Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XXIV

FROBENIUS' THEORY

∘ distributions & their integrability

★ On the Frobenius theorem

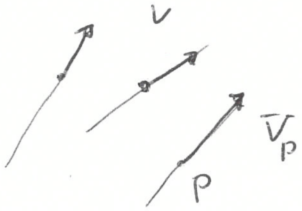
Prelude: Working in $\mathbb{R}^2$, for simplicity, the problem

$$\begin{cases} \dot{x} = X \\ \dot{y} = Y \end{cases} \qquad V = (X, Y) \text{ vector field}$$

consists in the following:

given a direction field, determined, pointwise, by a vector field $\vec{V}$ (one has $\langle \vec{V}_p \rangle \leq \mathbb{R}^2$)

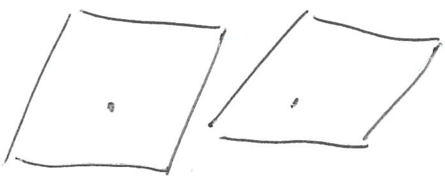
subspace



find its integral curves, i.e. curves such that whose tangents at each point have the given direction.

The Cauchy-Lipschitz theorem asserts that, at least locally, the problem admits a (unique) solution.

Let us generalize. Work for instance in $\mathbb{R}^3$, and assign a distribution of planes $p \longmapsto \Delta_p$ in $\mathbb{R}^3$



Does there exist, $\forall p$, an integral manifold of the above distribution, namely, a surface Σ_p such that $T_p \Sigma_p = \Delta_p \ \forall p$?

[In order to identify Δ_p, let us assign a pair of vector fields (X_1, X_2) such that, locally $(X_1(q), X_2(q))$ yield a basis of Δ_q:

$\mathrm{span}(X_1(q), X_2(q)) = \Delta_q$

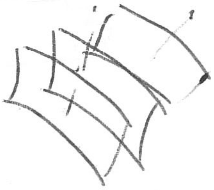
To be concrete, let $\vec{V} \in \mathfrak{X}(\mathbb{R}^3)$ be a vector field. Consider, at each point $p \in \mathbb{R}^3$, the plane π_p through p perpendicular to $\vec{V}(p)$.

One obtains a distribution of planes. We are going to discuss a <u>necessary</u> and <u>sufficient condition</u> ensuring that

$$\pi_p = T_p \Sigma_p$$

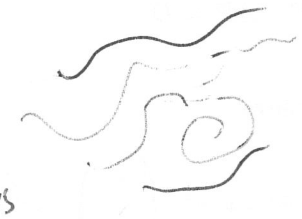
surface

In <u>fluid mechanics</u>, this is the issue of <u>laminarity</u>.



laminar flow

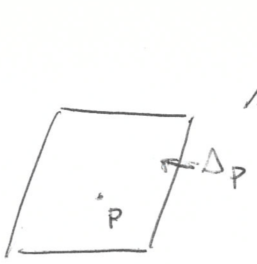
vs



turbulent flow

The answer is NO, in general

Let us formalize the preceding discussion

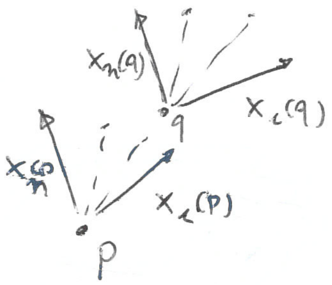


Let M be a smooth manifold, $\dim M = n+k$.

For all $p \in M$, let $\Delta_p \leq T_pM$, $\dim \Delta_p = n$. Consider the map

$$\Delta : \quad M \ni p \longmapsto \Delta_p \leq T_pM$$

Assume that $\forall p \in M$, $\exists$ $U \ni p$ and $X_1, \dots, X_n \in \mathfrak{X}(U)$ such that, $\forall q \in U$, $(X_1(q) \dots X_n(q))$ provide a basis for Δ_q. Δ is then termed smooth distribution of dimension n (smooth n-dimensional distribution) and $(X_1 \dots X_n)$ gives a local basis thereof.



Def. Δ is called involutive if, in a suitable
involutoria

neighbourhood of each point, there exists a local basis $(X_1 \dots X_n)$ of Δ such that, for suitable $C^k_{ij} \in C^\infty(M)$,

functions

one has
$$\boxed{[X_i, X_j] = C^k_{ij} X_k}$$
Einstein

Lie bracket

i.e. Δ is closed (locally) with respect to the Lie bracket of vector fields.

Remark The above concept is basis-independent, i.e.

given a local basis $(X_1 \dots X_n)$ for Δ with the property

$$[X_i, X_j] = c^k_{ij} X_k \qquad (*)$$

(i.e. Δ is involutive), then the same holds for any other local basis:

let $\quad Y_i = \beta^j_i X_j$; then

$$[Y_i, Y_j] = [\beta^l_i X_l, \beta^h_j X_h] =$$

$$\beta^l_i X_l(\beta^h_j X_h) - \beta^h_j X_h(\beta^l_i X_l) =$$

$$= \beta^l_i X_l(\beta^h_j) X_h - \beta^h_j X_h(\beta^l_i) X_l$$

$$+ \beta^l_i \beta^h_j [X_h, X_l] = \quad (\text{use } (*))$$

$$= \beta^l_i X_l(\beta^h_j) X_h - \beta^h_j X_h(\beta^l_i) X_l$$

$$+ \beta^l_i \beta^h_j c^p_{hl} X_p \quad = \text{l.c.} (X_1 \dots X_n)$$

function coefficients

(Y_i) is a local basis

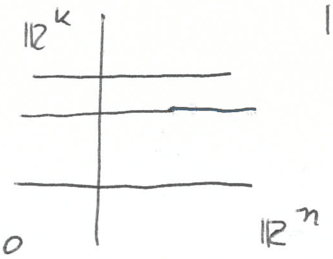
$$= \text{l.c.} (Y_1 \dots Y_n)$$

i.e. $\quad [Y_i, Y_j] = d^k_{ij} Y_k$

for suitable smooth functions d^k_{ij}

Typical example : in $\mathbb{R}^m$ $\qquad m = n + k$

$$\Delta = \left\langle \frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right\rangle$$

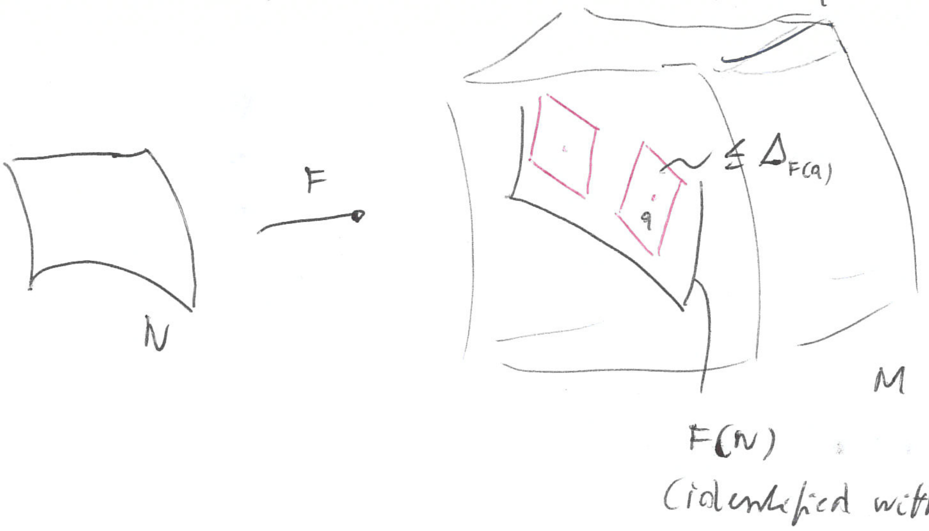


Notice that in this case

$$c^k_{ij} \equiv 0 \qquad \left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right] = 0$$

the $\frac{\partial}{\partial x^i}$ commute

Def. (i) An integral submanifold of Δ is a manifold N such that, if $F : N \to M$ is an injective immersion, one has, $\forall q \in N$, $F_*(T_q(N)) \leq \Delta_{F(q)}$

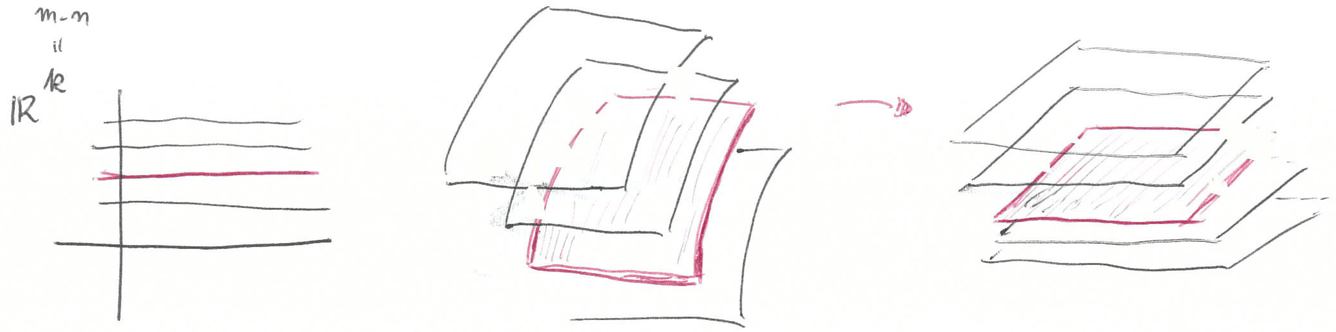


(ii) Δ is called completely integrable if, $\forall$ p M, $\exists$ a local coordinate system (defined on $U \ni p$)

$$\left(\frac{\partial}{\partial x^1}, \dots, \frac{\partial}{\partial x^n} \right) \text{ giving a local basis for } \Delta.$$

This being the case, there exists N integral submanifold with $T_q N = \Delta_q \quad \forall q \in U$, defined by $\{ x^i = a^i, \ i = n+1, \dots m \}$

i.e. N is an n-slice of U



Notice that Δ is clearly involutive.
The basic result governing the theory is the following:

** Theorem (Frobenius) Δ is involutive if and only if it is completely integrable

Comment: ($\Leftarrow$) is clear in view of the preceding discussion.
Also notice that if $N \subset M$ is a submanifold of M, $\mathcal{X}(N)$ is a Lie algebra. So the condition is necessary.
The crucial point is to prove sufficiency, which can be ascertained by induction and using the fact that an involutive distribution can be generated pointwise by commuting vector fields.

see chapter on symplectic geometry

*** Application. The Liouville - Arnol'd theorem in mechanics.

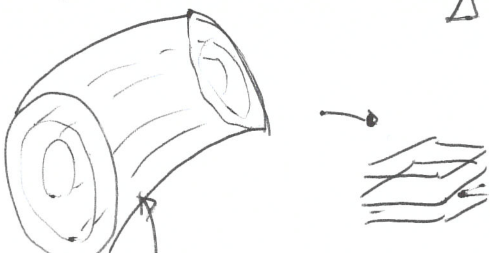
Given a Hamiltonian system (M, ω, H) $\dim M = 2n$
$\{f_i\}_{i=1..n}$ first integrals in involution ($\{f_i, f_j\} = 0$
($\Rightarrow [X_{f_i}, X_{f_j}] = 0$) which are
also functionally independent, the X_{f_i} yield a completely integrable distribution, whose integral submanifolds, under suitable conditions, are tori (Liouville or Lagrangian tori)

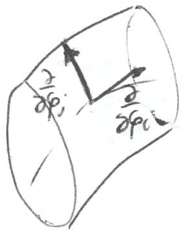
one can find local action-angle variables

N ~ $(\varphi_1 \dots \varphi_n, I_1 \dots I_n)$
angles actions

$\omega = d\varphi \wedge dI \quad (= \sum d\varphi_i \wedge dI_i)$

$\Delta = \left\{ \frac{\partial}{\partial \varphi_i} \right\}_{i=1\dots n}$





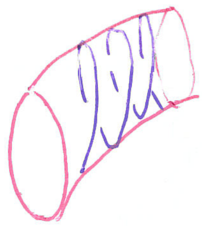
torus labelled by I, with angle coordinates on it (Lagrangian, or Liouville torus)

Hamilton's equations become: $\left\{ \begin{array}{l} \dot{I}_i = 0 \Rightarrow I_i = \text{constant} \\ \dot{\varphi}_i = c_i \quad (\text{constant}) \end{array} \right.$

and are indeed immediately integrated → $\varphi_i = \varphi_0^i + t c_i$

linearization of the flow

Example: the harmonic oscillator.



the trajectory winds on a fixed Lagrangian (or Liouville) torus

"quasiperiodicity"

Another very important example   See Chapter on Lie groups

Let G be a Lie group, with Lie algebra $\mathfrak{g}$

H a Lie subgroup of G, with Lie algebra $\mathfrak{h}$
(it is enough to require H closed, then it will be automatically a Lie group)

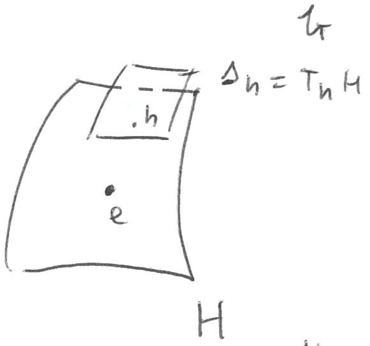
$\mathfrak{h} = \{$ left invariant vector fields on G and tangent to H in e $\}$

neutral element

$\mathfrak{h}$ is a Lie subalgebra of $\mathfrak{g}$

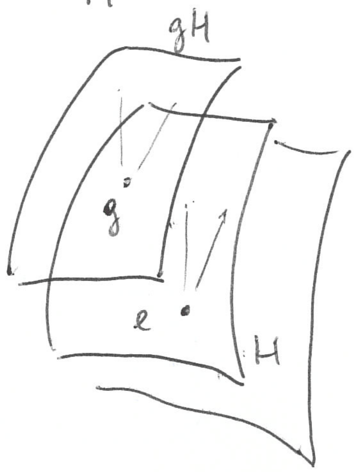
So there exists Δ, distribution on G such that $\Delta_h = T_h H$ and, in general,

$$\Delta_g = T_g(gH)$$



By Frobenius' theorem, Δ is completely integrable and its integral submanifolds are precisely the cosets $gH = \{ gh \,/\, h \in H, \ g \in G \text{ fixed} \}$

classi laterali



Note: if H is closed, gH is an embedded submanifold of G and the quotient space G/H will be a manifold as well.