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Lecture XXV

FROBENIUS' THEORY (à la Pfaff)

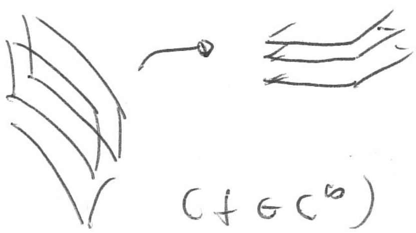
* Integrability and differential forms

Let us apply, in a very special case, Cartan's calculus to the integrability problem, in order to illustrate its tremendous power.

Let us work in $\mathbb{R}^3$, and let a plane distribution be given; it can be described as the kernel of a 1-form θ: $\theta = 0$ i.e.

$$\boxed{\theta_1\, dx + \theta_2\, dy + \theta_3\, dz = 0}$$

In view of the Frobenius theorem, the above distribution is integrable $\iff$ there exist, in a neighbourhood of any point, local coordinates (ξ, η, ζ) such that the integral manifolds through every point of the neighbourhood have the form $\zeta = c$



The condition $\theta = 0$ translates to $f(\xi, \eta, \zeta)\, d\zeta = 0$ ($f \in C^\infty$). Now compute:

$$d\theta = \frac{\partial f}{\partial \xi}\, d\xi \wedge d\zeta + \frac{\partial f}{\partial \eta}\, d\eta \wedge d\zeta$$

, and observe that

$$\theta \wedge d\theta = f\, d\zeta \wedge \left(\frac{\partial f}{\partial \xi} \cdot d\xi \wedge d\zeta + \frac{\partial f}{\partial \eta} \cdot d\eta \wedge d\zeta \right) = 0$$

$$\boxed{\theta \wedge d\theta = 0}$$

This condition is intrinsic, and translates this particular instance of Frobenius theorem. Indeed, it is also a sufficient condition for integrability

Reverting to Cartesian coordinates, one has:

$$(\theta_1\, dx + \theta_2\, dy + \theta_3\, dz) \wedge \Big[\left(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z} \right) dy \wedge dz + \left(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x} \right) dz \wedge dx + \left(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y} \right) dx \wedge dy \Big] = 0$$

namely

$$\theta_1 \left(\frac{\partial \theta_3}{\partial y} - \frac{\partial \theta_2}{\partial z} \right) + \theta_2 \left(\frac{\partial \theta_1}{\partial z} - \frac{\partial \theta_3}{\partial x} \right) + \theta_3 \left(\frac{\partial \theta_2}{\partial x} - \frac{\partial \theta_1}{\partial y} \right) = 0$$

in vector analysis notation: $\theta \rightsquigarrow \underline{v}$

one gets $\boxed{\underline{v} \cdot \mathrm{curl}\, \underline{v} = 0}$

(velocity $\perp$ vorticity) : this ensures the laminar character of a fluid (which proceeds with "fronts")

Remark

Also notice that, upon changing w to $\lambda \cdot w$, λ smooth, $\lambda \neq 0$ one still has

$$\lambda w \wedge d(\lambda w) = \lambda w \wedge (d\lambda \wedge w) + \lambda\, dw)$$

$$= \lambda \underbrace{w \wedge d\lambda}_{1\text{ forms}} \wedge w + \lambda^2 \underbrace{w \wedge dw}_{=0}$$

$$= -\lambda\, d\lambda \wedge \underbrace{w \wedge w}_{=0} = 0$$

This is of course as it should be, since w and λw have the same kernel, and give rise to the same distribution.

(the condition $w \wedge dw = 0$ is intrinsic)

✧ The integrability condition for a distribution of planes in ordinary space can be also obtained in the following manner:

Start from $\theta = 0$ $\qquad \theta \in \Lambda^1(\mathbb{R}^3)$

Integrability amounts to the existence of $g \neq 0$ ("integrating factor") - a smooth function - such that

$$(*) \qquad g\theta = d\mathcal{Y} \qquad \mathcal{Y} \in C^\infty(\mathbb{R}^3)$$

(whence the integral submanifolds will be given by the level surfaces $\mathcal{Y} = c$)

[Equivalently $\quad \theta = h\, d\mathcal{Y} \quad$ (for $h \neq 0$; $h = g^{-1}$)]

Differentiating $(*)$ we get

$$d(g\theta) = dg \wedge \theta + g\, d\theta = d^2\mathcal{Y} = 0, \text{ that is}$$

$$dg \wedge \theta + g\, d\theta = 0$$

multiplying on the right by θ, we have

$$dg \wedge \underbrace{\theta \wedge \theta}_{0} + g\, d\theta \wedge \theta = 0$$

whence $(g \neq 0)$ $\qquad \boxed{d\theta \wedge \theta = 0 \qquad (**)}$

Thus $(**)$ is a necessary condition for integrability. Its sufficiency is tantamount to Frobenius' theorem

("contact structure")

★ Example $\omega = x\,dy + dz$

kernel: $(x\,dy + dz)\left(\underbrace{\alpha}_{\text{functions}} \frac{\partial}{\partial x} + \beta \frac{\partial}{\partial y} + \gamma \frac{\partial}{\partial z}\right) = 0$

$\Rightarrow \quad x\beta + \gamma = 0 \quad \Rightarrow \quad \gamma = -\beta x$

$\Rightarrow$ a local basis for the distribution Δ is, for instance $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right)$

Let us compute $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y} - x\frac{\partial}{\partial z}\right] =$

$$= \underbrace{\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right]}_{= 0} + \left[\frac{\partial}{\partial x}, -x\frac{\partial}{\partial z}\right]$$

$\ldots = \frac{\partial(-x)}{\partial x}\frac{\partial}{\partial z} = -\frac{\partial}{\partial z}$ which is not

shorthand computation!

a linear combination (with function coefficients) of $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y} - x\frac{\partial}{\partial z}$

$\Rightarrow \Delta$ is not integrable. Let us obtain the same conclusion via Cartan's calculus (it is immediate).

$d\omega = dx \wedge dy \quad ; \quad \omega \wedge d\omega = (x\,dy + dz) \wedge dx \wedge dy$

$= dz \wedge dx \wedge dy = dx \wedge dy \wedge dz \neq 0$. (!)

* Generalization (Pfaff systems)

Let Δ_k be a distribution of order k, in M, $\dim M = n$ given by the ~~in~~ intersection of kernels of $n-k$ 1-forms

$$\Delta_k : \begin{cases} \omega_1 = 0 \\ \omega_2 = 0 \\ \omega_{n-k} = 0 \end{cases}$$

The differential form version of the Frobenius theorem states that Δ_k is integrable if and only if

$$\boxed{(\heartsuit) \qquad d\omega_j \wedge \omega_1 \wedge \dots \wedge \omega_{n-k} = 0 \qquad \forall\, j = 1, \dots, n-k}$$

If $\dim M = n = 3$, $k = 2$, we recover the previous condition

Let us check the necessity of $(\heartsuit)$, for $n = 4$, $k = 2$, ~~in~~ for simplicity.

If $\Delta = \Delta_2$ is integrable, let (x_1, x_2, ξ_1, ξ_2) be a local coord. system such that the integral submanifolds are given by $\xi_1 = c_1$, $\xi_2 = c_2$ (and described by coordinates x_i)

Then ω_1 is of the form

smooth functions

$$\omega_1 = f_1^{(1)}(x,\xi)\, d\xi_1 + f_2^{(1)}(x,\xi)\, d\xi_2$$

(the kernel of w_1 containing, at each point $\frac{\partial}{\partial x^i}$, $i = 1, 2$.)

Similarly $\omega_2 = f_1^{(2)}(x,\xi)\, d\xi_1 + f_2^{(2)}(x\, \xi_2)\, d\xi_2$

Also $\quad d\omega_1 = df^{(1)}_1 \wedge d\xi_1 + df^{(1)}_2 \wedge d\xi_2$

$\quad\quad\quad d\omega_2 = df^{(2)}_1 \wedge d\xi_1 + df^{(2)}_2 \wedge d\xi_2$

Therefore:

$$d\omega_1 \wedge \omega_1 \wedge \omega_2 = \boxed{\begin{array}{c}(df^{(1)}_1 \wedge d\xi_1 + df^{(1)}_2 \wedge d\xi_2) \\ \wedge \\ (f^{(1)}_1 d\xi_1 + f^{(1)}_2 d\xi_2) \\ \wedge \\ (f^{(2)}_1 d\xi_1 + f^{(2)}_2 d\xi_2)\end{array}}$$

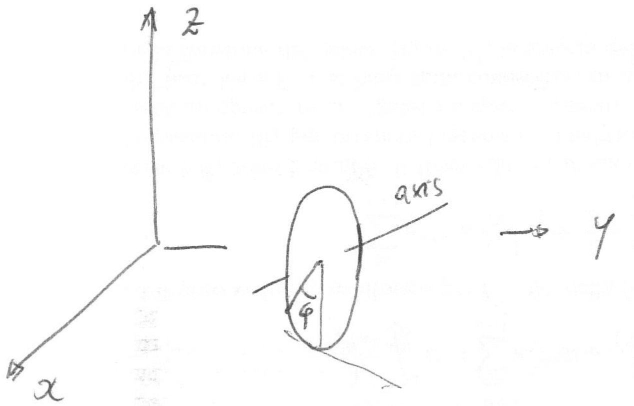
$\ldots = 0$

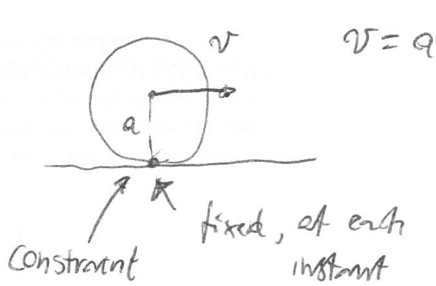
(in each summand, at least two equal differentials $d\xi_i$ occur)

Similarly $\quad d\omega_2 \wedge \omega_1 \wedge \omega_2 = 0$

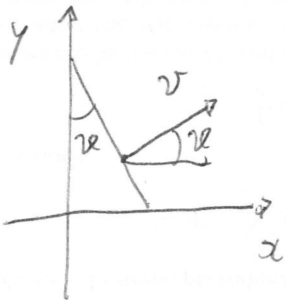
(and it is clear that the argument works in general).

7 A mechanical example: a vertical disc rolling on a plane without sliding





$v = a\dot\varphi$



$\dot x = v\cos\vartheta = a\cos\vartheta\,\dot\varphi$

$\dot y = v\sin\vartheta = a\sin\vartheta\,\dot\varphi$

$$\begin{cases} \overbrace{dx - a\cos\vartheta\, d\varphi}^{w_1} = 0 \\ \underbrace{dy - a\sin\vartheta\, d\varphi}_{w_2} = 0 \end{cases}$$

Pfaff system

$\begin{cases} w_1 = 0 \\ w_2 = 0 \end{cases}$ in $\mathbb{R}^4$ (coordinates: x, y, ϑ, φ)

$dw_1 = a\sin\vartheta\, d\vartheta \wedge d\varphi$

$dw_2 = -a\cos\vartheta\, d\vartheta \wedge d\varphi$

$w_1 \wedge w_2 = dx \wedge dy - a\cos\vartheta\, d\varphi \wedge dy - a\sin\vartheta\, dx \wedge d\varphi$

$dw_1 \wedge w_1 \wedge w_2 = a\sin\vartheta\, d\vartheta \wedge d\varphi \wedge dx \wedge dy \neq 0$

$dw_2 \wedge w_1 \wedge w_2 = -a\cos\vartheta\, d\vartheta \wedge d\varphi \wedge dx \wedge dy \neq 0$

This is an example of an anholonomic constraint (not integrable).