Lectures on
DIFFERENTIAL GEOMETRY AND TOPOLOGY

Prof. M. Spera
UCSC Brescia

Lecture XXVI

INTEGRABILITY OF ODE & PDE REVISITED

Euler's integrating factor
(where it all began...)

consider $\boxed{y' = f(x,y)}$ (★)

$\frac{dy}{dx}$ Rewrite it as $' = \frac{d}{dx}$

$$\omega = dy - f\,dx = 0$$

in general ω is not closed

$$d\omega = -df \wedge dx = -f_y\, dy \wedge dx = f_y\, dx \wedge dy$$

$\neq 0$ if $f_y \neq 0$

However, multiplication by $\xi = \xi(x,y) \neq 0$ does not change the field of directions:

$$\tilde{\omega} = \overset{Q}{\tilde{\xi}}\, dy - \overset{P}{\widetilde{\xi f}}\, dx = 0$$

this being the case, $y' = f(x)$ would be integrated immediately: $y = \int f(x)\,dx$

now, the requirement $d\tilde{\omega} = 0$ ($\Rightarrow \tilde{\omega} = d\mathcal{Y}$ — primitive ★ Poincaré lemma

i.e. $\boxed{\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}}$ yields a partial differential equation for ξ

which in many cases can be solved explicitly

(ξ is called integrating factor)

Given ξ and given a primitive $\mathcal{Y}$, the solutions of (★) are given by the level curves $\mathcal{Y} = c$

Example

$$\boxed{y' = P(x)y + Q(x)}$$

inhomogeneous linear equations (or non-homogeneous)

$$w = dy - [P(x)y + Q(x)]dx = 0$$

$$\tilde{w} = \xi w = \underbrace{\xi}_{Q} dy - \underbrace{[P(x)y + Q(x)]\xi}_{P} dx = 0$$

$w = P dx + Q dy$

$dw = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\wedge dy$

integrability: $d\tilde{w} = 0$, yielding

$$\frac{\partial (P(x)y + Q(x))\xi}{\partial y} = -\frac{\partial \xi}{\partial x}$$

★ look for $\xi = \xi(x)$ $\left(\Rightarrow \frac{\partial \xi}{\partial y} = 0\right)$

we find

$$-P(x) = \frac{\partial \xi}{\partial x} \Rightarrow \xi(x) = e^{-\int P(x)\,dx}$$

then $\xi = \mathcal{Y}_y$ $\qquad \mathcal{Y}(x,y) = e^{-\int P(x)dx} \cdot y + \psi(x)$

$(-Py - Q(x))\xi = \mathcal{Y}_x$ $\qquad \mathcal{Y}_x = -e^{-\int P(x)dx} P \cdot y + \psi'$

$$\parallel$$

$$-Py\xi - Q(x)\xi$$

$\Rightarrow \psi' = -Q(x)\xi$

$\psi' = -Q(x)e^{-\int P(x)dx}$ $\qquad \psi(x) = -\int Q(x)e^{-\int P(x)dx}dx$

So, eventually

$$f(x,y) = C = e^{-\int P(x)dx} y - \int Q(x) e^{-\int P(x)dx} dx$$

$$\Rightarrow \quad y = \underbrace{C e^{\int P(x)dx}}_{\uparrow} + \underbrace{\int \{ Q(x) e^{-\int P(x)dx} \} dx \cdot e^{\int P(x)dx}}$$

Solution of the <u>homogeneous</u> equation

<u>Special</u> Solution

$$\left(\text{Check: } y' = Q(x) \cancel{e^{-\int P(x)dx}} \cdot \cancel{e^{\int P(x)dx}} + y \cdot P(x) \quad \checkmark \right)$$

$\ast$ Pfaff system approach to

$$\boxed{\begin{cases} z_x = f(x,y,z) \\ z_y = g(x,y,z) \end{cases}} \qquad z = z(x,y)$$

$$dz = z_x\,dx + z_y\,dy = f\,dx + g\,dy$$

$$\boxed{\omega := f\,dx + g\,dy - dz = 0}$$

$$\boxed{d\omega} = df \wedge dx + dg \wedge dy =$$

$$= f_y\,dy \wedge dx + f_z\,dz \wedge dx + g_x\,dx \wedge dy + g_z\,dz \wedge dy$$

$$= \boxed{(g_x - f_y)\,dx \wedge dy + f_z\,dz \wedge dx + g_z\,dz \wedge dy}$$

$$\boxed{\omega \wedge d\omega} = -(g_x - f_y)\,dz \wedge dx \wedge dy + g f_z\,dy \wedge dz \wedge dx + f g_z\,dx \wedge dz \wedge dy =$$

$$= \boxed{(f_y - g_x + g f_z - f g_z)\,\underbrace{dz \wedge dx \wedge dy}_{= dx \wedge dy \wedge dz}}$$

integrability: $\boxed{\omega \wedge d\omega = 0}$

$$\Rightarrow \quad f_y - g_x + g f_z - f g_z = 0$$

$$(\ast) \quad \boxed{f_y + g f_z = g_x + f g_z}$$

$\ast$) $\boxed{z_{xy} = z_{yx}}$ yields $\quad f_y + f_z \cdot \underset{\overset{\|}{g}}{z_y} = g_x + g_z \cdot \overset{\overset{f}{\|}}{z_x}$ $\quad$ that is $\boxed{(\ast) = (\ast')}$!

↑ $\ast$ Schwarz

XXVI - 4