Prof. Mauro Spera, UCSC Brescia

TENSOR BUNDLES

Lecture XVIII

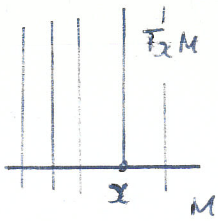
★ Tangent bundle

Let M be a (smooth) manifold

Let $TM = \bigsqcup_{x \in M} T_x M$ ← tangent space at x

disjoint union

★ tangent bundle of M



★ cotangent bundle of M $\quad T^*M = \bigsqcup_{x \in M} T_x^* M$

$= (T_x M)^*$ dual of $T_x M$

$\equiv$ cotangent space at x

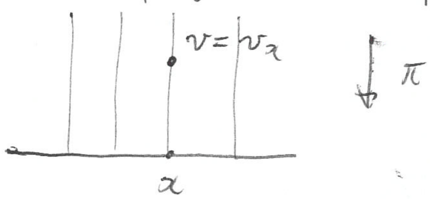
★ TM and T^*M are naturally endowed with a manifold structure. Let us focus on TM (T^*M is treated similarly)

Let $\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha \in \mathcal{R}}$ be an atlas for M

some index set

Let $\pi : TM \longrightarrow M$ be the natural projection map

$v \equiv v_x \longmapsto x$



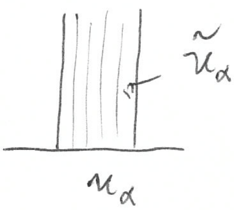
any $v \in TM$ belongs

to one and only one $T_x M$

for $x \in M$

Let $\tilde{\mathcal{U}}_\alpha = \pi^{-1}(\mathcal{U}_\alpha) \simeq \mathcal{U}_\alpha \times \mathbb{R}^n$

bijection



Set

$$\tilde{\varphi}_\alpha : \left(x, \sum_{i=1}^n b^i \frac{\partial}{\partial x^i}\right) \longmapsto (x^j, b^j)$$

$\cap$ $T_x M$ $\qquad$ $\| \ \ x^j \circ \varphi$ local coordinates

$\tilde{\mathcal{A}} = \{(\tilde{\mathcal{U}}_\alpha, \tilde{\varphi}_\alpha)\}_{\alpha \in \mathcal{R}}$ becomes an atlas for TM

★ TM can be topologized according to the second definition; it is clear that it has a countable basis and is Hausdorff if M is such.

The transition functions are readily obtained via the following computation (abridged notation, together with Einstein's convention)

$y = y(x)$ coordinate change

$y = f(x)$

insert argument

$$b^i \frac{\partial g}{\partial x^i} = b^i \frac{\partial g}{\partial y^j}\frac{\partial y^j}{\partial x^i}$$

$$= \underbrace{\left(b^i \frac{\partial y^j}{\partial x^i}\right)}_{(b'^j)} \frac{\partial g}{\partial y^j}$$

$$(b^i) \longmapsto (b'^i)$$

$$b'^i = b^k \frac{\partial y^i}{\partial x^k} = \frac{\partial y^i}{\partial x^k} b^k$$

just a relabeling

$f = \psi \circ \varphi^{-1}$ (see below)

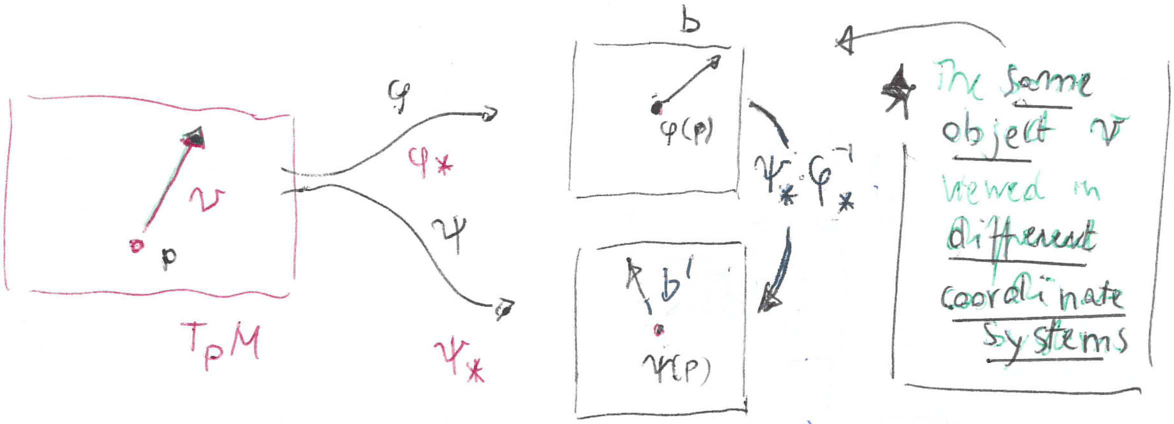
$$\underset{b'}{\big[\ \big]} = \boxed{f_*} \underset{b}{\big[\ \big]}$$

$f_* = (\psi\circ\varphi^{-1})_* = \psi_* \circ \varphi_*^{-1}$

this is fully consistent with the interpretation of the b's as velocity vectors of curves

transition maps:

$$(x, b) \longmapsto (y, b')$$



$$f = \psi \circ \varphi^{-1}$$

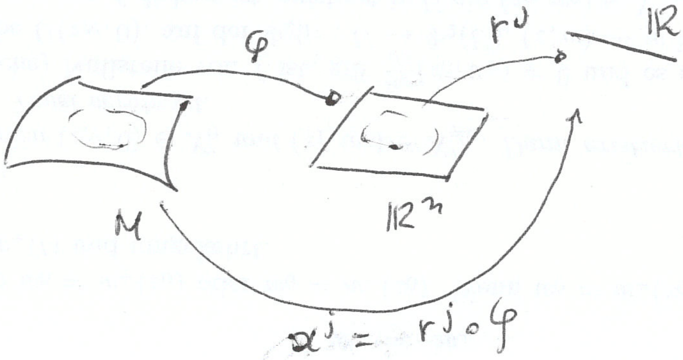
$$f_* = (\psi \circ \varphi^{-1})_* = \psi_* \circ \varphi_*^{-1}$$

★ The cotangent bundle

A similar treatment can be devised for the cotangent bundle T^*M. Charts (given an atlas for M):

$$\tilde{\varphi}_\alpha : \Big(x, \underbrace{\sum_{i=1}^{n} b_i \, dx^i}_{\in\, T_x^*M}\Big) \longmapsto (x^j, b^j)$$

notice: dx^i is the differential of the jth-coordinate function x^j



in accordance with the general definition

$\psi : X \to Y$

$\psi_*|_x : T_x X \to T_{\psi(x)} Y$

Remark: Cotangent spaces are fundamental in mechanics, being examples of phase spaces (symplectic manifolds), i.e. receptacles of positions and momenta of point particles.

momenta $\equiv$ duals of velocities

remember that identification with dual space is not canonical

Again work out the transition functions

quick recipe:

$$b_i\, dx^i = \overbrace{b_i \frac{\partial x^i}{\partial y^j}}^{b'_j} dy^j$$

$$b \longmapsto b'$$

$$(b_i) \longmapsto \left(b_k \frac{\partial x^k}{\partial y^i}\right) = \left(\frac{\partial x^k}{\partial y^i} b_k\right) \equiv (b'_i)$$

relabeling

★ again notice the different behaviours (contravariance vs covariance)

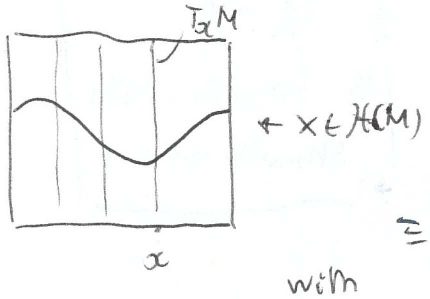
transition maps

$$(x, b) \longmapsto (y, b')$$

covector

— • — • —

★ vector fields on M (notation: $\mathcal{X}(M)$) are the smooth sections of TM, i.e given $\pi : TM \to M$ (canonical projection: it is a smooth map), $X : M \to TM$ (smooth) such that $\pi \circ X = id_M$, i.e. $X(x) \in T_xM$ $\forall x \in M$



★ Differential 1-forms (notation: $\Lambda^1(M)$) = smooth sections of T^*M; $\omega : M \to T^*M$ with $\pi \circ \omega = id_M$ ($\pi : T^*M \to M$ canonical projection)

This generalizes to generic tensor field

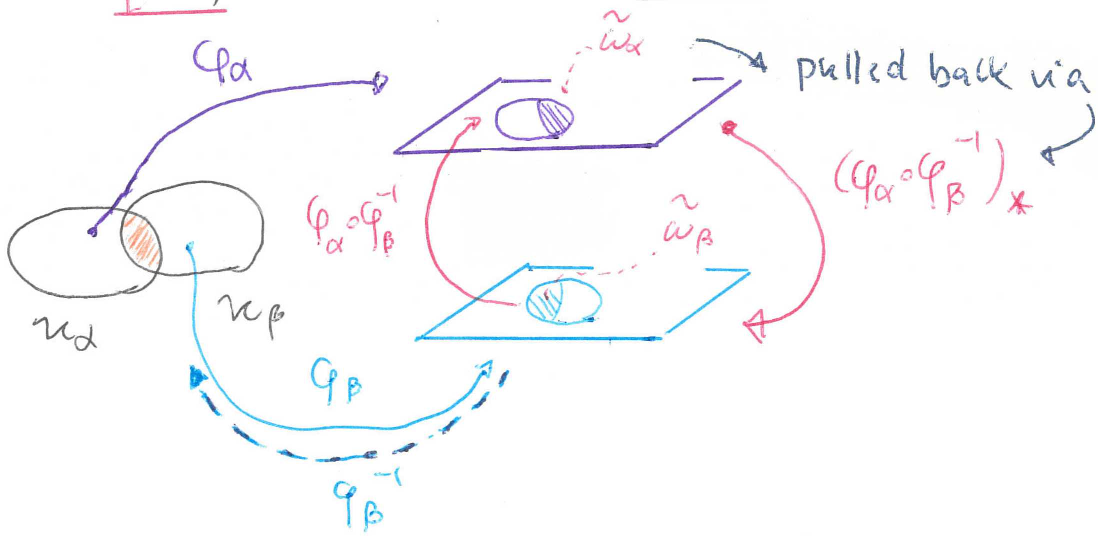
★ Local representation of differential k-forms (elements of $\Lambda^k(M)$)

$\mathcal{A} = \{(\mathcal{U}_\alpha, \varphi_\alpha)\}_{\alpha\in\mathfrak{A}}$ atlas for M

$\omega \in \Lambda^k(M)$ can be represented by a collection $\{\tilde{\omega}_\alpha\}$ of k-forms on $\mathbb{R}^n$ such that

$$\boxed{\tilde{\omega}_\beta = (\varphi_\alpha \circ \varphi_\beta^{-1})^* \tilde{\omega}_\alpha}$$

⚠



The exterior differential and the wedge product can be defined locally and their definition is well posed (by "functoriality" of pull-back)

E.g. $d\tilde{\omega}_\beta = d\left((\varphi_\alpha\circ\varphi_\beta^{-1})^*\tilde{\omega}_\alpha\right) = (\varphi_\alpha\circ\varphi_\beta^{-1})^* d\tilde{\omega}_\alpha$

d commutes with pull-back

$\wedge$ commutes with pull back

$\tilde{\omega}_\beta \wedge \tilde{\varphi}_\beta = (\varphi_\alpha\circ\varphi_\beta^{-1})^*\tilde{\omega}_\alpha \wedge (\varphi_\alpha\circ\varphi_\beta^{-1})^*\tilde{\varphi}_\beta = (\varphi_\alpha\circ\varphi_\beta^{-1})^*\tilde{\omega}_\alpha\wedge\tilde{\varphi}_\alpha$

and all their properties persist, in particular

$\boxed{d^2 = 0}$ $\qquad \boxed{d(\underset{\Lambda^k}{\omega} \wedge \varphi) = d\omega\wedge\varphi + (-1)^k \omega\wedge d\varphi}$

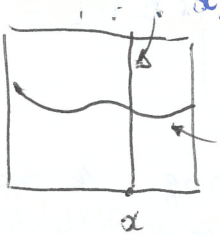
★ Tensor bundles

One can similarly define tensor bundles, whose sections are tensor fields

$\pi : T^{(p,q)}(M) \to M$

$$T^{(p,q)}(M) \ni \Big(x,\ t^{\overset{p}{I}}_{\underset{q}{J}} \frac{\partial}{\partial x^I} \otimes dx^J\Big)$$

$$\underbrace{T^*_xM \otimes \dots T^*_xM}_{q} \otimes \underbrace{T_xM \otimes \dots T_xM}_{p}$$



tensor field

notation: $\Gamma^{(p,q)}(M)$

$\downarrow$ local chart

$$(x,\ t^I_J)$$

↖ components

★ transition maps: $y = y(x)$

$$t^{i_1 \dots i_p}_{j_1 - j_q} \frac{\partial}{\partial x^{i_1}} \otimes \frac{\partial}{\partial x^{i_p}}\, dx^{j_1} \otimes \dots dx^{j_q}$$

$$\|$$

insert f fictitiously

$$\frac{\partial f}{\partial x^{i_1}} = \frac{\partial f}{\partial y^{l_1}} \frac{\partial y^{l_1}}{\partial x^{i_1}}$$

etc...

$$dx^{j_1} = \frac{\partial x^{j_1}}{\partial y^{h_1}} dy^{h_1}$$

Einstein

et cetera

$$t^{i_1 - i_p}_{j_1 - j_q} \frac{\partial y^{l_1}}{\partial x^{i_1}} \cdot \frac{\partial y^{l_p}}{\partial x^{i_p}} \frac{\partial x^{j_1}}{\partial y^{h_1}} \cdots \frac{\partial x^{j_q}}{\partial y^{h_q}} \frac{\partial}{\partial y^{l_1}} \otimes \dots \otimes \frac{\partial}{\partial y^{l_p}}\, dy^{h_1} \otimes \dots \otimes dy^{h_q}$$

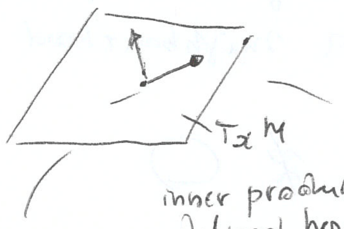
$$\| \quad t'^{\,l_1 - l_p}_{\ h_1 - h_q}$$

Concisely:
$$t^I_J \frac{\partial}{\partial x^I} \otimes dx^J = \underbrace{t^I_J \frac{\partial y^L}{\partial x^I} \frac{\partial x^J}{\partial y^H}}_{t'^L_H} \frac{\partial}{\partial y^L} \otimes dy^H$$

$$\boxed{t'^L_H = t^I_J \frac{\partial y^L}{\partial x^I} \frac{\partial x^J}{\partial y^H}}$$

↖ sum over I and J

★ Example : A Riemannian metric on M is a smoothly varying family of inner products on $T_x M$, $x \in M$; it is a symmetric, positive definite (at each point) element of $\mathcal{T}^{(0,2)}(M)$



locally :

$$x \longmapsto g_{ij}\, dx^i dx^j$$

$dx^i dx^j = \frac{1}{2}(dx^i \otimes dx^j + \frac{1}{2} dx^j \otimes dx^i)$:

★ symmetric tensor product

Let us check its transformation law :

$$g_{ij}\, dx^i dx^j = g_{ij} \frac{\partial x^i}{\partial y^k} dy^k \frac{\partial x^j}{\partial y^l} dy^l$$

as a function of x

$$= \underbrace{g_{ij}}_{\text{as a function of } y} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l} dy^k dy^l$$

$$\equiv g'_{kl}\, dy^k dy^l$$

$dx \wedge dy = \frac{1}{2}(dx \otimes dy - dy \otimes dx)$

★ antisymmetric tensor product

$\boxed{dx \otimes dy = dx dy + dx \wedge dy}$

$$\boxed{g'_{kl} = g_{ij} \frac{\partial x^i}{\partial y^k} \frac{\partial x^j}{\partial y^l}}$$

(this is a function of y)

Once one gets used to this kind of computations they are easily performed automatically.

★ Tensoriality

In view of future use, let us address the following question: given a map

$$\mathfrak{X}(M) \times \mathfrak{X}(M) \dots \mathfrak{X}(M) \longmapsto \mathscr{C}^\infty(M)$$
$$(X_1 \quad X_2 \dots X_k) \longmapsto T(X_1 \dots X_n)$$

how can one ascertain its tensor character, i.e. whether it defines, in the specific case, a tensor (field) in $\mathcal{T}^{(0,k)}(M)$?

Answer: check whether multilinearity over $\mathscr{C}^\infty(M)$ holds that is, whether

$$T(\dots, \alpha X_j^{(1)} + \beta X_j^{(2)}, \dots) = \alpha T(\dots X_j^1 \dots) + \beta T(\dots X_j^2 \dots)$$

with $\alpha, \beta \in \mathscr{C}^\infty(M)$. (One has pointwise multilinearity)

|| This also holds in general, for assessing tensoriality of prospective objects: check multilinearity over functions

Examples. Elements in $\mathfrak{X}(M)$ and $\Lambda^k(M)$ themselves are obviously tensors. A metric $g \in \mathcal{T}^{(0,2)}(M)$ has tensor character. We shall build up several objects which turn out to yield tensors, and others which will not give tensors.

Remark. It is quite common to call tensor fields simply tensors.