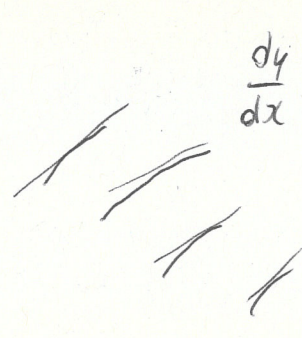


★ Euler's integrating factor
(where it all began...)

consider $y' = f(x, y)$ (*)

Lecture
XXVI



Rewrite it as $\frac{dy}{dx} = f(x, y)$

$$\omega = dy - f dx = 0$$

INTEGRABILITY OF
ODE & PDE
REVISITED

in general ω is not closed

$$d\omega = -df \wedge dx = -f_y dy \wedge dx = f_y dx \wedge dy$$

$$\neq 0 \text{ if } f_y \neq 0$$

However, multiplication by $\xi = \xi(x, y) \neq 0$ does not change the field of directions:

$$\tilde{\omega} = \sum \xi^Q dy - \sum \xi^P f dx = 0$$

This being the case,
 $y' = f(x)$ would be
integrated $y = \int f(x) dx$
immediately:

primitive

now, the requirement $d\tilde{\omega} = 0 \Rightarrow \tilde{\omega} = d\psi$

i.e. $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ yields a partial differential equation for ξ

★ Poincaré lemma

which in many cases can be solved explicitly

(ξ is called integrating factor)

Given ξ and given a primitive ψ , the solutions of (*) are given by the level curves $\psi = C$

Example

$$\boxed{y' = P(x)y + Q(x)}$$

In homogeneous
linear equations

(or non-homogeneous)

$$w = dy - [P(x)y + Q(x)]dx = 0$$

$$\tilde{w} = \xi w = \int \left(dy - [P(x)y + Q(x)] \right) \xi dx = 0$$

$$w = P dx + Q dy$$

$$dw = \frac{\partial Q}{\partial x} dx - \frac{\partial P}{\partial y} dy$$

integrability: $d\tilde{w} = 0$, yielding

$$\frac{\partial (P(x)y + Q(x))\xi}{\partial y} = - \frac{\partial \xi}{\partial x}$$

★ look for $\xi = \xi(x)$ ($\Rightarrow \frac{\partial \xi}{\partial y} = 0$)

we find

$$-P(x) = \frac{\partial \xi}{\partial x} \Rightarrow \xi(x) = e^{-\int P(x) dx}$$

then $\xi = \frac{1}{f(x)}$

$$(-P y - Q(x))\xi = \frac{1}{f(x)}$$

$$f(x, y) = e^{-\int P(x) dx} \cdot y + \psi(x)$$

$$f_x = -e^{-\int P(x) dx} P(x) y + \psi'(x)$$

"

$$-P y \xi - Q(x) \xi$$

$$\Rightarrow \psi' = -Q(x) \xi$$

$$\psi' = -Q(x) e^{-\int P(x) dx}$$

$$\psi(x) = - \int Q(x) e^{-\int P(x) dx} dx$$

So, eventually

$$y'(x, y) = c = e^{-\int P(x) dx} y - \int Q(x) e^{-\int P(x) dx} dx$$

$$\Rightarrow y = \underbrace{c e^{\int P(x) dx}}_{\substack{\uparrow \\ \text{solution of the} \\ \text{homogeneous equation}}} + \underbrace{\int \left\{ Q(x) e^{-\int P(x) dx} \right\} dx}_{\text{special solution}} \cdot e^{\int P(x) dx}$$

$$\left(\text{check: } y' = \cancel{Q(x) e^{-\int P(x) dx}} \cdot \cancel{e^{\int P(x) dx}} + y \cdot P(x) \quad \checkmark \right)$$

* Pfaff system approach to

$$\begin{cases} z_x = f(x, y, z) \\ z_y = g(x, y, z) \end{cases} \quad z = z(x, y)$$

$$dz = z_x dx + z_y dy = f dx + g dy$$

$$w := f dx + g dy - dz = 0$$

$$dw = df \wedge dx + dg \wedge dy =$$

$$= f_y dy \wedge dx + f_z dz \wedge dx$$

$$+ g_x dx \wedge dy + g_z dz \wedge dy$$

$$= (g_x - f_y) dx \wedge dy + f_z dz \wedge dx + g_z dz \wedge dy$$

$$w \wedge dw = -(g_x - f_y) dz \wedge dx \wedge dy + g f_z dy \wedge dz \wedge dx + f g_z dx \wedge dz \wedge dy =$$

$$= (f_y - g_x + g f_z - f g_z) dz \wedge dx \wedge dy$$

" $dx \wedge dy \wedge dz$

integrability:

$$w \wedge dw = 0$$

$$\Rightarrow f_y - g_x + g f_z - f g_z = 0$$

$$(*) \quad f_y + g f_z = g_x + f g_z$$

that is

$$(*) = (*')$$

* $z_{xy} = z_{yx}$ yields

* Schwarz

$$f_y + \underset{g}{f_z} \cdot \underset{g}{z_y} = g_x + g_z \cdot \overset{f}{z_x}$$

XXVI.1=4