Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY

V2

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Lecture XXIX

ADJOINT REPRESENTATION
STRUCTURE CONSTANTS OF A LIE ALGEBRA
SO(3)

* Adjoint representation of a Lie group G on its Lie algebra $\mathfrak{g}$

It is defined as follows:

$$\boxed{\underset{\underset{G}{\cap}}{\mathrm{Ad}\, h} \cdot \underset{\underset{\mathfrak{g}}{\cap}}{X} := \underset{\underset{\mathfrak{g}}{\cap}}{(R_{h^{-1}})_* X}}$$

← to be checked!

Let us check that, indeed, the r.h.s. belongs to $\mathfrak{g}$ (i.e. it is left-invariant)

Recall:

$R_h g = g \cdot h$

$L_h g = h \cdot g$

$(R_h \cdot R_g)x = R_h(xg) = (xg)h = x(gh)$

$= R_{gh} \cdot x$

$L_g R_h x = g x h = R_h L_g x$

(left and right translations commute)

$$(L_g)_* \left[(R_{h^{-1}})_* X \right] = (L_g \circ R_{h^{-1}})_* X$$

$$= (R_{h^{-1}} \circ L_g)_* X = (R_{h^{-1}})_* [\underbrace{(L_g)_* X}_{X}] = (R_{h^{-1}})_* X .$$

Also,

$$\mathrm{Ad} : G \longrightarrow \mathrm{Aut}(\mathfrak{g})$$

← invertible endomorphisms of $\mathfrak{g}$

is indeed a group representation (i.e. a group homomorphism)

namely:

$$\boxed{\mathrm{Ad}(g_1 \cdot g_2) = \mathrm{Ad}\, g_1 \cdot \mathrm{Ad}\, g_2}$$

product in G — operator product (composition)

Indeed:

$$\mathrm{Ad}(g_1 g_2)\, \overset{\substack{\mathfrak{g}\\ \cup}}{X} = \left(R_{(g_1 g_2)^{-1}}\right)_* X =$$

$$= \left(R_{g_2^{-1} g_1^{-1}}\right)_* X \underset{\text{beware!}}{=} \left(R_{g_1^{-1}} \cdot R_{g_2^{-1}}\right)_* X$$

$$= \underset{\text{(Chain rule)}}{\left(R_{g_1^{-1}}\right)_* \cdot \left(R_{g_2^{-1}}\right)_*} X = \mathrm{Ad}\, g_1 \cdot \mathrm{Ad}\, g_2 \cdot X$$

At matrix level, everything is simpler:
(G matrix group)

$$\mathrm{Ad}\, g \underset{\substack{\cap\\ \mathfrak{g} \cong T_e G}}{X} = \underbrace{g X}_{X^\#(g)} g^{-1} \qquad X^\#(e) = X$$

Also notice that

$$\mathrm{Ad}\, g\ X = \frac{d}{dt}\left(g \cdot \exp tX \cdot g^{-1}\right)\Big|_{t=0}$$

* <u>Ad</u>joint <u>re</u>pre<u>sentation</u> of $\mathfrak{g}$ <u>on</u> <u>itself</u>:

$$\mathrm{ad}\, \underset{\substack{\cap\\ \mathfrak{g}}}{X} \cdot \underset{\substack{\cap\\ \mathfrak{g}}}{Y} := \underset{\substack{\cap\\ \mathfrak{g}}}{[X, Y]} \qquad \mathrm{ad}\, X \in \mathrm{End}(\mathfrak{g})$$

ad X is linear (obvious) and it is indeed a Lie algebra morphism, i.e.

$$\underset{\substack{\uparrow \\ \text{Lie algebra} \\ \text{bracket}}}{\mathrm{ad}\,[X,Y]} = [\mathrm{ad}\,X \,,\, \mathrm{ad}\,Y]$$

$\nwarrow$ operator commutator
(concretely: matrix commutator)

This is straightforward: $\forall\, Z \in \mathfrak{g}$, we have

$$① := \mathrm{ad}\,[X,Y]\,Z \underset{\text{def}}{=} [[X,Y],Z] = -[Z,[X,Y]].$$

But

$$② := [\mathrm{ad}\,X \,,\, \mathrm{ad}\,Y]\,Z = \mathrm{ad}\,X \cdot \mathrm{ad}\,Y \cdot Z - \mathrm{ad}\,Y \cdot \mathrm{ad}\,X \cdot Z$$

$$= [X,[Y,Z]] - [Y,[X,Z]]$$

$$= [X,[Y,Z]] + [Y[Z,X]] = ①$$ by the Jacobi identity.

Notice that (for matrix groups)

$$\underset{\substack{\| \\ [X,Y]}}{\mathrm{ad}\,X} \cdot Y = \frac{d}{dt}\left(\mathrm{Ad}\,e^{tX} \cdot Y\right)\Big|_{t=0} \qquad X, Y \in \mathfrak{g}$$

In fact $\displaystyle \frac{d}{dt}\left(\mathrm{Ad}\,e^{tX}\,Y\right)\Big|_{t=0} = \frac{d}{dt}\left(e^{tX}\,Y\,e^{-tX}\right)\Big|_{t=0}$

$$= XY - YX = [X,Y] = \mathrm{ad}\,X \cdot Y$$

that is:

☆ <u>ad</u> <u>is</u> <u>the</u> <u>differential</u> of <u>Ad</u> (at e).

More generally, the calculation runs as follows:

$$\mathrm{Ad}\,(\exp tx)\,Y_e = (R_{\exp -tx})_* \underbrace{(L_{\exp tx})_* Y_e}_{Y_{\exp tx}} \qquad (\text{l. invariance})$$

$$= (R_{\exp(-tx)})_* Y_{\exp tx}$$

$$= \underbrace{(F^X_{-t})_* Y_{F^X_t(e)}}_{\diamond_t}$$

$$\begin{aligned} F^X_t(g) &= g\cdot F^X_t(e) \\ &= R_{F^X_t(e)}\, g \end{aligned}$$

But $$\frac{d}{dt}(\diamond_t)\Big|_{t=0} \underset{\text{def}}{=} (\mathcal{L}_X Y)_e = [X,Y]_e$$

whence the conclusion.

* structure constants of a Lie algebra

Given a Lie algebra $(L, [\,,\,])$ (finite dimensional)

and given a basis $(X_i)_{i=1,\dots n}$ $n = \dim L$ (as a vector space)

one has

$$[X_i, X_j] = c^k_{ij} X_k \qquad \text{(Einstein's convention)}$$

the c^k_{ij}'s are called structure constants of L

Given a Lie algebra automorphism of L

$(T \in gl(L) , \quad T[X,Y] = [TX, TY])$

it is immediately checked that the structure constants are preserved! if $Y_i = TX_i$ (one gets a new basis for L)

(and this justifies their name):

$$[Y_i, Y_j] = [TX_i, TX_j] = T([X_i, X_j]) = T(c^k_{ij} X_k)$$

$$= c^k_{ij} TX_k = c^k_{ij} Y_k .$$

The Jacobi identity reads, in terms of c^k_{ij}:

$$[X_i[X_j, X_k]] + [X_j, [X_k, X_i]] + [X_k[X_i, X_j]] = 0$$

$$[X_i, c^l_{jk} X_l] + [X_j, c^m_{ki} X_m] + [X_k, c^h_{ij} X_h] = 0$$

$$c^l_{jk} c^p_{il} X_p + c^m_{ki} c^p_{jm} X_p + c^h_{ij} c^p_{kh} X_p = 0$$

$\forall p = 1..n$

$$\Rightarrow \boxed{c^l_{jk} c^p_{il} + c^m_{ki} c^p_{jm} + c^h_{ij} c^p_{kh} = 0}$$

$\forall p, i, j, k$ fixed.

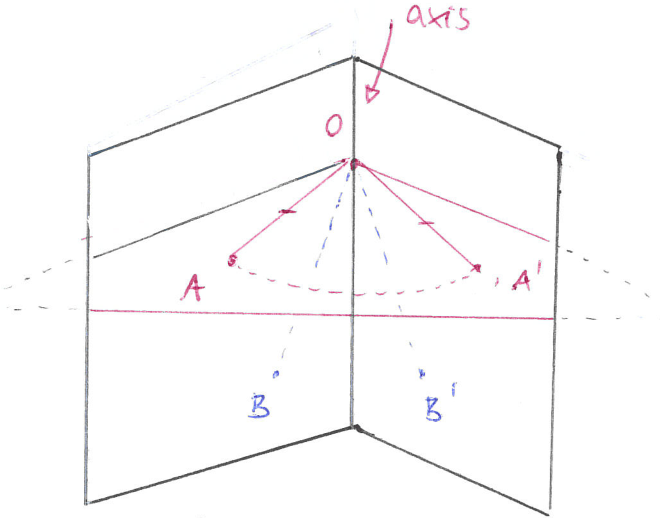
(sum over l, m, h)

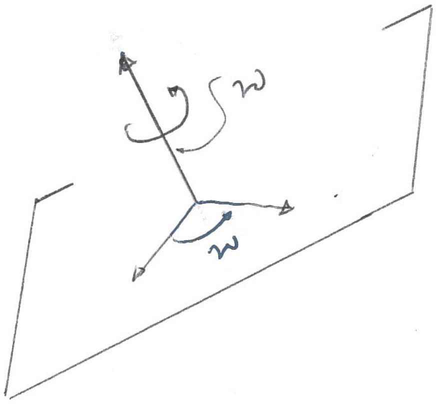
★ A digression on SO(3)

Recall that $SO(3) = \{ O \in M_3(\mathbb{R}) \,/\, O^T O = O O^T = I \,/\, \det O = 1 \}$

special orthogonal group

It is well known that $\lambda = 1$ is an eigenvalue of any $O \in SO(3)$. As a consequence, any $O \in SO(3)$ is a rotation around an axis (this is also straightforward via a direct geometric argument: a special orthogonal matrix sends planes (through 0) to planes through 0, and preserves distances; it must then fix the intersection of corresponding planes, which is then the axis of the rotation.





Consequently SO(3), as a topological space(+), can be viewed as a ball of radius π in $\mathbb{R}^3$, with antipodal points of the boundary 2-sphere identified, since they give rise to the same rotation. But the latter is homeomorphic to the real projective space $\mathbb{R}\mathbb{P}^3$ (cf. the analogous assertion for $\mathbb{R}\mathbb{P}^2$, realised as a disc with antipodal boundary points identified).

(+) the radius can then be set to be 1.

In more detail:

$$S^3 = \{ x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \}$$

$$\mathbb{R}\mathbb{P}^3 = S^3/\sim \quad + \quad \text{identification of} \quad \underset{(x_0, x_1, x_2, x_3)}{\underset{\parallel}{x}} \quad \text{and} \quad -x$$

Consider

$$x_0^2 + x_1^2 + x_2^2 = 1 - x_3^2 \equiv R^2 \leq 1$$

If $x_3 = 0$, you have a sphere of radius 1 and identification of antipodal points.

If $x_3 \neq 0$, then $x_0^2 + x_1^2 + x_2^2 = R^2 < 1$, and no further identification.

$$\star \quad \boxed{(\mathbb{R}^3, \times) \cong (so(3), [\,,\,])}$$

as Lie algebras ↖ skew-symmetric 3×3 matrices

The correspondence is given as follows

$$(\omega_1, \omega_2, \omega_3) \longleftrightarrow \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}$$

example: $\underline{i} = (1, 0, 0) \longmapsto X_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$

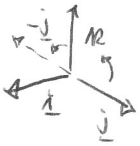
$$\underbrace{\frac{d}{dt} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix} \Bigg|_{t=0}}_{\text{rotations around } \underline{i}} = \underbrace{X_1}_{\text{infinitesimal generator}}$$

corresponds to

axis: $\underline{i} \mapsto \underline{0}$

$\underline{j} \mapsto \underline{k}$

$\underline{k} \mapsto -\underline{j}$



One easily checks that

$\underline{i} \times \underline{j} = \underline{k}$ becomes $[X_1, X_2] = X_3$

etc.

(via Dirac's notation: $X_1 = -|2\rangle\langle 3| + |3\rangle\langle 2|$

$X_2 = |1\rangle\langle 3| - |3\rangle\langle 1|$

$X_1 X_2 = |2\rangle\langle 1|$; $X_2 X_1 = |1\rangle\langle 2|$

$[X_1, X_2] = |2\rangle\langle 1| - |1\rangle\langle 2| = X_3$

In general $[X_i, X_j] = \varepsilon_{ijk} X_k$

↖ Levi-Civita tensor

structure constants of $so(3)$

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{even perm. of } (1\,2\,3) \\ -1 & \text{odd perm.} \\ 0 & \text{otherwise} \end{cases}$$

see below