Lectures on:

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

Prof. Mauro Spera, UCSC, Brescia

Lecture XVII

MAPS BETWEEN MANIFOLDS

- Smooth maps between manifolds
- differential
- example
- Tangent & cotangent bundles
- Tensor bundles
- Riemannian metrics
- Behaviour under smooth maps.

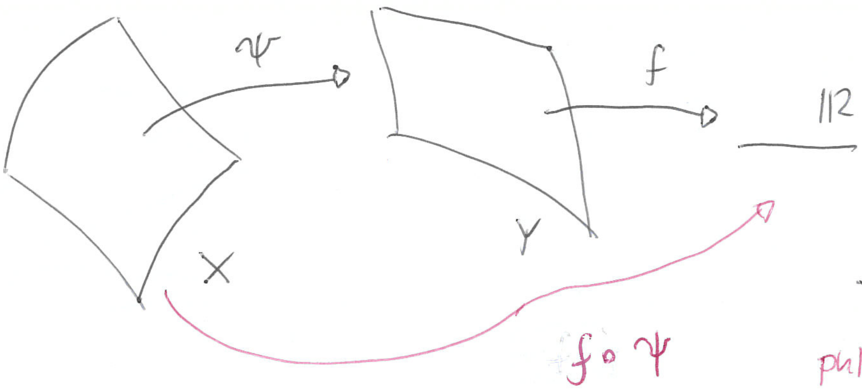
★ Smooth maps between manifolds

Let X, Y be differentiable manifolds of dimension n and m respectively

★ A map $\psi : X \longrightarrow Y$ is said to be **smooth**

if, whenever $f \in C^\infty(Y, \mathbb{R})$ (smooth function on Y), then $f \circ \psi \in C^\infty(X, \mathbb{R})$

" smooth is who smooth does "



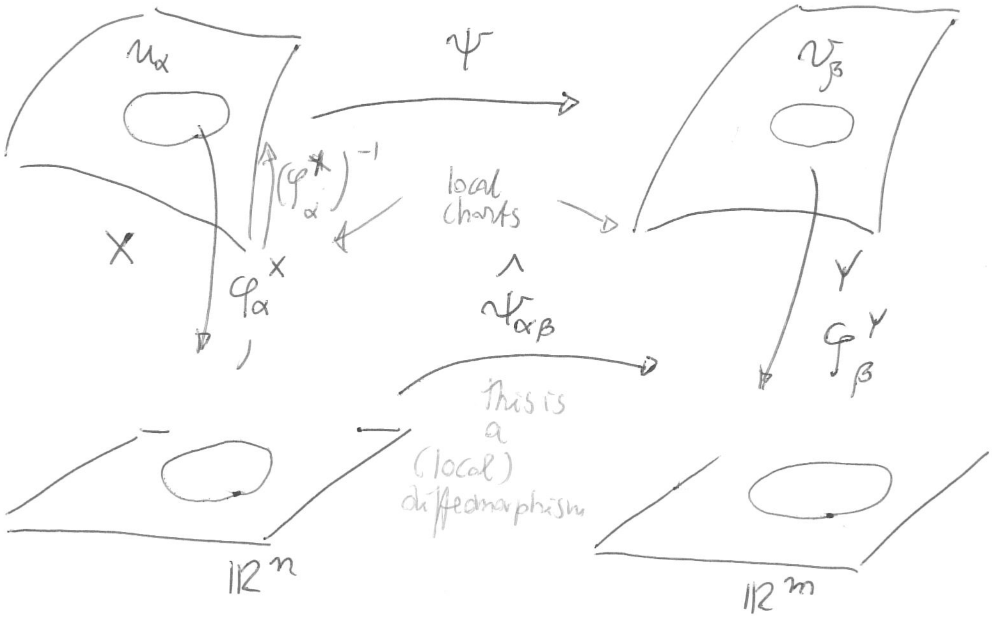
$f \circ \psi =: \psi^* f$ pull-back of f via ψ

" knowing functions on X one knows X "

Def. $\psi : X \longrightarrow Y$ is said to be a (smooth) diffeomorphism

if
- ψ is bijective (and smooth)
- ψ^{-1} is smooth

Locally, $\psi : X \to Y$ induces smooth maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ (upon applying the definition to local coordinate functions)



$$\hat{\Psi}_{\alpha\beta} = \varphi^Y_\beta \circ \Psi \circ \left(\varphi^X_\alpha\right)^{-1}$$

if Ψ is smooth, and $f \in C^\infty(Y, \mathbb{R})$, then $f \circ \Psi = \Psi^* f \in C^\infty(X, \mathbb{R})$ by definition

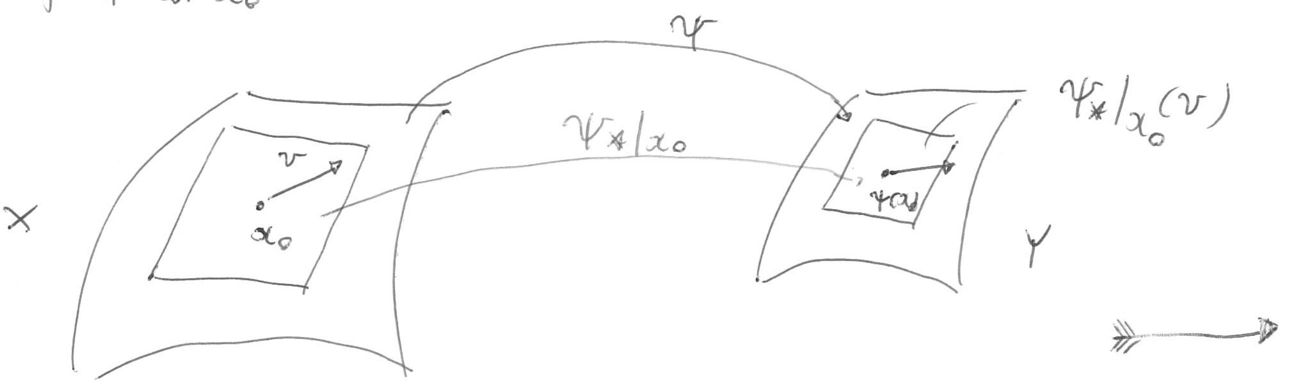
* Differential of a smooth map

Let $\Psi : X \to Y$ smooth

and $x_0 \in X$. One can define

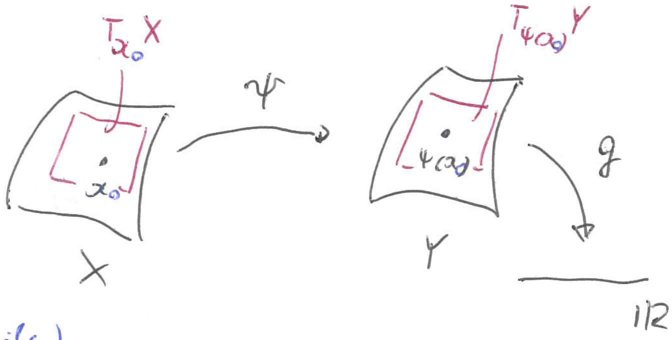
$$d\Psi|_{x_0} \equiv \Psi_*|_{x_0} : T_{x_0} X \longrightarrow T_{\Psi(x_0)} Y$$
$$\quad\quad\quad\quad\quad\quad\quad\quad \upsilon \longmapsto \Psi_*|_{x_0}(\upsilon)$$

differential (push-forward) of Ψ at x_0



via the position

$$\boxed{\Big(\underbrace{\psi_*|_{x_0}}_{\substack{\text{This is a derivation,}\\ \text{acting on functions}\\ \text{defined on a neighbourhood}\\ \text{of } \psi(x_0)}}\big(\underset{\underset{T_{x_0}X}{\cap}}{v}\big)\Big)\big(\underset{\underset{C^\infty(Y,\psi(x_0),\mathbb{R})}{\cap}}{g}\big) := v\big(\underset{\underset{C^\infty(X,x_0,\mathbb{R})}{\cap}}{\overset{\psi^*g}{g\circ\psi}}\big)}$$



One finds that (See below for details)

$$\boxed{\left(\psi_*|_{x_0}(v)\right)(g) = \sum_{j=1}^{m}\left[v(y^j\circ\psi)\frac{\partial}{\partial y^j}\right](g)}$$

In particular $\quad \psi_*|_{x_0}\left(\frac{\partial}{\partial x^i}\right) = \sum_{j=1}^{m}\frac{\partial}{\partial x^i}(y^j\circ\psi)\frac{\partial}{\partial y^j}$

$$|||$$

$$\frac{\partial}{\partial x^i} \longmapsto \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j} \qquad\qquad \frac{\partial y^j}{\partial x^i}\frac{\partial}{\partial y^j}$$

$$|||$$

$$\frac{\partial}{\partial x} \longmapsto J^T\frac{\partial}{\partial y} \qquad\qquad J^T$$

$$\begin{tikzcd} M \arrow[r, "\psi"] \arrow[rr, bend left, "f\circ\psi"] & N \arrow[r, "f"] & \mathbb{R} \end{tikzcd}$$

A more compact notation:

$$\boxed{\big(\underset{\underset{T_{\psi(x_0)}N}{\cap}}{\psi_* X}\big)(f)\big(\underset{\underset{N}{\cap}}{\psi(x_0)}\big) = \underset{\underset{T_{x_0}N}{\cap}}{X}\big(\underset{\psi^*f}{f\circ\psi}\big)\big(\underset{\underset{M}{\cap}}{x_0}\big)}$$

⚠ In general, vector fields are not mapped to vector fields
See below

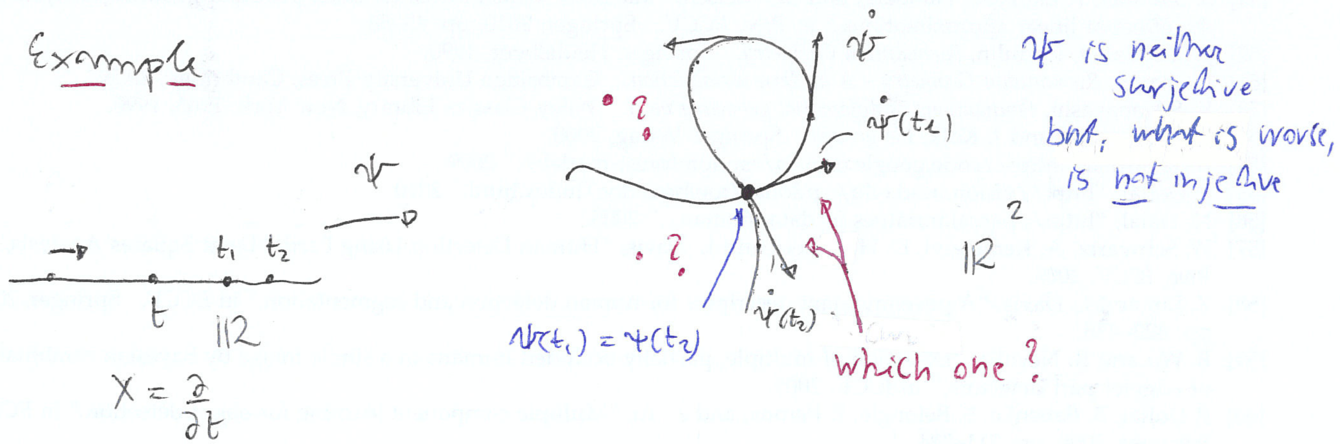
★ Behaviour of vector fields and differential forms under smooth maps

★ $\mathfrak{X}(M)$ Notice that in general, the push-forward

$$\psi_* : TM \to TN \quad \text{of} \quad \psi : M \to N$$

does NOT map vector fields on M to vector fields on N, in general (although, of course, ψ_* sends tangent vectors to M to tangent vectors to N) ⚠

Example



No problem arises if ψ is a diffeomorphism:

$$\underset{\mathcal{C}^\infty(N)}{(\psi_* X)(\underset{\cap}{f})}\,(\underset{\underset{N}{\cap}}{y}) = X(\underset{\underset{\mathcal{C}^\infty(M)}{\cap}}{\psi^* f})(\underset{\underset{M}{\cap}}{x}) = X(\psi^* f)(\psi^{-1}(y)) \qquad x = \psi^{-1}(y)$$

★ $\Lambda^k(M)$ On the other hand, given $\omega \in \Lambda^k(N)$, the pulled-back form $\psi^*\omega \in \Lambda^k(M)$ is always defined (for any ψ smooth). One has:

$$(\psi^*\omega)(x)(\underset{\cap}{X_1} \dots \underset{\cap}{X_k}) := \omega(\psi(x))(\underset{\cap}{\psi_* X_1} \dots \underset{\cap}{\psi_* X_k})$$

$\quad X_i \in T_xM, \qquad \psi_* X_i \in T_{\psi(x)}N$

(this definition extends the one given in the $\mathbb{R}^n$-case)

Let us provide some details $\qquad \psi : X \to Y$

φ : coordinate system $(x^1 \dots x^n)$ around $x_0 \in X$
local chart

τ : $\quad$ = $\quad (y^1 \dots y^m)$ around $y_0 \in Y$

Compute:

$$\left(\psi_*\big|_{x_0}(v)\right)(g) = v(g\circ\psi) = \sum_{i=1}^{n} a^i \frac{\partial}{\partial x^i}(g\circ\psi)$$

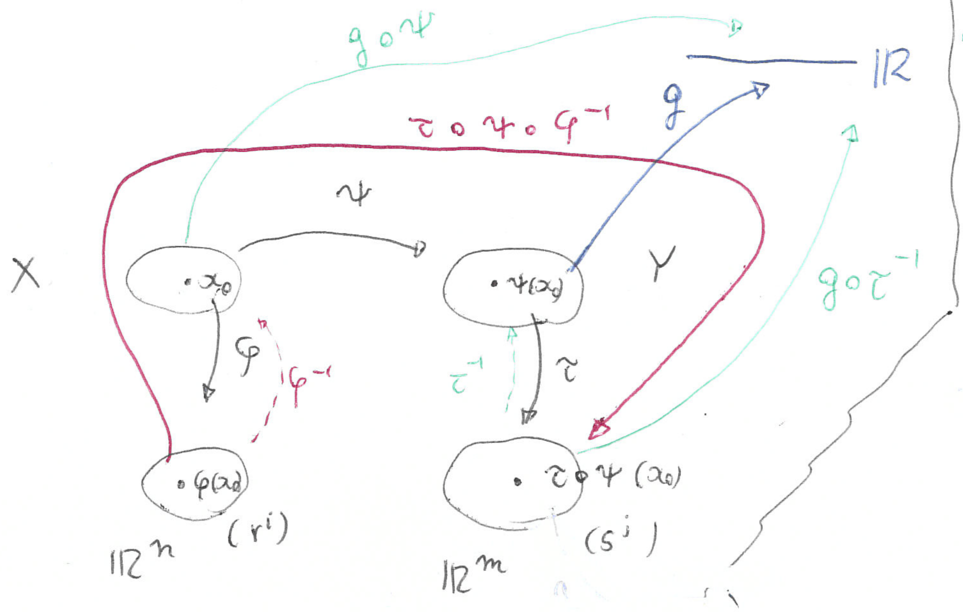
$$= \sum_{i=1}^{n} a^i \frac{\partial}{\partial r^i}\left(\underbrace{g\circ\tau^{-1}}\circ\underbrace{\tau\circ\psi\circ\varphi^{-1}}\right)\Big|_{\varphi(x_0)}$$

chain rule

$$= \sum_{i=1}^{n} a^i \sum_{j=1}^{m} \frac{\partial}{\partial s^j}(g\circ\tau^{-1})\Big|_{\tau\circ\psi(x_0)} \cdot \frac{\partial}{\partial r^i}(s_j\circ\tau\circ\psi\circ\varphi^{-1})\Big|_{\varphi(x_0)}$$

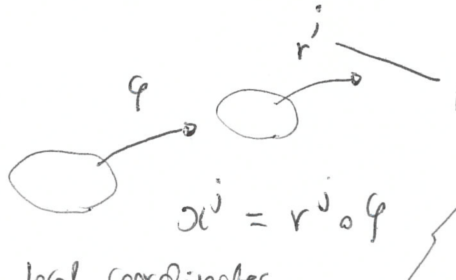
coordinate maps on $\mathbb{R}^m$

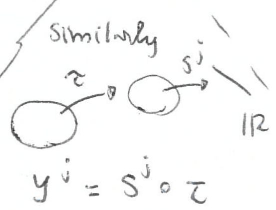
$$= \sum_{i=1}^{n}\sum_{j=1}^{m} a^i \frac{\partial g}{\partial y^j}\cdot\frac{\partial}{\partial x^i}(y_j\circ\psi)$$



$$= \left[\sum_{j=1}^{m} v(y_j\circ\psi)\frac{\partial}{\partial y^j}\right](g)$$

this is indeed a tangent vector at $\psi(x_0)$





We find, in particular

$$\psi_*\Big|_{x_0} : T_{x_0}X \longrightarrow T_{\psi(x_0)}Y$$

$$\frac{\partial}{\partial x^i} \longmapsto \sum_{j=1}^{m} \underbrace{\frac{\partial}{\partial x^i}(y_j \circ \psi)}_{(\psi_*|_x)_{ji}} \frac{\partial}{\partial y_j} \quad \sim J^T$$

Let us also check that if $X \xrightarrow{\psi} Y \xrightarrow{\Phi} Z \xrightarrow{h} \mathbb{R}$

then $d(\Phi \circ \psi) = d\Phi \circ d\psi$ generalized chain rule

↖ as homomomorphisms no matrix product.

$$\boxed{(\Phi \circ \psi)_* = \Phi_* \circ \psi_*}$$

associativity of $\circ$

$$\left[(\Phi \circ \psi)_*(v)\right](h) \underset{\text{def}}{=} v(h \circ (\Phi \circ \psi)) = v((h \circ \Phi) \circ \psi)$$

$$= \psi_*(v)(h \circ \Phi) = \Phi_*(\psi_*(v))(h)$$

$$= \left[(\Phi_* \circ \psi_*)(v)\right](h) \qquad \square$$

* **Example.** Let $M = \mathbb{R}^2 \setminus \{(x,0) \mid x \geq 0\}$

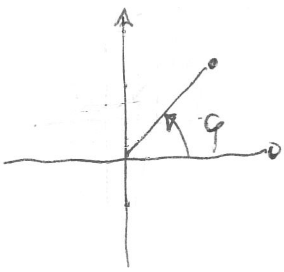
$$\Psi : \underset{\substack{\cup \\ M}}{(\rho, \varphi)} \longmapsto (x, y) \in \mathbb{R}^2 \qquad \Psi : \begin{cases} x = \rho \cos\varphi \\ y = \rho \sin\varphi \end{cases}$$

$\rho > 0$

$\varphi \in (0, 2\pi)$

$$d\Psi : \begin{cases} dx = d\rho \cos\varphi - \rho \sin\varphi \, d\varphi \\ dy = d\rho \sin\varphi + \rho \cos\varphi \, d\varphi \end{cases}$$

$d\Psi \equiv \Psi_*$



$$\begin{pmatrix} \cos\varphi & -\rho \sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{pmatrix} \qquad \frac{\partial}{\partial \rho} \leftrightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{\partial}{\partial \varphi} \leftrightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Psi_*\left(\frac{\partial}{\partial \rho}\right) = \cos\varphi \frac{\partial}{\partial x} + \sin\varphi \frac{\partial}{\partial y} = \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y}$$

$$\Psi_*\left(\frac{\partial}{\partial \varphi}\right) = -\rho \sin\varphi \frac{\partial}{\partial x} + \rho \cos\varphi \frac{\partial}{\partial y} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y}$$

This is of course in agreement with the general definition

$$\boxed{(\Psi_* X)(f) = X(f \circ \Psi)}$$

$$\begin{tikzcd} & \mathbb{R} \\ M \arrow[r, "\Psi"] \arrow[ur, bend right, "f \circ \Psi"'] & N \arrow[u, "f"'] \end{tikzcd}$$

Take $X = \frac{\partial}{\partial \rho}$. Compute:

$$\underset{\text{"}f = f(x,y)\text{"}}{\Psi_*\left(\frac{\partial}{\partial \rho}\right)(f)} = \underset{\text{"}f = f(\rho,\varphi)\text{"}}{\frac{\partial}{\partial \rho}(f \circ \Psi)} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \rho}$$

$$= \frac{\partial f}{\partial x} \underset{\substack{\| \\ \frac{x}{\sqrt{x^2+y^2}}}}{\cos\varphi} + \frac{\partial f}{\partial y} \underset{\substack{\| \\ \frac{y}{\sqrt{x^2+y^2}}}}{\sin\varphi} \underset{\text{"remove } f\text{"}}{\Rightarrow} \boxed{\Psi_*\left(\frac{\partial}{\partial \rho}\right) = \frac{x}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y}} \ \checkmark$$

(♦)

Similarly:

$$\Psi_*\left(\frac{\partial}{\partial\varphi}\right)(f) = \frac{\partial}{\partial\varphi}(f\circ\Psi) = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi}$$

$$= \frac{\partial f}{\partial x}\underbrace{(-\rho\sin\varphi)}_{-y} + \frac{\partial f}{\partial y}\underbrace{(\rho\cos\varphi)}_{x} \Rightarrow \quad (\blacklozenge\blacklozenge) \quad \boxed{\Psi_*\left(\frac{\partial}{\partial\varphi}\right) = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}}$$

— • — • — • — •

Let us examine this "classical" computation as well (useful in general)

$$\frac{\partial f}{\partial\rho} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\rho} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\rho}$$

$$\frac{\partial f}{\partial\varphi} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial\varphi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial\varphi}$$

$$\begin{pmatrix} \frac{\partial x}{\partial\rho} & \frac{\partial y}{\partial\rho} \\ \frac{\partial x}{\partial\varphi} & \frac{\partial y}{\partial\varphi} \end{pmatrix} = J^t$$

that is:
(remove f)

$$\begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\varphi} \end{pmatrix} = \begin{pmatrix} \cos\varphi & \sin\varphi \\ -\rho\sin\varphi & \rho\cos\varphi \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$$

$$\begin{cases} \frac{\partial}{\partial\rho} = \overbrace{\cos\varphi}^{\frac{x}{r}}\frac{\partial}{\partial x} + \overbrace{\sin\varphi}^{\frac{y}{r}}\frac{\partial}{\partial y} \\ \frac{\partial}{\partial\varphi} = \underbrace{-\rho\sin\varphi}_{-y}\frac{\partial}{\partial x} + \underbrace{\rho\cos\varphi}_{x}\frac{\partial}{\partial y} \end{cases}$$

... but this is to be interpreted ... as $(\blacklozenge)$ & $(\blacklozenge\blacklozenge)$

$\frac{\partial}{\partial\rho}$ is actually $\Psi_*\left(\frac{\partial}{\partial\rho}\right)$

and similarly for $\frac{\partial}{\partial\varphi}$. !