Lectures on

# DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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## Lecture XVI

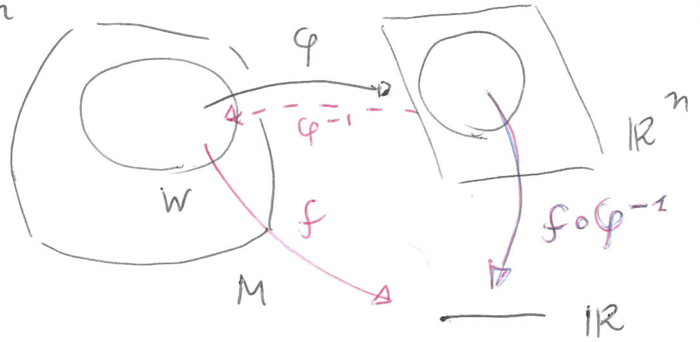
### FUNCTIONS ON MANIFOLDS — TANGENT SPACE

- Functions on manifolds
- Tangent vectors and derivations (on $\mathbb{R}^n$)
- Tangent vectors and derivations (on a manifold)
- Tangent vectors as velocities

### 1. Smooth functions on manifolds

Let $M$ be a differentiable manifold (smoothness assumed throughout) [We omit the explicit specification of an atlas]

**Def.** A function $f : M \to \mathbb{R}$ is said to be **smooth** if $\forall$ chart $\varphi : U \to \mathbb{R}^n$, the function $f \circ \varphi^{-1} : W \to \mathbb{R}$, $W = \varphi(U)$, is smooth




Notice that this concept is intrinsic, i.e. independent of the choice of a chart: in a non-empty intersection of the domains of two charts $\varphi$ and $\psi$, one has

$$f \circ \varphi^{-1} = f \circ \psi^{-1} \circ \psi \circ \varphi^{-1} = (f \circ \psi^{-1}) \circ (\psi \circ \varphi^{-1})$$

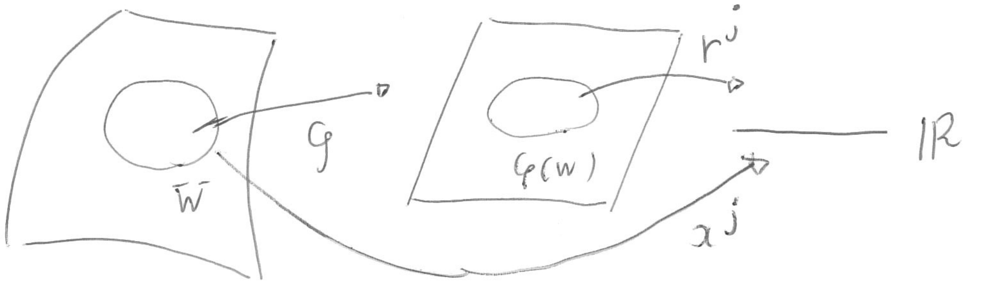
which is smooth if and only if $f \circ \psi^{-1}$ is smooth (being $\psi \circ \varphi^{-1}$ smooth)

Let us now define local coordinates (or local coordinate functions). Let $r^j : \mathbb{R}^n \to \mathbb{R}$ the standard coordinate functions on $\mathbb{R}^n$: $r^j(a) = a^j \quad (a = (a^i))$

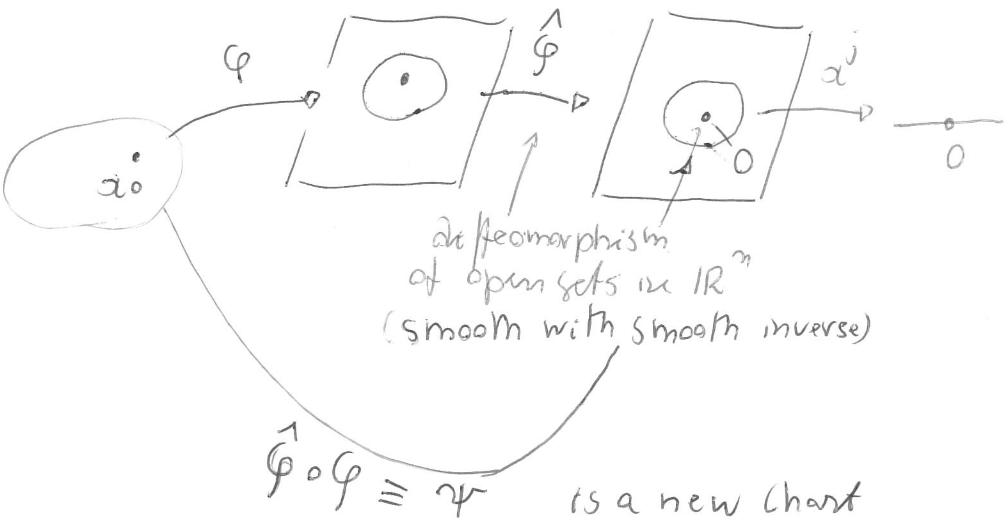
Then, given a chart $\varphi : W \longrightarrow \mathbb{R}$, $W \subset M$

set $x^j : W \longrightarrow \mathbb{R}$

$$x^j := r_j \circ \varphi$$ ← local coordinate functions




Given $x_0 \in M$, one can devise a local coordinate system centred at $x_0$ (i.e. $x^j(x_0) = 0 \quad \forall j = 1..n$)




This can be achieved in view of maximality of the atlas
(one can add to a fixed atlas any chart compatible with all others).

## ✶ Tangent vectors as derivations. The $\mathbb{R}^n$ case

**Def.** A derivation on an algebra $A$ is a map $D : A \to A$ such that

1. $D$ is linear (A is in particular a vector space)
2. $D(a \cdot b) = D(a)\cdot b + a\, D(b)$ (Leibniz rule)

**Def.** A derivation of the algebra $\mathcal{C}^\infty(\mathbb{R}^n)$ at $0 \in \mathbb{R}^n$ is a map $v : \mathcal{C}^\infty(\mathbb{R}^n) \to \mathbb{R}$ such that

1. $v$ is linear
2. $v(f \cdot g) = v(f)\, g(0) + f(0)\, v(g)$ (Leibniz)

✶ **Theorem** $$v = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}\Big|_0$$ for a unique vector $(a^i) \in \mathbb{R}^n$

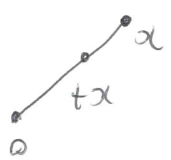
Namely, $T_0 \mathbb{R}^n$ can be identified with the space of derivations of $C^\infty(\mathbb{R}^n)$ at $0$ (true in general...)

**Proof.** One needs the following

**Lemma** (Willmore). Let $f \in C^\infty(\mathbb{R}^n)$. Then

$$f(x) - f(0) = \sum_{j=1}^{n} x^j h_j(x), \quad h_j \in C^\infty(\mathbb{R}^n).$$

Indeed: $$f(x) - f(0) = \underbrace{\int_0^1 \frac{d}{dt} f(tx)\, dt = \sum_{j=1}^{n} \int_0^1 x^j \frac{\partial f}{\partial x^j}(tx)\, dt}_{\text{(chain rule)}}$$



$$= \sum_{j=1}^{n} x^j \underbrace{\int_0^1 \frac{\partial f}{\partial x^j}(tx)\, dt}_{h_j(x)}$$

which proves the lemma

Notice that $h_j(x) = \frac{\partial f}{\partial x^j}(\xi)\cdot 1$ (Lagrange)

$$\Rightarrow h_j(0) = \frac{\partial f}{\partial x^j}(0)$$

Now, given a derivation at $0$, say $v$, we find,

subsequently

Willmore

$$v(f) = v\left(f(0) + \sum_j x^j h_j(x)\right)$$

(linearity)

$$= \underbrace{v(f(0))}_{=0} + \sum_j v(x^j h_j(x))$$

viewed as a constant function

$v(1) = v(1 \cdot 1) =$
$= 1 \cdot v(1) + v(1) \cdot 1$
$= 2\, v(1)$
$\Rightarrow v(1) = 0$
Also: $v(c) = 0$

$$= \sum_j \underbrace{v(x^j)}_{\equiv a^j} \underbrace{h_j(0)}_{=\frac{\partial f}{\partial x^j}(0)} + \underbrace{\sum_j 0 \cdot v(h_j)}_{=0}$$

$$= \sum_{j=1}^{n} a^j \frac{\partial f}{\partial x^j}(0)$$

Conversely, any tangent vector at $0$ fulfils the two properties of a derivation at $0$.

This completes the proof of the theorem. $\square$

## 4. The tangent space $T_{x_0}M$

⚠ A bit abstract

**Def.** $T_{x_0}M$ (tangent space to $M$ at $x$)

consists of maps

$$v : \mathcal{C}^\infty(M, x_0, \mathbb{R}) \longrightarrow \mathbb{R}$$

smooth functions on a neighbourhood $\mathcal{U}$ of $x_0$, domain of a chart $\varphi$ ····▸ we have already introduced the notion of smooth function on a manifold (or on an open subset thereof)

such that
$$v(f) = \sum_{i=1}^{n} a^i \frac{\partial}{\partial r^i} (f \circ \varphi^{-1})\Big|_{\varphi(x_0)}$$

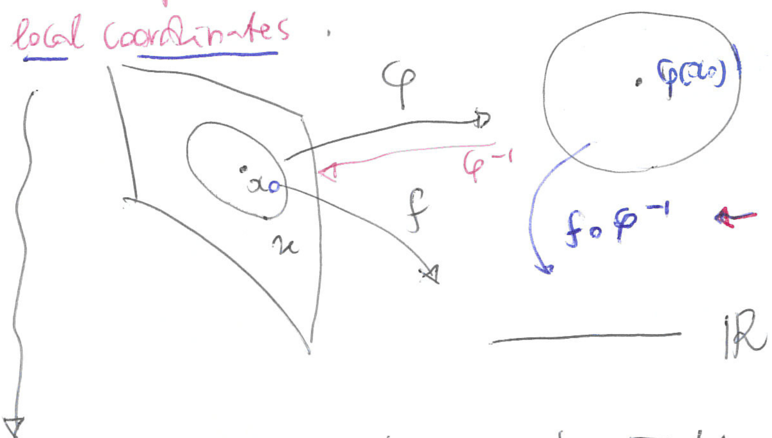
for some $(a^i) \in \mathbb{R}^n$. $v$ is called a tangent vector at $x_0$

Namely: tangent vectors are derivations (at $x_0$) of the above algebra of functions.

One uses the suggestive notation $v = \sum a^i \frac{\partial}{\partial x^i}\Big|_{x_0}$

Then define:
$$\frac{\partial f}{\partial x^j}(x_0) := \frac{\partial (f \circ \varphi^{-1})}{\partial r^j} (\varphi(x_0))$$

4 partial derivatives with respect to local coordinates.




← a smooth function on an open set of $\mathbb{R}^n$

They give rise to a basis of $T_{x_0}M$, $\left(\frac{\partial}{\partial x^j}\Big|_{x_0}\right)_{j=1\ldots n}$

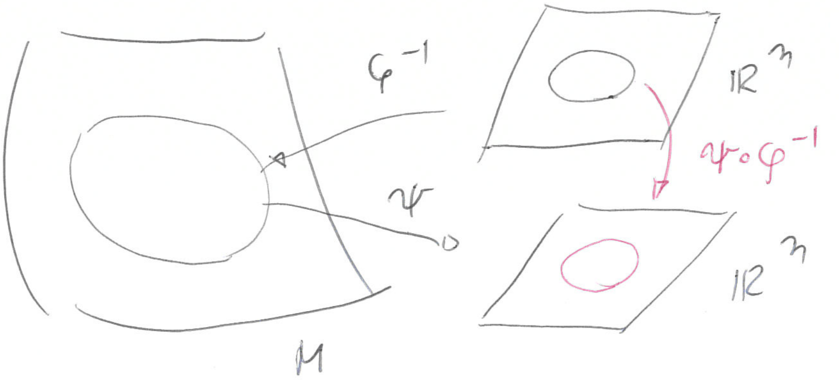
or $\left(\partial_j\big|_{x_0}\right)_{j=1\ldots n}$

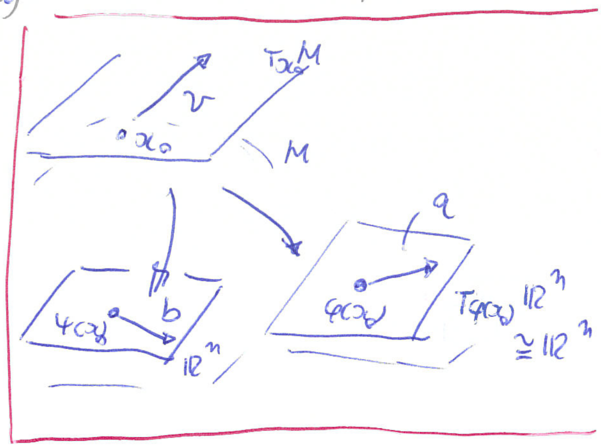
It is necessary, and instructive, to check chart-independence

$$v(f) = \sum_i a^i \frac{\partial}{\partial r^i}(f\circ\varphi^{-1})\Big|_{\varphi(x_0)} =$$

$$= \sum_i a^i \frac{\partial}{\partial r^i}\Big(f\circ\psi^{-1}\circ \overset{\text{another chart}}{\overset{\downarrow}{\psi}}\circ\varphi^{-1}\Big)\Big|_{\varphi(x_0)}$$

$$= \sum_i a^i \sum_j \frac{\partial}{\partial r^j}(f\circ\psi^{-1})\Big|_{\psi(x_0)} \cdot \overset{(J^T)}{\overset{\shortparallel}{J_{ji}}}(\psi\circ\varphi^{-1})\Big|_{\varphi(x_0)}$$




$$= \sum_j \frac{\partial}{\partial r^j}(f\circ\psi^{-1})\Big|_{\psi(x_0)} \underbrace{\Big(\sum_i a^i J_{ji}(\psi\circ\varphi^{-1})\Big)}_{b^j}$$

$$= \sum_j b^j \frac{\partial}{\partial r^j}(f\circ\psi^{-1})\Big|_{\psi(x_0)}$$

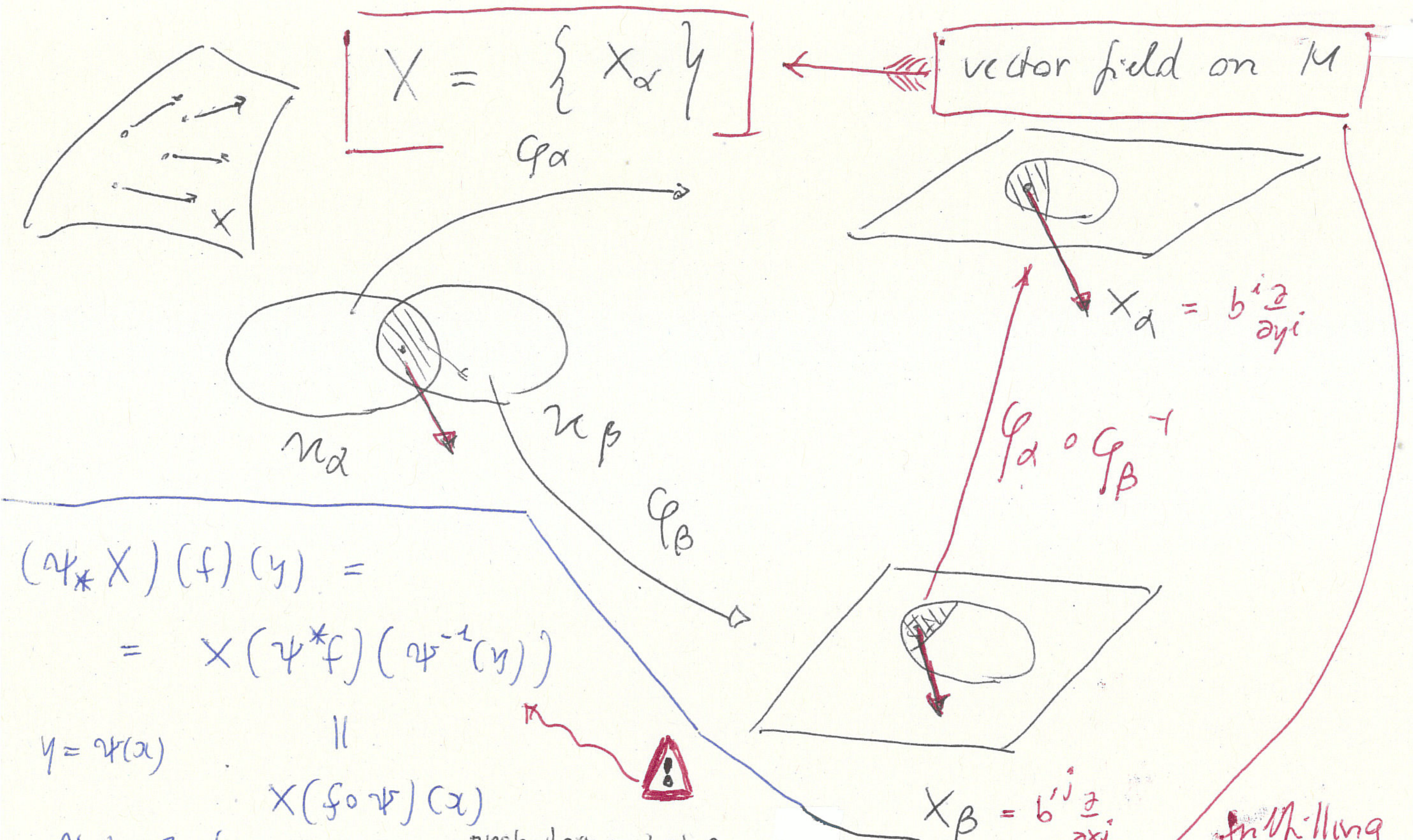
We shall use the stenographic notation already employed for $\mathbb{R}^n$, omitting explicit indication of charts $v(f) = \sum_i a^i \frac{\partial}{\partial x^i}$ (or simply $a^i\frac{\partial}{\partial x^i}$ (Einstein)). We still have for $y = y(x)$

$$\frac{\partial}{\partial x^j} = \frac{\partial y^i}{\partial x^j}\frac{\partial}{\partial y^i} \qquad \frac{\partial}{\partial x} = J^t\frac{\partial}{\partial y} \qquad \frac{\partial}{\partial y} = (J^{-t})\frac{\partial}{\partial x}$$

coordinate change

contravariance

* that is, we still employ the $\mathbb{R}^n$-notation, with a generalized meaning




$$(\psi_* X)(f)(y) =$$

$$= X(\psi^* f)(\psi^{-1}(y))$$

$$\parallel$$

$y = \psi(x)$

$$X(f \circ \psi)(x)$$

Abstract formulation: push-forward of a vector field under a diffeomorphism

to be discussed later on in full generality

$$\boxed{X_\alpha = (\varphi_\alpha \circ \varphi_\beta^{-1})_* X_\beta}$$

(compatibility condition) local chart formulation

⋆ Concretely: $\varphi_\alpha \circ \varphi_\beta^{-1} \rightsquigarrow y = y(x)$

$$\boxed{X_\alpha = b^i \frac{\partial}{\partial y^i}}$$

$$X_\alpha(f) = b^i \frac{\partial f}{\partial y^i} = b^i \frac{\partial f}{\partial x^j} \frac{\partial x^j}{\partial y^i}$$

$$= \underbrace{b^i \frac{\partial x^j}{\partial y^i}}_{b'^j} \frac{\partial f}{\partial x^j}$$

$$b'^j = \frac{\partial x^j}{\partial y^i} b^i$$

$$\frac{\partial y^k}{\partial x^j} b'^j = \frac{\partial y^k}{\partial x^j} \frac{\partial x^j}{\partial y^i} b^i$$

$$= \delta^k_i b^i = b^k$$

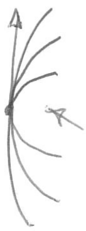
vector fields on M: résumé

$$\boxed{b^k = \frac{\partial y^k}{\partial x^j} b'^j} \qquad \boxed{X_\beta = b'^j \frac{\partial}{\partial x^j}}$$

Jacobian matrix (as expected)

## ★ Another approach to the tangent space (Hint)

$\boxed{\text{On } \mathbb{R}^n}$, two curves passing through a point $x$ said to be tangent at $x$ if they have the same velocity vector. This defines an equivalence relation. This notion is clearly diffeomorphism-invariant



same velocity

same velocity

same velocity

$y = y(x)$ Jacobian matrix

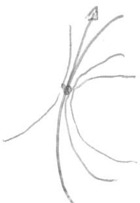
Schematically:

$$y(t) = y(x(t)) \qquad \dot{y}(t) = J \cdot \dot{x}(t)$$

So the tangent space at $p \in \mathbb{R}^n$ is the space of velocity vectors of curves passing through it (an equivalence class of curves)

$\boxed{\text{On a manifold } M}$ two curves are said to be tangent at a point $x$ if they are such for a chart (whose domain contains $x$), i.e. if their images through the chart are tangent in $\mathbb{R}^n$. This notion is indeed chart-independent (by the observed diffeomorphism invariance).

A tangent vector at $x$ is then an equivalence class of tangent curves at $x$ (all curves containing whose images in a (and hence in all) chart have the same velocity at the image point



We have already appreciated the importance of this point of view.