Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture XX

DINI'S THEORY FOR MANIFOLDS

- k-slices
- inverse function theorem
- Rank theorem
- Implicit function theorem
- Immersed and embedded submanifolds
- examples (see prologue as well)

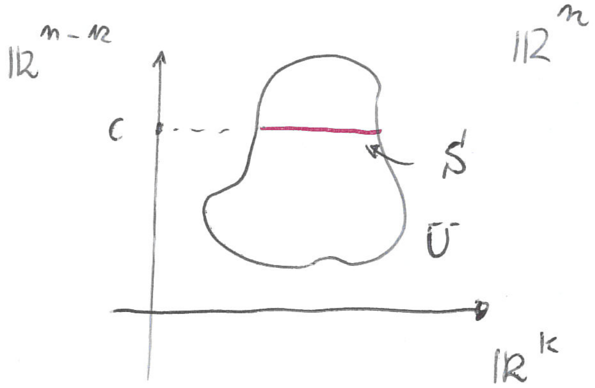
★ k-slices — recalled from prologue

The sets of the form

$$S = \{(x^1 \dots x^k, x^{k+1}, \dots x^n) \in U \;/\; x^{k+1} = c^{k+1}, \dots x^n = c^n\}$$

U open set in $\mathbb{R}^n$; $c^{k+1}, \dots, c^n$ fixed

are called k-slices of U



★ Extension of the fundamental theorems of analysis in several variables to manifolds.

★ The inverse function theorem

$$\psi : X \to Y \qquad \psi \text{ smooth}$$
$$\dim X = \dim Y = n$$

Let $\psi_*|_{x_0} : T_{x_0}X \to T_{y_0}Y$ be an isomorphism, $y_0 = \psi(x_0)$

Then $\exists\ \mathcal{U}_0 \ni x_0$ (neighbourhood of x_0) such that

$$\Psi|_{\mathcal{U}_0} : \mathcal{U}_0 \to \Psi(\mathcal{U}_0)$$

is a diffeomorphism

In coordinates, the proof reduces to the standard one, via the contraction lemma (Banach-Caccioppoli theorem) see also prologue

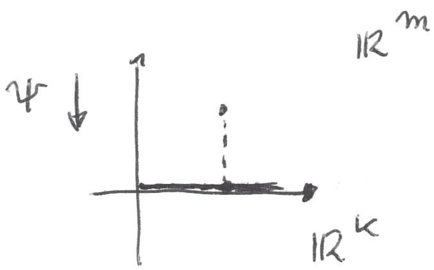
* The rank theorem

Let $\Psi : M \to N$ ($\dim M = m$, $\dim N = n$) be smooth, with constant rank k, $k \le m$. Then $\forall\, p \in M$, there exist coordinates $(x^1 \dots x^m)$ centred at p and $(y^1, \dots, y^n)$ centred at $\Psi(p)$ such that

$$\Psi(x^1 \dots x^k, x^{k+1} \dots x^m) = (x^1 \dots x^k, 0, \dots 0)$$

$k \le \min(m,n)$

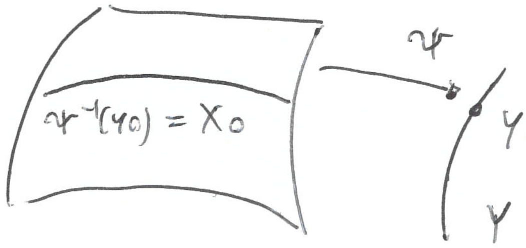
a slice in $\mathbb{R}^n$



* The implicit function theorem

Let $\Psi : X \to Y$ smooth, $\dim X = n >$ $\dim Y = m$

Let $y_0 \in Y$ and $X_0 = \Psi^{-1}(y_0) = \{x \in X \,/\, \Psi(x) = y_0\}$



Assume that $\Psi_*|_x : T_x X \to T_{\Psi(x)} Y$ is surjective $\forall x \in X_0$.

That is, ψ is submersive $(\forall x \in X_0)$.

Then X_0 is a manifold (equipped with the relative topology inherited from X), and $X_0 \hookrightarrow X$ (inclusion) is smooth. Moreover $\dim X_0 = \dim X - \dim Y$
X_0: level manifold (of ψ) $= n - m$.
varietà di livello

<u>Proof</u>. (Sketch) Let $V \ni y_0$ a coordinate neighbourhood (of y_0) with local coordinates $(y^1 .. y^m)$. Let $x_0 \in X_0$ and $U \ni x_0$ (coord. neighbourhood), with local coordinates $(x^1 ... x^n)$ centred at x_0 $(x^i(x_0) = 0, i = 1..n)$
Since $\psi_*|_{x_0}$ is surjective, the Jacobian matrix

$$\left(\frac{\partial}{\partial x^j} (y^i \circ \psi)\Big|_{x_0} \right)_{\substack{i=1...m \\ j=1..n}}$$

has rank m, so, up to a coordinate relabelling one can assume it to be of the form

$$(* \mid J)$$

$m \times m$, non singular

Define $\tilde{\psi} : U \to \mathbb{R}^{n-m} \times \tilde{V}$ in the following manner:

$$\tilde{\psi}(x) = (x^1(x) ... x^{n-m}(x), \psi(x))$$

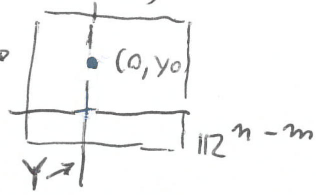
$$\Rightarrow \quad \tilde{\psi}_*|_{x_0} \sim \begin{pmatrix} I_{n-m} & 0 \\ * & J \end{pmatrix}$$

examples on pages 6,7

which is an isomorphism. Therefore, by virtue of the inverse function theorem, $\exists\, U_0 \ni x_0$ such that $\tilde{\psi}|_{U_0}$ is injective, $\tilde{\psi}(U_0)$ is open in $\mathbb{R}^{n-m} \times V$ and $\tilde{\psi}^{-1} : \tilde{\psi}(U_0) \to U_0$ is smooth.

Without loss of generality (w.l.o.g.), $\tilde{\psi}(U_0)$ can be taken of the form $\underset{\substack{\Psi \\ 0}}{W_0} \times \underset{\substack{\Psi \\ y_0}}{V_0} \rightsquigarrow$

open hyperparallelepipeds $\to$



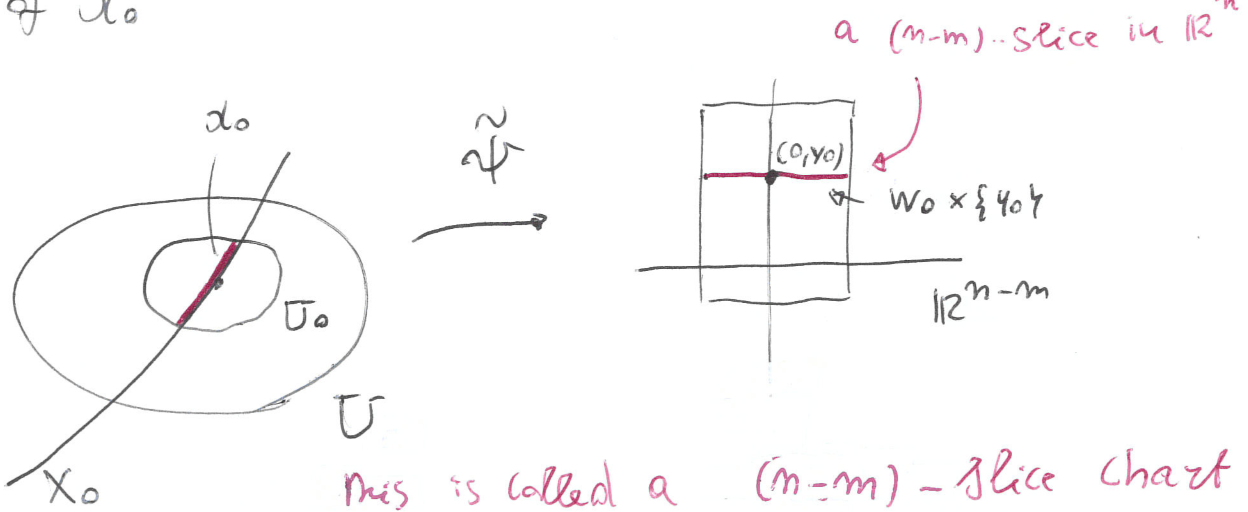
(open sets of this type yield a basis for the topology of $\mathbb{R}^{n-m} \times V$.

Now $\tilde{\psi}^{-1}(W_0 \times \{y_0\}) = X_0 \cap U_0$.

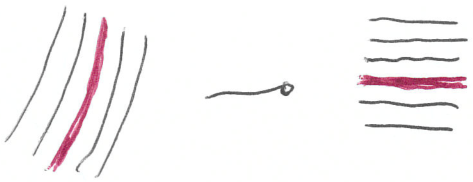
Since $\tilde{\psi}|_{U_0}$ is a homeomorphism, the map

$$\tilde{\psi}\Big|_{X_0 \cap U_0} : X_0 \cap U_0 \xrightarrow{\text{homeom.}} \begin{array}{c} W_0 \times \{y_0\} \\ \cong \\ W_0 \subset \mathbb{R}^{n-m} \end{array}$$

is a coordinate system in a neighbourhood of x_0



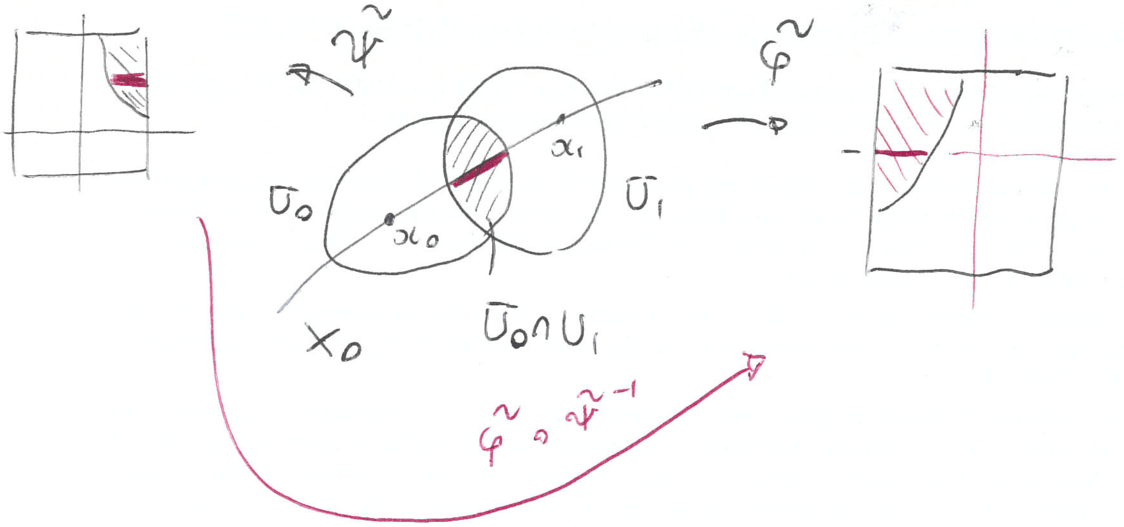
This is called a $(n-m)$-slice chart



level sets of ψ

One is left with checking the behaviour on non empty intersections

$$\tilde{\psi} : \bar{U}_0 \to W_0 \times V_0$$
$$\tilde{\varphi} : \bar{U}_1 \to W_1 \times \bar{V}_1$$



$$\tilde{\varphi} \circ \tilde{\psi}^{-1} \Big|_{\tilde{\psi}(\bar{U}_0 \cap \bar{U}_1)} \quad \text{is } \mathcal{C}^\infty, \text{ and so is}$$

$$\tilde{\varphi} \circ \psi^{-1} \Big|_{\tilde{\psi}(X_0 \cap U_0 \cap \bar{U}_1)} : \tilde{\psi}(X_0 \cap \bar{U}_0 \cap \bar{U}_1) \hookrightarrow \tilde{\varphi}(X_0 \cap \bar{U}_0 \cap \bar{U}_1)$$

$$\tilde{\psi}(X_0 \cap U_0 \cap \bar{U}_1) = (\tilde{\psi}(\bar{U}_0 \cap \bar{U}_1) \cap \mathbb{R}^{n-m}) \times \{y_0\}$$

(the same being true for $\tilde{\psi} \circ \varphi^{-1}$),

and this yields the desired conclusion.

sottovarietà immersa

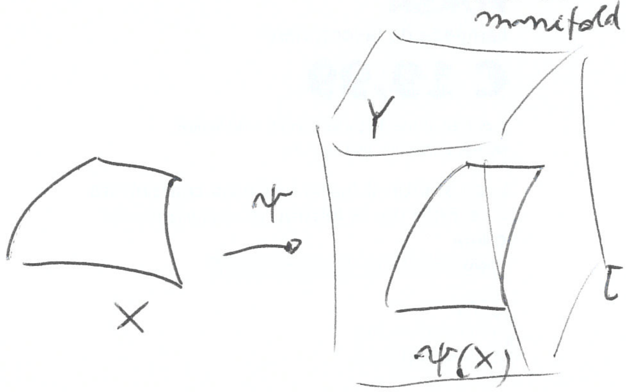
Def. (i) An <u>immersed submanifold</u> of a manifold Y (smooth)

is a pair (X, ψ), $\psi : X \to Y$, smooth
(X: smooth manifold)

with ψ and ψ_*

injective

(ψ: injective immersion)



[with an abuse of language one identifies then X with its image $\psi(X) \subset Y$]

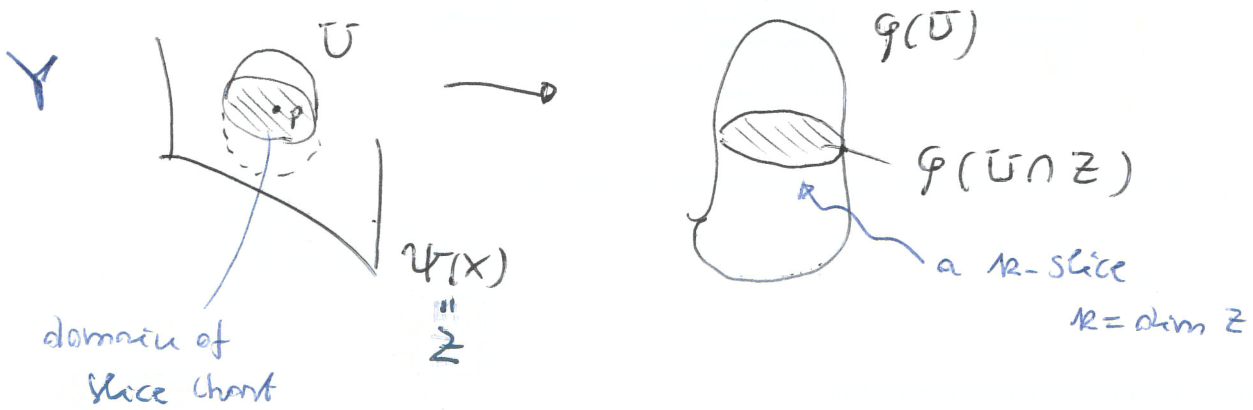
(ii) One has an <u>embedding</u> if, in addition to (i),
inclusione

$\psi : X \to \psi(X)$ is a <u>homeomorphism</u>

(with $\psi(X)$ equipped with the relative topology induced from Y)

*** Let us check that, if ψ is an <u>embedding</u>, then $\psi(X)$ is a <u>submanifold of</u> Y in the <u>following</u> <u>sense</u>:

Every point of $\psi(X)$ admits a coordinate neighbourhood $U \subset Y$ such that $\psi(X) \cap U$ is the domain of a slice-chart



Remark. The level submanifolds previously discussed are indeed submanifolds of X in the above sense.

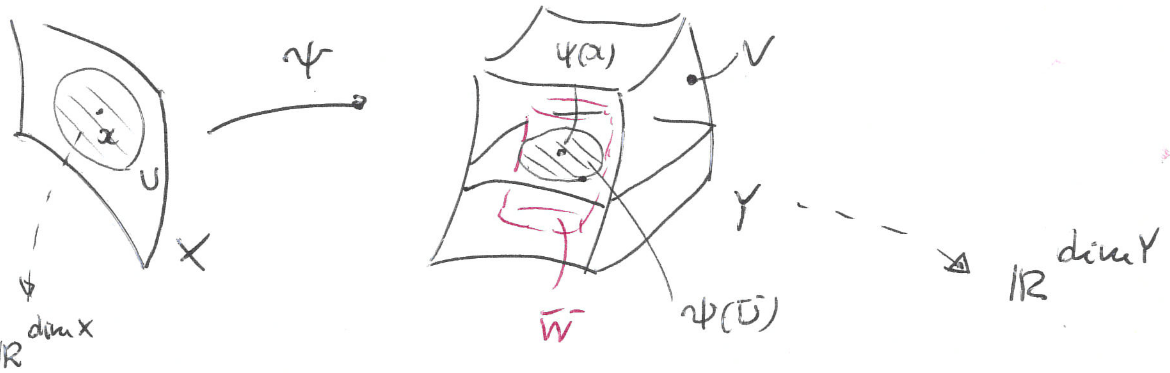
Proof. Let $x \in X$. Since an embedding has constant rank, by the rank theorem one can find local coordinate systems centred at x and $\psi(x)$, respectively, with

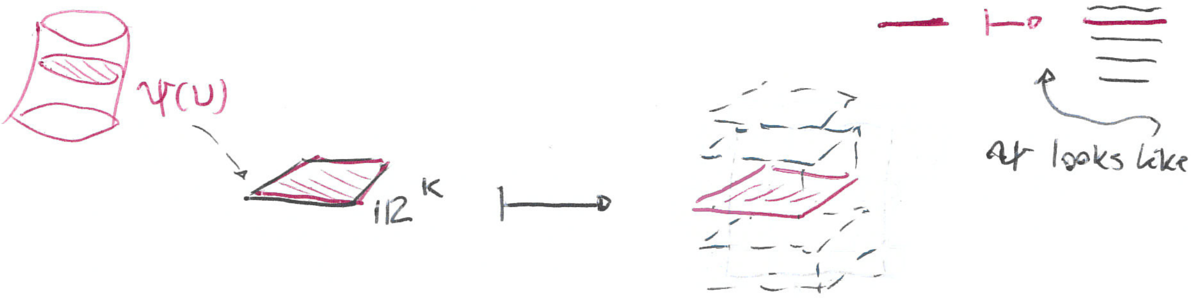
abuse of notation ⇝ $$\psi : \overset{U}{\underbrace{(x^1 \dots x^k)}_{\dim X}} \longmapsto \overset{V}{(x^1 \dots x^k, \underbrace{0 \dots 0}_{\dim Y - \dim X})}$$

upon possible restriction of V, $\psi(U)$ becomes a slice in Y. Now $\psi(U)$ is open in $\psi(X)$ (ψ is a homeomorphism onto $\psi(X)$) $\Rightarrow \exists\, W \subset Y$ such that $\psi(U) = W \cap \psi(X)$

⚠ this is the crucial point

Let $\hat{V} = V \cap W$. One produces a slice chart such that $\hat{V} \cap \psi(X) = \hat{V} \cap \psi(U)$ is a slice in $\hat{V}$





Notice that if X is compact, ψ is automatically a homeomorphism onto $\psi(X)$ (since $\psi(X)$ is Hausdorff and ψ is injective)

In any case, in view of Dini's theory,
ψ injective + ψ_* injective $\Rightarrow$ ψ homeomorphism (locally)

i.e. an injective immersion is locally an embedding

★ Summary F_* (differential, or push-forward)

slogan " F_* behaves locally like F "

★ crucial tools
$\boxed{\text{inverse function theorem}}$
⇓
- rank theorem
- implicit function theorem

important case: surjective submersions

$F: M \to N$

rank of F at $p \in M$ $\equiv$ rank of $F_*: T_pM \to T_{F(p)}N$

if this does not vary with p, we say F of constant rank

		rank
F: submersion :	F_* surjective $\forall p \in M$	$r(F) = \dim N$
F: immersion :	F_* injective	$r(F) = \dim M$

F: embedding : F immersion + F homeomorphism onto $F(M)$

"topological embedding"

equivalently :

F injective
F_* injective
$F: M \to F(M)$ homeomorphism

equipped with the relative topology inherited from N

For examples see prologue

Remark

According to the celebrated Whitney theorem, every finite dimensional manifold can be embedded in a suitable $\mathbb{R}^N$. However, this property is more effective in theoretical issues than in practice.

John Nash extended Whitney's results to isometric embeddings of Riemannian manifolds in $\mathbb{R}^N$ (so that the metric is inherited from the standard one), a major "tour-de-force"!

However, we close our discussion at this point.