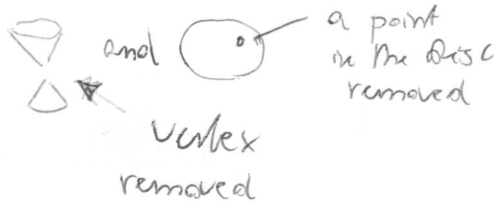
2. A "non-example" : $x^2+y^2-z^2=0$

(cone in $\mathbb{R}^3$, endowed with relative topology)

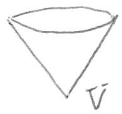


This is not a smooth manifold, and not even a topological manifold: $\bar{v}$ does not possess a neighbourhood homeomorphic to an open disc!

Why? Were it, then



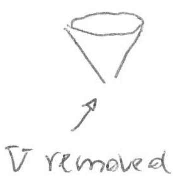
would be homeomorphic, but this is false (the latter space is connected, the former is not)

2'. 

is a topological manifold (C^0)

$x^2+y^2-z^2=0$

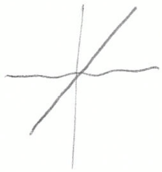
$z \geq 0$

2''. 

is a smooth manifold (and a smooth submanifold of $\mathbb{R}^3$ as well, as we have already seen)

3. A remark on the concept of differentiable structure

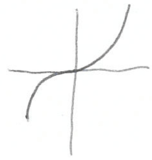
$M_1 = (\mathbb{R}, t)$ $\qquad \varphi_1(t) = t$



atlas

consisting of a single chart

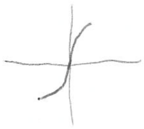
$M_2 = (\mathbb{R}, t^3)$ $\qquad \varphi_2(t) = t^3$



* The two atlases are not compatible (upon requiring $k > 0$) degree of differentiability

$\varphi_1^{-1}\varphi_2 : \quad t \xrightarrow{\varphi_1^{-1}} t \xrightarrow{\varphi_2} t^3$ $\qquad$ is smooth

$\varphi_2^{-1}\varphi_1 : \quad t \longmapsto t^{\frac{1}{3}} \xrightarrow{\varphi_1} t^{\frac{1}{3}}$ $\qquad$ is not smooth (nor C^k $\;k \geq 1$)



Therefore one has $\mathbb{R}$ equipped with different differentiable structures (they are however equivalent in a suitable sense. * The situation is really complicated in general:

Jungle of topological manifolds:

* For dim $M \leq 3$ $\exists!$ differentiable structure (Munkres, Moise)

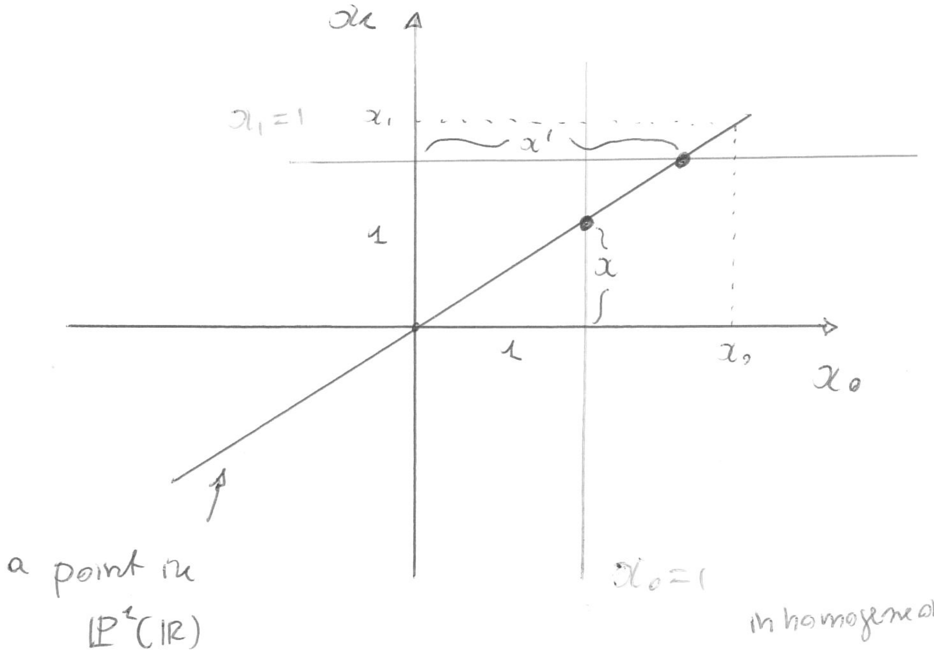
* In dimension > 3, $\exists$ M which do not admit any differentiable structure.

* Exotic spheres (Milnor, Kervaire): on S^7 there exist 28 inequivalent differentiable structures

* Fake $\mathbb{R}^4$: on $\mathbb{R}^4$, there exists an uncountable set of inequivalent differentiable structures (Freedman)

We refrain from further delving into these fascinating topics.

4. The real projective line $\mathbb{P}^1(\mathbb{R})$: lines in $\mathbb{R}^2$ passing through the origin



inhomogeneous (affine) coordinate

$$\mathcal{U}_0 \ni [\underset{\neq 0}{x_0}, x_1] \xmapsto{\varphi_0} \left(1, \overset{x}{\overset{\parallel}{\frac{x_1}{x_0}}}\right) \equiv x$$

removed

$$\mathcal{U}_1 \ni [x_0, \underset{\neq 0}{x_1}] \xmapsto{\varphi_1} \left(\underbrace{\frac{x_0}{x_1}}_{x'}, 1\right) \equiv x'$$

in $\mathcal{U}_0 \cap \mathcal{U}_1$, $x_0 \neq 0$, $x_1 \neq 1$

$$x \xmapsto{\varphi_1 \circ \varphi_0^{-1}} x' = \frac{1}{x}$$

$x' = \frac{1}{x}$

smooth with smooth inverse

5. Projective spaces (real & complex)

$$\mathbb{P}^n(\mathbb{R}) \equiv \mathbb{P}(\mathbb{R}^{n+1})$$
$$\quad\ \ \mathbb{C} \qquad\qquad \mathbb{C}^{n+1}$$

$$\varphi_i([x_0 \dots x_n]) = \left(\frac{x_0}{x_i}, \dots \underbrace{\widehat{\frac{x_i}{x_i}}}_{=1}, \dots \frac{x_n}{x_i}\right) \in \begin{matrix}\mathbb{R}^n \\ \mathbb{C}^n\end{matrix}$$

homogeneous coordinates

omitted

defined on

$$\mathcal{U}_i = \{[x], x_i \neq 0\}$$

Let us calculate the transition maps, for $\mathcal{U}_i \cap \mathcal{U}_j \neq \emptyset$

(i.e. $x_i \neq 0$, $x_j \neq 0$)

jth position ith-position

$$\boxed{\varphi_j \circ \varphi_i^{-1}(x_1 \dots x_n) = \left(\frac{x_1}{x_j} \dots \underbrace{\widehat{\frac{x_j}{x_j}}}_{=1,\ \text{omitted}}, \frac{1}{x_j}, \dots \frac{x_n}{x_j}\right)}$$

in detail:

insert at ith-pos.

$$(x_1 \dots x_n) \xmapsto{\varphi_i^{-1}} [x_1 \dots 1 \dots x_n] \in \mathbb{P}^n$$

$$\left(= \left[\frac{X_1}{X_i}, \dots \frac{X_i}{X_i} \dots \frac{X_n}{X_i}\right]\right)$$

X_i : coordinates in $\mathbb{R}^{n+1}$ $(\mathbb{C}^{n+1})$

ith-pos.

$$\xmapsto{\varphi_j} \left(\frac{x_1}{x_j} \dots \underbrace{\widehat{\frac{x_j}{x_j}}}_{=1}, \dots \frac{1}{x_j}, \dots \frac{x_n}{x_j}\right)$$

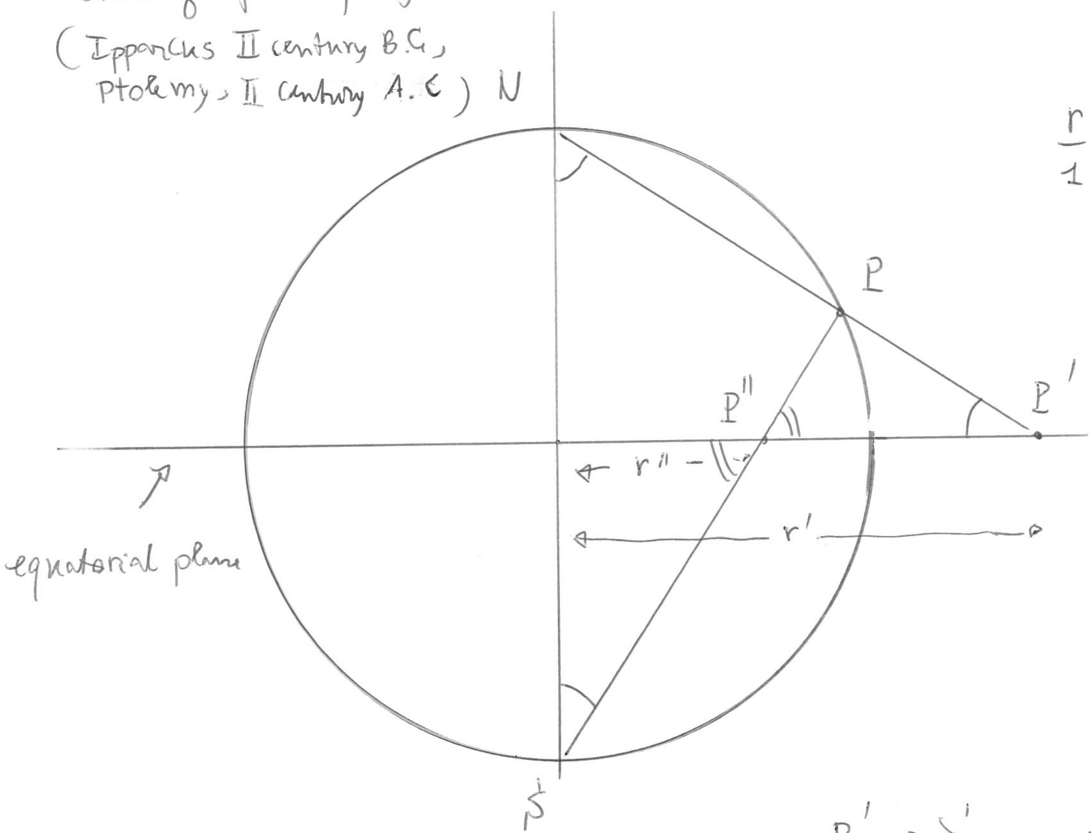
(at jth position)

<u>remove it</u>

These maps are smooth (and holomorphic in the complex case)

5'. The Riemann sphere (the complex projective line as a Riemann surface)

stereographic projections
(Ipparchus II century B.C., Ptolemy, II century A.C.)



$$\frac{r'}{1} = \frac{1}{r''}$$

$$r' = \frac{1}{r''}$$

$P' \leftrightarrow \zeta'$ $\quad r' = |\zeta'|$

$P'' \leftrightarrow \zeta''$ $\quad r'' = |\zeta''|$

$U_N = S^2 \setminus \{N\}$ 

$U_S = S^2 \setminus \{S\}$ 

$S^2 = U_N \cup U_S$ $\qquad \varphi_N : P \mapsto P'$

$\qquad\qquad\qquad\quad \varphi_S : P \mapsto P''$

If $P \notin \{N, S\}$ they are both defined

$$\varphi_N \circ \varphi_S^{-1} : \begin{array}{ccccc} P'' & \xmapsto{\varphi_S^{-1}} & P & \xmapsto{\varphi_N} & P' \\ \cap\!\!\mid & & & & \cap\!\!\mid \\ \mathbb{C} & & & & \mathbb{C} \end{array}$$

equatorial plane $\quad \zeta'' \longmapsto \zeta'$ $\qquad \zeta' = \frac{1}{\zeta''}$

for suitable orientations (see further on)

real form

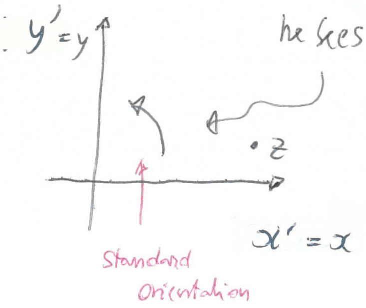
$$x' + iy' = \frac{1}{x'' + iy''} =$$

$$= \frac{x'' - iy''}{x''^2 + y''^2} \Rightarrow$$

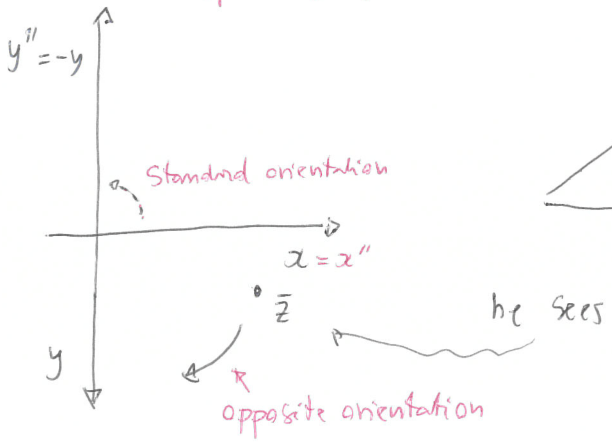
$$\begin{cases} x' = \dfrac{x''}{x''^2 + y''^2} \\ y' = \dfrac{-y''}{x''^2 + y''^2} \end{cases}$$

Remark on orientation

take two copies of the equatorial plane $\cong \mathbb{C}$



* conjugation is involved

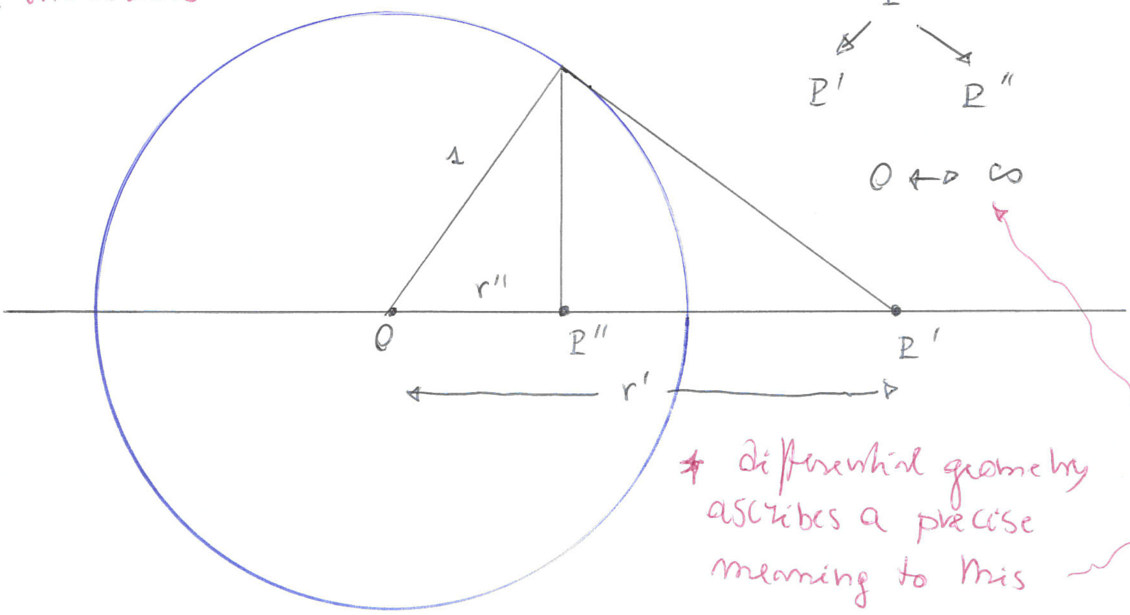


notice this

$$\underbrace{x''+iy''}_{\xi''} = \frac{1}{\underbrace{x'+iy'}_{\xi'}} = \frac{x'-iy'}{x'^2+y'^2}$$

Further remark: P' and P'' are related by a circular inversion

$r' \cdot r'' = 1$



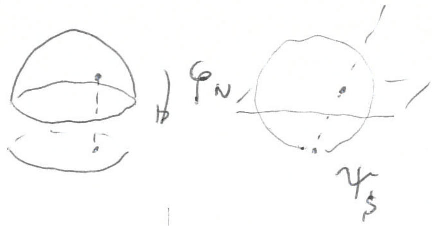
$P \to P'$, $P \to P''$

$0 \leftrightarrow \infty$

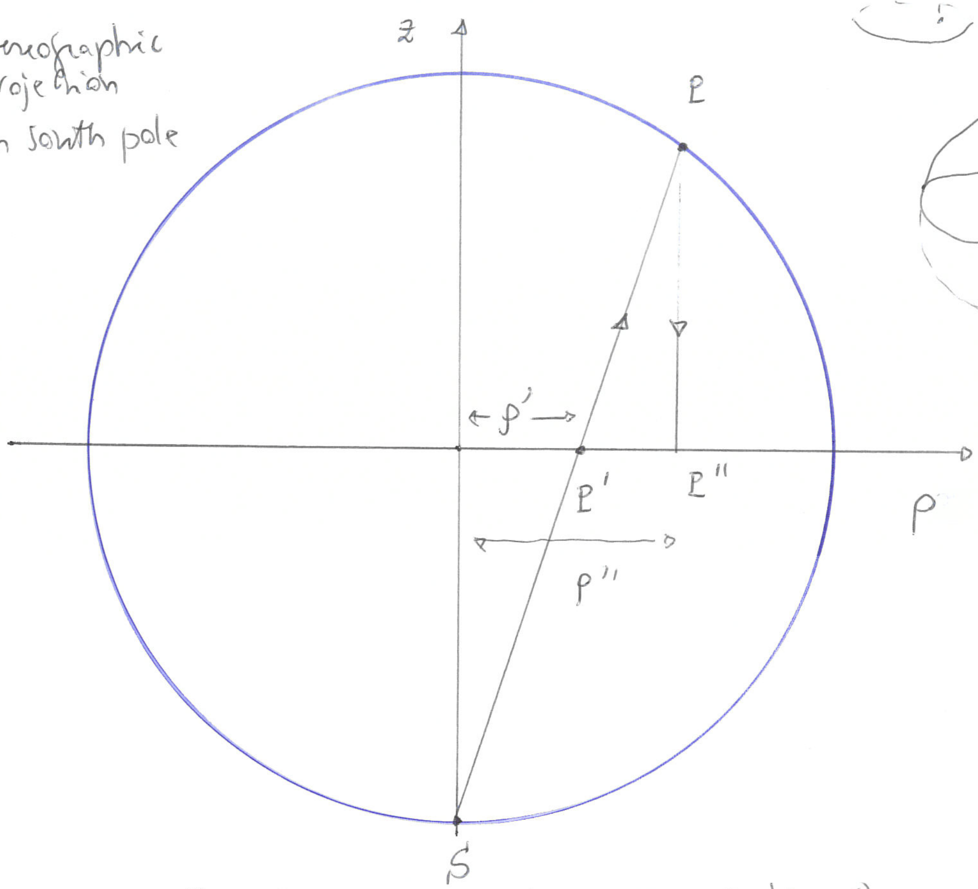
* differential geometry ascribes a precise meaning to this

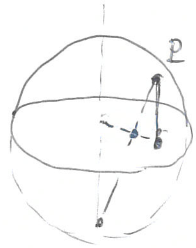
6. Notice that we exhibited two atlases for the sphere. Let us check that they are compatible; it is then enough to show that the two charts below are compatible

$\varphi_N : U_N \to \mathbb{R}^2$ considered previously



ψ_S : stereographic projection from south pole





This is geometrically clear. Find the relationship between ρ' and ρ'' ....

They define the same differentiable structure