Lectures on
DIFFERENTIAL GEOMETRY AND TOPOLOGY
V2

Lecture XXIII

- Lie derivative for k-forms.
- Cartan's magic formula

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LIE DERIVATIVES
(FORM & GENERAL TENSOR CASE)

Remark. Using a partition of unity, a form defined on $U \subset M$ (open) can be extended to a form defined on M.

We wish to give an intrinsic formulation of $d\omega$, $\omega \in \Lambda^1(M)$

$$\boxed{d\omega(X,Y) = X\omega(Y) - Y\omega(X) - \omega([X,Y])}$$

Proof. We work locally, so it is enough to take $\omega = u\,dv$,

$u, v \in \mathcal{C}^\infty(M)$.

l.h.s. $\quad d\omega = du \wedge dv \quad$ and

$$d\omega(X,Y) = (du \wedge dv)(X,Y) = (du)(X)(dv)(Y) - (du)(Y)(dv)(X)$$

$$= X(u)Y(v) - X(v)Y(u).$$

r.h.s. $\quad X\omega(Y) = X[u\,dv(Y)] = X[u\,Y(v)]$

$$= X(u)Y(v) + u\,X(Y(v)).$$

$$-Y\omega(X) = -Y(u\,dv(X)) = -Y(u\,X(v))$$

$$= -Y(u)X(v) - u\,Y(X(v)).$$

they add to $u \cdot [X,Y](v)$

$$-\omega([X,Y]) = -u\,dv([X,Y]) = -u\,[X,Y](v)$$

Thus r.h.s. = ... l.h.s

In general, one shows that, for $\omega \in \Lambda^k(M)$

$$d\omega(X_1 \dots X_{k+1}) = \sum_{i=1}^{k+1} (-1)^{i+1} X_i\, \omega(X_1 \dots \hat{X}_i, \dots X_{k+1})$$

$$+ \sum_{i<j} (-1)^{i+j} \omega([X_i, X_j], X_1 \dots \hat{X}_i \dots \hat{X}_j \dots X_{k+1})$$

first argument

$\wedge$: omission

★ Interior multiplication (or contraction)

$$i_X : \Lambda^k(M) \longrightarrow \Lambda^{k-1}(M)$$

$$\omega \longmapsto i_X \omega$$

extended to $i : \Lambda \to \Lambda$

$X \in \mathcal{X}(M)$

actually, i is a purely algebraic operation.

$$(i_X \omega)(X_1 \dots X_{k-1})$$

$$:= \omega(X, X_1, \dots X_{k-1})$$

first slot

(other notation: $X \lrcorner \omega$

i_X is linear and, for $X, Y \in \mathcal{X}(M)$, $\alpha, \beta \in \mathcal{C}^\infty(M)$

$$i_{\alpha X + \beta Y}\, \omega = \alpha\, i_X \omega + \beta\, i_Y \omega$$

($\diamond$) $\boxed{i_X^2 = 0}$

and i is an antiderivation

($\diamond\diamond$) $$\boxed{i_X(\underset{\Lambda^k}{\omega} \wedge \eta) = i_X \omega \wedge \eta + (-1)^k \omega \wedge i_X \eta}$$

i behaves like d

Let us check that $i_X^2 = 0$ $\qquad w \in \Lambda^k$

$$i_X(i_X w)(X_1 \dots X_{k-2}) =$$

$$= i_X w(X, X_1 \dots X_{k-2}) = w(X, X, X_1 \dots X_{k-2}) = 0$$

As for $(\diamond\diamond)$, it is enough to check the formula

$$i_X(w^1 \wedge \dots w^k) = i_X w^1 \wedge w^2 \wedge \dots w^k$$
$$- i_X w^2 \wedge w^1 \wedge \dots w^k$$
$$+ i_X w^3 \wedge w^1 \wedge \dots w^k$$

1-forms

quite useful in explicit computations

and this is true by Laplace's formula:

if $X_1 = X$

$$(w^1 \wedge w^2 \dots w^k)(\overset{X}{\overset{\shortparallel}{X_1}}, X_2 \dots X_k) = \det((w^i(X_j)))$$

$$= \sum_{i=1}^{k} (-1)^{i-1} w^i(\underset{\underset{X}{\shortparallel}}{X_1}) \underbrace{(w^1 \wedge \dots \hat{w}^i \wedge \dots w^k)(X_2, \dots X_k)}_{\det \mathbb{X}^{\hat{i}}_{\hat{1}}}$$

* Lie derivative of a k-form

Let $X \in \mathfrak{X}(M)$, $\omega \in \Lambda^k(M)$

The $\boxed{\text{Lie derivative of } \omega \text{ along } X}$, $\mathcal{L}_X\omega \in \Lambda^k(M)$ is defined as:

recall that the pull-back is defined algebraically

$F^X_\cdot$: flow of X

$(F^X_t)^*\omega(F^X_t(p))$

$$\boxed{(\mathcal{L}_X\omega)(p) = \frac{d}{dt}\left[(F^X_t)^*\omega\right]\Big|_{t=0} = \lim_{t\to 0}\frac{[(F^X_t)^*\omega](p) - \omega(p)}{t}}$$

One may prove that $\mathcal{L}_X(\omega\wedge\tau) = \mathcal{L}_X\omega\wedge\tau + \omega\wedge\mathcal{L}_X\tau$

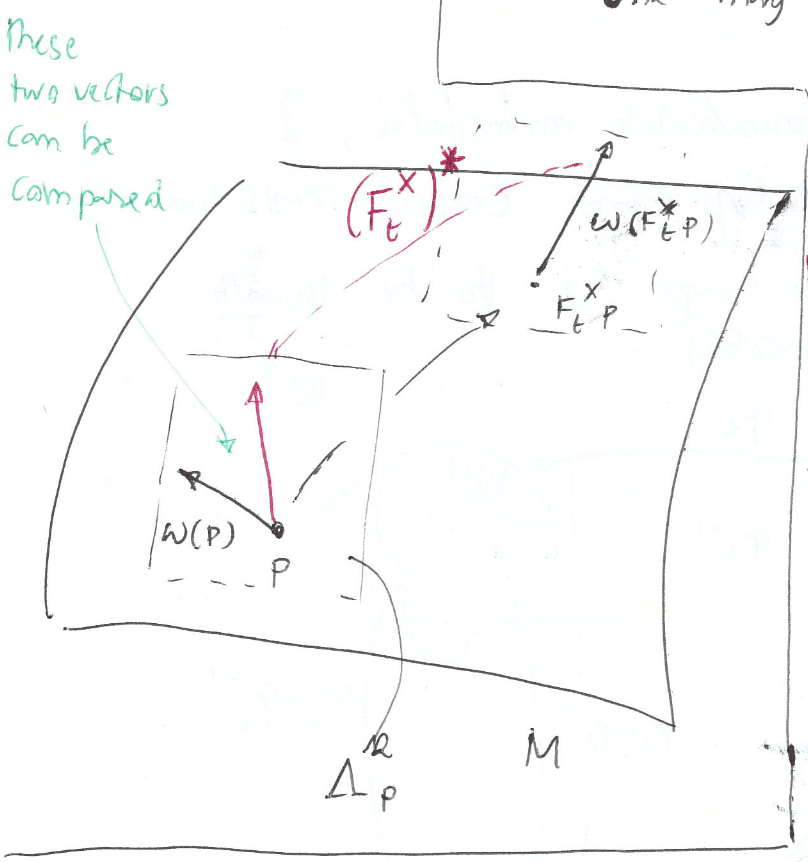
and, more generally

$$\boxed{\mathcal{L}_X(T\otimes S) = \mathcal{L}_X T\otimes S + T\otimes\mathcal{L}_X S} \quad (\star)$$

for any tensor fields

(generalized Leibniz rule)

Also, for a $(0,k)$-tensor one has:



$$(\mathcal{L}_X\sigma)(Y_1\ldots Y_k) = X(\sigma(Y_1\ldots Y_k)) - \sigma([X,Y_1], Y_2\ldots Y_k) - \ldots - \sigma(Y_1\ldots Y_{k-1}, [X,Y_k]),$$

This, in turn, coming from

$$\underbrace{X(\sigma(Y_1\ldots Y_k))}_{\mathcal{L}_X(\sigma(Y_1\ldots Y_k))} = (\mathcal{L}_X\sigma)(Y_1\ldots Y_k) + \sigma(\overset{[X,Y_1]}{\mathcal{L}_X Y_1}, Y_2\ldots Y_k) + \ldots + \sigma(Y_1\ldots \mathcal{L}_X Y_k)$$

Leibniz rule: all arguments are varied one at a time

Let us check (☆) in the case of a tensor product of covariant tensors, just to pinpoint the (quite simple) basic idea.

$$\mathcal{L}_X(\sigma\otimes\tau)(p) = \lim_{t\to 0}\frac{(F_t^X)^*\left[(\sigma\otimes\tau)(F_t^X(p))\right] - (\sigma\otimes\tau)(p)}{t}$$

$$= \lim_{t\to 0}\frac{\overbrace{(F_t^X)^*\sigma(F_t(p))\otimes(F_t^X)^*\tau(F_t^X(p))}^{A} - \overbrace{\sigma(p)\otimes\tau(p)}^{B}}{t}$$

$$= \lim_{t\to 0}\left[\frac{A - \overbrace{\sigma(p)\otimes(F_t^X)^*\tau(F_t^X(p))}^{C}}{t} + \frac{C - B}{t}\right]$$

[This is the same idea one uses to prove $(fg)' = f'g + fg'$ in calculus]

$$f(x+h)g(x+h) - f(x)g(x)$$
$$= f(x+h)g(x+h) - f(x)g(x+h) + f(x)g(x+h) - f(x)g(x)$$
$$= \{f(x+h) - f(x)\}g(x+h) + f(x)\{g(x+h) - g(x)\}$$
$$\vdots$$

$$= \lim_{t\to 0}\underbrace{\frac{(F_t^X)^*\sigma(F_t(p)) - \sigma(p)}{t}}_{\to\,\mathcal{L}_X\sigma}\otimes\underbrace{(F_t^X)^*\tau(F_t^X(p))}_{\xrightarrow{t\to 0}\,\tau(p)}$$

$$+\ \sigma(p)\otimes\underbrace{\lim_{t\to 0}\frac{(F_t^X)^*\tau(F_t^X(p)) - \tau(p)}{t}}_{\mathcal{L}_X\tau}$$

$$= \mathcal{L}_X\sigma\otimes\tau + \sigma\otimes\mathcal{L}_X\tau$$

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also recall that in general if $S = \sigma^I_J$, $T = \tau^K_L$

$S\otimes T = \sigma^I_J\cdot\tau^K_L$ ← no summation, just take all products

numerical product

✱ **Theorem** Let $\omega \in \Lambda(M)$, $X \in \mathfrak{X}(M)$.

One has $$\boxed{\mathcal{L}_X \omega = d\, i_X \omega + i_X d\omega}$$

✱✱✱ (Cartan's magic formula)

⚠ It holds for forms, not for general tensors (d is not defined ...)

Pf. We prove it for $\omega = u\, dv$. The general case follows by induction.

First of all, notice that

$$(\mathcal{L}_X dv)(Y) \underset{\text{general formula}}{=} X[dv(Y)] - dv([X,Y])$$

$$= X[Y(v)] - [X,Y](v) = (XY - XY + YX)(v) = Y\underbrace{X(v)}_{\mathcal{L}_X v}$$

$$= d(\mathcal{L}_X v)(Y)$$

namely $\boxed{\mathcal{L}_X dv = d\mathcal{L}_X v}$

As a corollary, if $(x^1 \dots x^n)$ are local coordinates and $X = \frac{\partial}{\partial x^1}$, just to fix ideas, then $\mathcal{L}_X (dx^{i_1} \wedge \dots dx^{i_k}) = 0$,

this following from $\mathcal{L}_X dx^j = d\, \mathcal{L}_X x^j = d\left(\frac{\partial x^j}{\partial x^1}\right)$

and from the general Leibniz rule. $= d(\delta_{j1}) = 0$

Now compute:

$$\mathcal{L}_X (u\,dv)(Y) = (\mathcal{L}_X u . dv + u\, \mathcal{L}_X dv)(Y)$$
$$= (\mathcal{L}_X u\, dv + u\, d\, \mathcal{L}_X v)(Y)$$
$$= (X(u)\, dv + u\, d(X(v)))(Y)$$
$$= X(u) Y(v) + u Y(X(v)) \qquad \mathrm{I}$$

compute II =

$$= (d\, i_X + i_X d)(u\,dv)(Y) =$$
$$= d(i_X(u\,dv))(Y) + i_X\, d(u\,dv)(Y)$$
$$= d(\underbrace{i_X u}_{=0}\, dv + u\, \underbrace{i_X dv}_{X(v)})(Y) + \underbrace{i_X(du \wedge dv)(Y)}_{(du\wedge dv)(X,Y)}$$
$$= d(u\, X(v))(Y) + X(u) Y(v) - X(v) Y(u)$$
$$= (du\, X(v) + u\, d(X(v)))(Y) + X(u) Y(v) - X(v) Y(u)$$
$$= Y(u) X(v) + u\, Y(X(v)) + X(u) Y(v) - X(v) Y(u)$$
$$= X(u) Y(v) + u\, Y(X(v)) = \mathrm{I}$$

□

Notice that, in general for forms, one has:

$$\boxed{\mathcal{L}_X d = d\, \mathcal{L}_X}$$

Indeed: $\mathcal{L}_X d = (d i_X + i_X d) d = d i_X d + i_X \overset{?}{\underset{0}{d^2}}$
$= d i_X d$

whereas
$$d\,\mathcal{L}_X = d(d i_X + i_X d) = \underbrace{d^2 i_X}_{0} + d i_X d = d i_X d$$

□

Examples

In $\mathbb{R}^3$ | 1. $X = \frac{\partial}{\partial x}$ $\qquad \omega = a\, dx \wedge dy \qquad a \in C^2(\mathbb{R}^3)$

$$d\omega = da \wedge dx \wedge dy = \frac{\partial a}{\partial z} dz \wedge dx \wedge dy = \frac{\partial a}{\partial z} dx \wedge dy \wedge dz$$

$$\boxed{i_X d\omega = \frac{\partial a}{\partial z} dy \wedge dz}$$

$$\Big(i_X d\omega\,(Y, Z) = \underbrace{\Big(\frac{\partial a}{\partial z} dx \wedge dy \wedge dz\Big)}\big(\underset{\overset{\parallel}{\frac{\partial}{\partial x}}}{X}, Y, Z\big) =$$

running through $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$= \frac{\partial a}{\partial z} \begin{vmatrix} \overbrace{dx(X)}^{1} & dx(Y) & dx(Z) \\ \underbrace{dy(X)}_{=0} & \cdot & \cdot \\ \underbrace{dz(X)}_{=0} & \cdot & \cdot \end{vmatrix} = \frac{\partial a}{\partial z} \begin{vmatrix} dy(Y) & dy(Z) \\ dz(Y) & dz(Z) \end{vmatrix}$$

$$= \frac{\partial a}{\partial z} (dy \wedge dz)(Y, Z)\Big)$$

One can proceed directly from the definition

$$\frac{\partial a}{\partial z} (dx \wedge dy \wedge dz)(X, \overset{Y}{\bullet}, \overset{Z}{\blacklozenge}) =$$

$$= \frac{\partial a}{\partial z}\Big(\overbrace{dx(\underset{\overset{\parallel}{\frac{\partial}{\partial x}}}{X})}^{1}\, dy(\bullet)\, dz(\blacklozenge) - \underbrace{dx(X)}_{1}\, dy(\blacklozenge)(\bullet) + \dots \Big)$$

all other terms are 0 since $dy(X) = dz(X) = 0$, $X = \frac{\partial}{\partial x}$

$$= \frac{\partial a}{\partial z} dy \wedge dz\,(\bullet, \blacklozenge)$$

* Actually, the one below is the quickest method:

$$i_X\Big(\frac{\partial a}{\partial z} dx \wedge dy \wedge dz\Big) = \frac{\partial a}{\partial z}\, i_X(dx \wedge dy \wedge dz)$$

$$= \frac{\partial a}{\partial z}\Big(\underset{\overset{dx(\frac{\partial}{\partial x})}{\overset{\parallel}{1}}}{(i_X dx)} \wedge dy \wedge dz - dx \wedge \underbrace{i_X dy}_{dy(X) = X(y) = \frac{\partial y}{\partial x} = 0} \wedge dz + \underbrace{\text{other terms}}_{=0} \Big)$$

$$= \frac{\partial a}{\partial z} dy \wedge dz$$

$$i_X \omega = a \underset{\frac{\partial}{\partial x}}{\overset{}{i_X}} (dx \wedge dy) = a\, dy$$

(proceed as before..)

$$\left[d(i_X \omega) = d(a\, dy) = da \wedge dy = \frac{\partial a}{\partial x} dx \wedge dy + \frac{\partial a}{\partial z} dz \wedge dy \right.$$

$$\left. = \frac{\partial a}{\partial x} dx \wedge dy - \frac{\partial a}{\partial z} dy \wedge dz \right]$$

Therefore $(d\, i_X + i_X d)\, \omega = \frac{\partial a}{\partial x} dx \wedge dy$

Compute $\mathcal{L}_X \omega$ directly:

Leibniz

$$\mathcal{L}_X (a\, dx \wedge dy) = \underbrace{(\mathcal{L}_X a)}_{X(a) = \frac{\partial a}{\partial x}} dx \wedge dy + a \underbrace{\mathcal{L}_X dx}_{d \mathcal{L}_{\frac{\partial}{\partial x}} x = d1 = 0} \wedge dy + a\, dx \wedge \underbrace{\mathcal{L}_X dy}_{=0}$$

$$= \frac{\partial a}{\partial x} dx \wedge dy \quad \checkmark$$

2. $X = \frac{\partial}{\partial x}$ $\qquad \omega = a\, dy \wedge dz$

$$d\omega = da \wedge dy \wedge dz = \frac{\partial a}{\partial x} dx \wedge dy \wedge dz$$

$$i_X d\omega = \ldots = \frac{\partial a}{\partial x} dy \wedge dz$$

$$i_X \omega = \ldots 0 \qquad (a\, dy \wedge dz)(X, \cdot) = a[\underbrace{dy(X)}_{0} dz(\cdot) - dy(\cdot) \underbrace{dz(X)}_{0}]$$

$$d i_X \omega = 0 \qquad (d i_X + i_X d)\, \omega = \frac{\partial a}{\partial x} dy \wedge dz$$

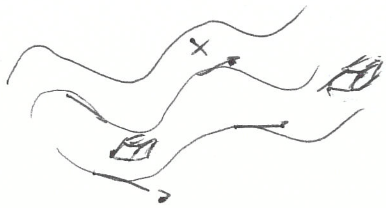
$$\mathcal{L}_X (a\, dy \wedge dz) = \mathcal{L}_X a \cdot dy \wedge dz + \ldots 0 \ldots = \frac{\partial a}{\partial x} dy \wedge dz \quad \checkmark$$

$\boxed{3.}$ in $\mathbb{R}^3$ $\quad \omega = dx \wedge dy \wedge dz$ standard volume form

Let $X = \underbrace{\alpha \frac{\partial}{\partial x}}_{X_1} + \underbrace{\beta \frac{\partial}{\partial y}}_{X_2} + \underbrace{\gamma \frac{\partial}{\partial z}}_{X_3}$ $\quad \alpha, \beta, \gamma \in \mathcal{C}^\infty(\mathbb{R}^3)$

Compute $\mathcal{L}_X \omega = \left(\frac{\partial \alpha}{\partial x} + \frac{\partial \beta}{\partial y} + \frac{\partial \gamma}{\partial z} \right) dx \wedge dy \wedge dz$

(computes variation of volume elements) $\quad$ divergence of X $\quad$ ⚠ a metric is involved



$\mathcal{L}_X \omega = 0$ if $\operatorname{div} X = 0$ $\quad$ X solenoidal (or divergence-free)

defines volume-preserving flows

= incompressible flows

This holds on Riemannian manifolds

Let us perform the calculation; directly:

$\mathcal{L}_X (dx \wedge dy \wedge dz) = \mathcal{L}_X dx \wedge dy \wedge dz + dx \wedge \mathcal{L}_X dy \wedge dz + dx \wedge dy \wedge \mathcal{L}_X dz$

$= d(\mathcal{L}_X x) \wedge dy \wedge dz$ + similar terms

$= d(X(x)) \wedge dy \wedge dz + \dots$ $\quad$ $\boxed{X(x) = \dots = \alpha}$

$= d\alpha \wedge dy \wedge dz + \dots$

$= \frac{\partial \alpha}{\partial x} dx \wedge dy \wedge dz + \dots$ ✓

Use Cartan $\quad \underbrace{d\omega = 0}_{\text{a 4-form on } \mathbb{R}^3} \Rightarrow \mathcal{L}_X \omega = d\, i_X \omega$

$i_{X_1} \omega = \dots = \alpha\, dy \wedge dz \Rightarrow d\, i_{X_1} \omega = \frac{\partial \alpha}{\partial x} dx \wedge dy \wedge dz$

$X_1 = \alpha \frac{\partial}{\partial x}$

Summing up, we achieve the conclusion. ✓

$\boxed{4.}$ Let (M, g) be a Riemannian manifold and $X \in \mathcal{X}(M)$. Compute $\mathcal{L}_X g$. Work locally:

$X = \xi^i \frac{\partial}{\partial x^i}$

(we cannot use Cartan's formula, g is not a 2-form; in fact, it is a (symmetric, positive definite) $(0,2)$-tensor field)

$$\mathcal{L}_X (g_{ij}\, dx^i dx^j) =$$

$$= \underbrace{\mathcal{L}_X (g_{ij})}_{= X(g_{ij})} dx^i dx^j + g_{ij} \underbrace{(\mathcal{L}_X dx^i)}_{= d\mathcal{L}_X x^i = d\, X(x^i) = d\xi^i} dx^j + g_{ij}\, dx^i \underbrace{\mathcal{L}_X dx^j}_{= d\xi^j}$$

$$= X(g_{ij}) dx^i dx^j + g_{ij}\, d\xi^i dx^j + g_{ij}\, dx^i d\xi^j$$

one wants ij

$$= \xi^k \frac{\partial g_{ij}}{\partial x^k} dx^i dx^j + g_{ij} \frac{\partial \xi^i}{\partial x^k} dx^k dx^j + g_{ij}\, dx^i \frac{\partial \xi^j}{\partial x^l} dx^l$$

so relabel $k \to i$, $i \to k$ relabel: $l \to j$, $j \to k$

$$= \xi^k \frac{\partial g_{ij}}{\partial x^k} dx^i dx^j + g_{kj} \frac{\partial \xi^k}{\partial x^i} dx^i dx^j + g_{ik} \frac{\partial \xi^k}{\partial x^j} dx^i dx^j$$

$$= \left(\xi^k \frac{\partial g_{ij}}{\partial x^k} + g_{kj} \frac{\partial \xi^k}{\partial x^i} + g_{ik} \frac{\partial \xi^k}{\partial x^j} \right) dx^i dx^j$$

X is a **Killing** vector field (or, simply, X is Killing) if $\mathcal{L}_X g = 0$ ($\Leftrightarrow X$ generates isometries of (M, g))

$$\boxed{\xi^k \partial_k g_{ij} + g_{kj} \partial_i \xi^k + g_{ik} \partial_j \xi^k = 0}$$

i, j fixed, summation over k

For instance, in $(\mathbb{R}^n, g = \sum (dx^i)^2)$

$$g_{ij} = \delta_{ij}$$

The Killing condition becomes:

$$\left(\delta_{kj} \frac{\partial \xi^k}{\partial x^i} + \delta_{ik} \frac{\partial \xi^k}{\partial x^j} \right) = 0$$

i.e.

$$\boxed{\frac{\partial \xi^j}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} = 0}$$

in particular

$$\frac{\partial \xi^i}{\partial x^i} = 0$$

Constant vector fields and generators of rotations in the plane (i,j), namely $x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$,
$i \neq j$

i, j fixed

are examples of Killing fields:

$$\xi^i = -x^j \qquad \xi^j = x^i$$

$$\frac{\partial \xi^j}{\partial x^i} + \frac{\partial \xi^i}{\partial x^j} = \frac{\partial x^i}{\partial x^i} - \frac{\partial x^j}{\partial x^j} = 1 - 1 = 0$$