Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY v2

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Lecture XXVIII

LIE GROUPS

Lie groups
The Lie algebra of a Lie group
Integral curves of l. inv. v. fields
Exponential map.
Examples

Def. A Lie group G is a group endowed with a smooth manifold structure such that the map

$$G \times G \longrightarrow G$$
$$(g, h) \longmapsto g \cdot h^{-1}$$

is smooth [The Cartesian product $M \times N$ of two differentiable manifolds M and N has a natural differentiable manifold structure ...]. The above condition is equivalent to requiring the maps $(g, h) \mapsto gh$ and $g \mapsto g^{-1}$ to be smooth.

Examples: $GL(n, \mathbb{R})$, $GL(n, \mathbb{C})$ (general linear groups), $O(n)$ (orthogonal groups), $U(n)$ (unitary groups), $SO(n)$ (special orthogonal groups), $SU(n)$ (special unitary groups) ($\det = +1$) et cetera, are Lie groups. The group operation is just matrix product. Local charts can be constructed by means of the exponential map (see below).

The left and right translations

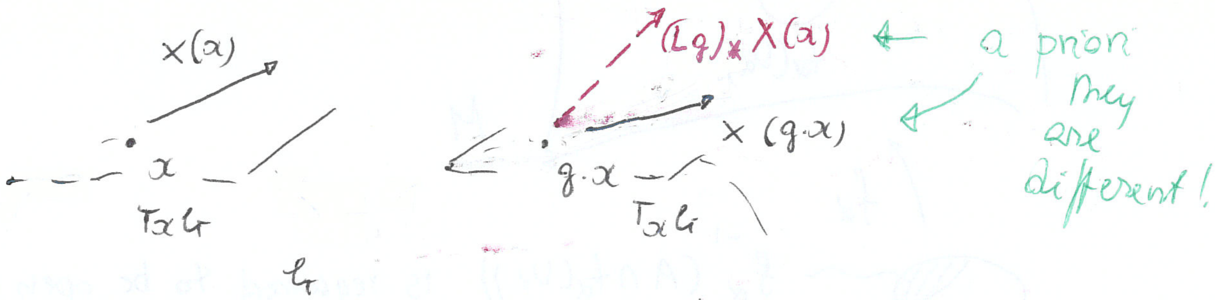
$$L_g : G \ni x \longmapsto g \cdot x \in G$$

$$R_g : G \ni x \longmapsto x \cdot g \in G$$

are diffeomorphisms

Def.: A vector field $X \in \mathfrak{X}(G)$ is called **left-invariant** if

$$(\diamond) \qquad X(g\cdot x) = (L_g)_* X(x)$$



and this amounts to the condition:

$$X(g) = (L_g)_* \underset{\substack{\cap \\ T_eG}}{X(e)}$$

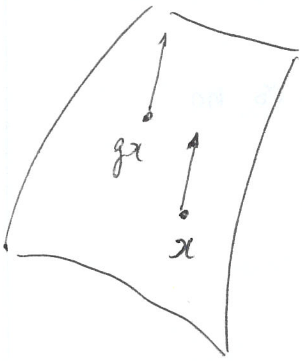
$\uparrow$ identity, or neutral element, of G

Notice that $(\diamond)$ can be reformulated, suggestively, as

$$(\diamond\diamond) \qquad \boxed{(L_g)_* X = X}$$

$\bullet\; y = g\cdot x$

$\bullet\; x = g^{-1}\cdot y$



Indeed

$$\underset{\substack{\cap \\ G}}{((L_g)_* X)(y)} \underset{\text{def}}{=} (L_g)_* (X(g^{-1}\cdot y))$$

$$\underset{\text{left invariance}}{=} X(g\cdot g^{-1}\cdot y) = X(y), \text{ which is } (\diamond\diamond).$$

Also, since L_g is a diffeomorphism, one has for l. inv. X, Y:

$$(L_g)_*([X,Y]) = [(L_g)_* X, (L_g)_* Y] = [X,Y],$$

that is, $[X,Y]$ is also left-invariant.

Similarly, one defines right-invariant v. fields

This entails that

$$\mathfrak{g} \equiv \{ \text{left invariant v. fields of } G \}$$

is a Lie algebra (with respect to $[\ ]$), termed

Lie algebra of G (older, and possibly better terminology, infinitesimal Lie group associated to G)

(it is actually a Lie subalgebra of all $\mathfrak{X}(G)$)

Every Lie algebra is the Lie algebra of a Lie group (Lie's theorem). We shall not prove this result.

Notice that, as vector spaces, $\mathfrak{g} \cong T_e G$

tangent space to G at e.

and $\dim \mathfrak{g} = \dim G$

Let us consider the flow of $X \in \mathfrak{g}$. One has the following

Theorem Let $X \in \mathfrak{g}$. Then

(a) $$F_t^X(g) = g \cdot F_t^X(e) \equiv L_g \cdot F_t^X(e)$$

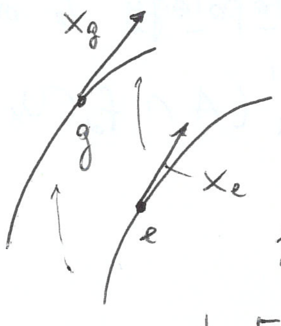
i.e. the integral curves of X are obtained simply by translating the integral curve passing through e.

(b) X is complete, i.e. its flow F_t^X is defined $\forall t \in \mathbb{R}$

Proof. Ad (a). For $t \in I$ (a suitable interval $\ni 0$), the two curves $t \longmapsto F_t^X(g)$

$t \longmapsto g \cdot F_t^X(e)$

recall $(\psi \circ \varphi)_* = \psi_* \cdot \varphi_*$ (chain rule)



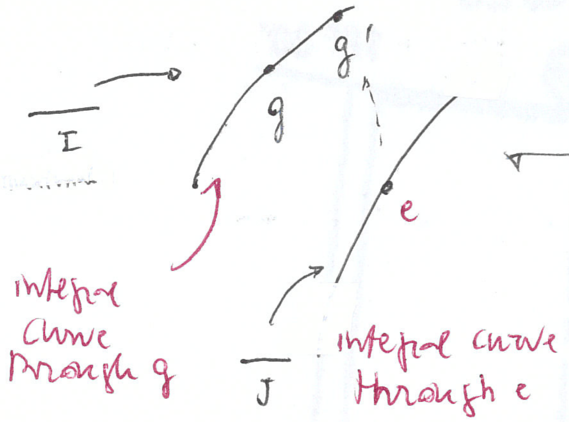
both pass through g (for $t=0$).

Let us compute their velocities at g.

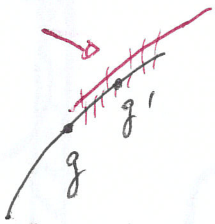
$$\left.\frac{d\, F_t^X(g)}{dt}\right|_{t=0} = X_g \ ; \quad \left.\frac{d\,(g \cdot F_t^X(e))}{dt}\right|_{t=0} = (L_g)_* X_e$$

$= X_g$ (by left invariance), hence they coincide.

Ad (b). As for the second property, resort to the following "pictorial" argument:



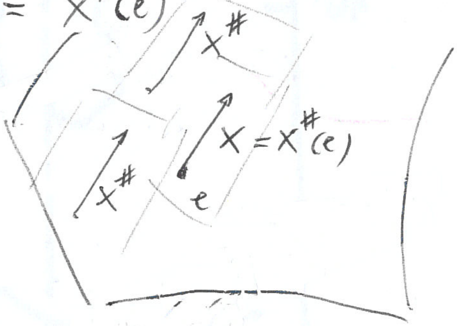
Let us transport over the former curve with e going to g'. We obtain an integral curve defined on a bigger interval



Therefore, if I is maximal, then it must coincide with $\mathbb{R}$ □

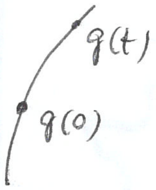
Let us write (a) with a different notation

$\mathfrak{g} \cong T_e G \ni X$ $\qquad X^{\#}$ = left inv. vector field induced by $X = X^{\#}(e)$



$$X^{\#}(g) = (L_g)_* \underbrace{X^{\#}(e)}_{= X}$$

Integral curves: $g = g(t)$



Then:

$$\boxed{\dot{g}(t) = (L_{g(t)})_* X} \quad (\ast)$$

$\uparrow$ velocity of $g(t)$

In particular, if G is a matrix group, $(*)$ becomes

$$\boxed{\dot{g} = g \cdot \underset{\text{matrix multiplication}}{X}}$$

$$\Rightarrow \quad \boxed{g(t) = g(0)\, e^{tX}}$$

$e^{Y} = \sum_{k=0}^{\infty} \frac{Y^k}{k!}$, convergent $\forall t \in \mathbb{R}$

matrix exponential (any norm on M_n)

In fact, if $\gamma = \gamma(t)$ is a smooth curve in G, a matrix group, with $\dot{\gamma}(0) = \Xi$ (a matrix), we have

$$\frac{d}{dt}\left(\underset{\text{fixed}}{g} \cdot \gamma(t)\right)\Big|_{t=0} = g \cdot \dot{\gamma}(0) = g \cdot \Xi$$

matrix product

We then set: $\mathbb{R} \ni t \longmapsto F_t^X(e) =: \exp(tX)$

integral curve of $X \in \mathfrak{g}$ through e

and we call it 1-parameter group generated by X.

$$\exp_X : \begin{array}{ccc} \mathbb{R} & \longrightarrow & G \\ t & \longmapsto & \exp(tX) \\ & & \| \\ & & F_t^X(e) \end{array}$$

is indeed a group homomorphism and $\{\exp(tX)\}_{t \in \mathbb{R}} = \exp_X(\mathbb{R})$ becomes an abelian subgroup of G

⚠ Notice: for matrices $e^{tX} e^{sX} = e^{(t+s)X} = e^{sX} e^{tX}$

but in general $e^X e^Y \neq e^{X+Y} \neq e^Y e^X$ $\rightsquigarrow$ CBH Formula

The map

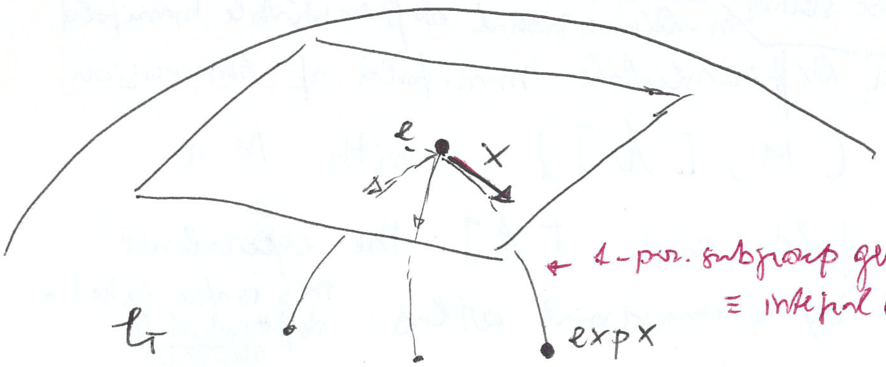
$$\exp : \mathfrak{g} \longrightarrow L_T$$
$$X \longmapsto \exp X = F_1^X(e)$$

is called <u>exponential map</u>. Locally, it is a diffeomorphism since

$$\exp_*|_0 = I_n \qquad \left(\frac{d \exp tX}{dt}\bigg|_{t=0} = X \right)$$

(by the inverse function theorem, which holds on manifolds)

Think of the matrix group case



⚠ The exponential map can be computed explicitly at any point, but its expression is quite <u>clumsy</u> (it involves the so-called Campbell-Baker-Hausdorff formula).

Examples

1. $\mathbb{R}^n$ $\qquad (x,y) \longmapsto x+y$ $\qquad$ (abelian group)

$$L_x y = x+y = R_x y$$

Let $x' = a+x$ $\qquad dx' = dx$ $\qquad \frac{\partial}{\partial x'} = \frac{\partial}{\partial x}$

$$(L_a)_* = I$$

$$(L_a)_* \frac{\partial}{\partial x^i} = \frac{\partial}{\partial x^i}$$

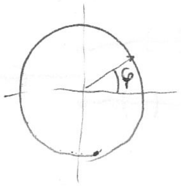
I claim that $\quad Lie(\mathbb{R}^n) = \mathbb{R}^n = \{\text{constant vector fields}\}$.

$$(L_a)_* \left(b^i(x)\frac{\partial}{\partial x^i}\right) = b^i(x+a)\frac{\partial}{\partial x^i} = b^i(x)\frac{\partial}{\partial x^i}$$

$$\Longleftrightarrow \quad b^i(x+a) = b^i(x) \quad \forall x \in \mathbb{R}^n, \ \forall a \in \mathbb{R}^n, \quad \text{i.e. } b^i(x) \equiv b^i \text{ constant}$$

integral curves : straight lines (translates of lines through the origin)

2. The circle S^1 $\qquad$ $\mathrm{Lie}(S^1) = \mathbb{R}\frac{\partial}{\partial\varphi}$



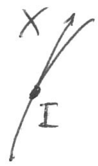
2'. The torus $\mathbb{T}^n = \underbrace{S^1 \times \ldots S^1}_{n}$ $\qquad$ $\mathrm{Lie}(\mathbb{T}^n) = \mathbb{R}^n$

3. $G = GL_n(\mathbb{R})$ $\qquad$ $\underset{\sim}{gl}_n(\mathbb{R}) = M_n(\mathbb{R})$ $\qquad$ $[\,,\,] =$ matrix commutator

all matrices

In fact every $X \in \underset{\sim}{g}$ generates the following 1-parameter group

$$g(t) = e^{tX} \quad (\equiv \exp(tX))$$



whose translates (left) $\tilde{g}(t) = g_0 \cdot e^{tX}$

yield the integral curves of $X^{\#}$,

the l. inv. vector field corresponding to X

Notice that, for $A \in G$, near I (use any norm),

then $A = e^X$ for a unique $X \in \underset{\sim}{g}$:

indeed, set $A = I + K$, with $\|K\| < 1$

then $A = \log(I+K) = K - \frac{K^2}{2} + \frac{K^3}{3} + \ldots$ (convergent) $(+)$

⚠ In order to complete the identification, one should prove that $[X^{\#}, Y^{\#}] = [X, Y]^{\#}$

Lie bracket $\qquad$ matrix commutator

This can be seen as follows

Let

$$X^\#\Big|_{A\in Gl(n,\mathbb{R})} = A\cdot X = A\cdot X^\#\Big|_I$$

$X^\#$ corresponding to $X\in gl(n,\mathbb{R})$

actually

$$X^\#\Big|_A = \underbrace{A^i{}_j \overset{X:\text{ matrix}}{X^j{}_k}}_{\xi^i{}_k} \frac{\partial}{\partial A^i{}_k} \qquad \text{akin to: } \left(\xi^I \frac{\partial}{\partial A^I}\right)$$

$$Y^\#\Big|_A = \underbrace{A^i{}_j Y^j{}_k}_{\eta^i{}_k} \frac{\partial}{\partial A^i{}_k} \qquad \frac{\partial A^i{}_r}{\partial A^j{}_l} = \delta^{ij}{}_{rl}$$

Then

$$[X^\#, Y^\#]\Big|_{I_n} = \dots = \underbrace{(X^i{}_k Y^k{}_r - Y^i{}_k X^k{}_r)}_{[X,Y]^i{}_r} \frac{\partial}{\partial A^i{}_r}\Big|_{I_n}$$

$$\Rightarrow [X^\#, Y^\#] = [X,Y]^\# \quad \text{(at all points)}$$

as claimed

(+) continues from preceding page

This illustrates the basic property of the exponential map of being a local diffeomorphism between suitable neighbourhoods of $0\in\mathfrak{g}$ and $e\in G$, respectively

4. $G = GL_n(\mathbb{C})$ (viewed as a real group)

idem $\mathfrak{g} = M_n(\mathbb{C})$

5. $U(n) = \{ U \in GL_n(\mathbb{C}) \,/\, U^* U = U U^* = I_n \}$

unitary group

$U^* = \overline{U^T} = \overline{U}^T \qquad U^{-1} = U^*$

(linear transformation leaving the standard hermitian inner product invariant)

$\mathfrak{g} = \mathfrak{u}(n) = \{ X \in M_n(\mathbb{C}) \,/\, X^* + X = 0 \}$

Indeed, let $U = U(t)$ a smooth curve passing through I_n (specifically $U(0) = I_n$; $t \in \mathcal{Y}$ interval) with velocity X, for instance $U(t) = e^{tX}$, $X \in M_n(\mathbb{C})$. We require it to lie in $U(n)$.

Therefore $U^*(t)\, U(t) = I \quad \forall t \in \mathcal{Y}$. Differentiating at $t=0$ yields

$$0 = \frac{d}{dt} U^*(t)\, U(t) = \dot{U}^*(t)\, U(t) + U^*(t)\, \dot{U}(t) \qquad \forall t \in \mathcal{Y}$$

$$\Rightarrow \quad \dot{U}^*(0) \underbrace{U(0)}_{= I_n} + \underbrace{U^*(0)}_{= I_n} \dot{U}(0) = 0 \qquad \dot{\overline{U^*}} = \dot{U}^*$$

$$X^* + X = 0$$

Slightly differently

$$U = I + tX + o(t)$$
$$U^* = I + tX^* + o(t)$$

$$U^* U = I + t\underbrace{(X + X^*)}_{= 0} + \ldots$$

That is, one "imposes $U^* U = I$ at first order".

$X + X^* = 0$ is the infinitesimal version of $U^* U = I$: in fact, anciently, Lie algebras where called "infinitesimal Lie groups" - a possibly better name.

$5'$. $G = SU(n)$ $\quad \mathfrak{g} = \{ X \in \mathfrak{u}(n) \,/\, \operatorname{tr} X = 0 \}$

We have to impose the extra condition $\det U(t) \equiv 1$

$U(t) = I + tX + \dots \qquad \det U(t) = \det(1 + tX + \dots)$

$0 = \frac{d}{dt} \det U(t) = \operatorname{tr} X$, either directly or via the argument in $\square$

$\det e^A = e^{\operatorname{Tr} A}$
true for diagonal matrices, then for diagonalizable matrices, via spectral theorem, then for all matrices, via density. Actually, unitary matrices are diagonalizable

6. $G = O(n)$; $\quad \mathfrak{g} = \{ X \in M_n(\mathbb{R}) \,/\, X^T + X = 0 \}$

$6'$. $G = SO(n)$; $\mathfrak{g}$ $\quad X^T + X = 0 \Rightarrow \operatorname{Tr} X = 0$

Hence $\boxed{\mathfrak{o}(n) = \mathfrak{so}(n)}$

This is not surprising, since $SO(n)$ is the connected component of $O(n)$ containing I_n, so their Lie algebras ($\cong$ tangent spaces at the identity) must coincide.

Or, one uses the fact that complex matrices can be cast into a triangular form

* On the interpretation of $[X, Y]$
(Special case)

take $G = GL_+(n, \mathbb{R})$ $\qquad e = I_n$

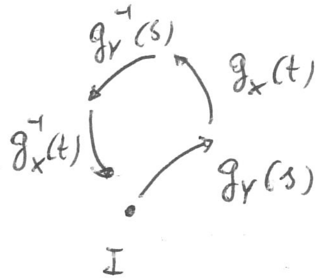
$\mathfrak{g} = \mathfrak{gl}(n, \mathbb{R}) = M_n(\mathbb{R})$

$[\,,\,] =$ matrix commutator

Let $g_X(t) = e^{tX}$ the 1-parameter group generated by $X \in \mathfrak{g}$. Let us evaluate

$$g_X^{-1}(t) \cdot g_Y^{-1}(s) \cdot g_X(t) \circ g_Y(s)$$

$$\left(= (g_Y \circ g_X)^{-1} g_X g_Y \right)$$



keep these terms

$$g_X \cdot g_Y = (I + tX + \dots)(I + sY + \dots)$$
$$= I + tX + sY + stXY + \dots$$

$$g_Y \cdot g_X = I + \underbrace{tX + sY + stYX}_{\xi} + \dots$$

$(1+\xi)^{-1} = 1 - \xi + \xi^2 + \dots$

geometric series

$$(g_Y \cdot g_X)^{-1} = I - tX - sY - stYX$$
$$+ stXY + stYX + \dots$$
$$= I - tX - sY + stXY$$

$I - tX - sY + stXY$

$I + tX + sY + st\,XY$

$\Rightarrow (g_Y g_X)^{-1} g_X g_Y =$

$$I + stXY + stXY - stXY - stYX + \dots$$
$$= I + st[X, Y] + \dots$$

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Thus $$\frac{\partial^2}{\partial s \partial t} \left(g_X^{-1}(t)\, g_Y^{-1}(s)\, g_X(t)\, g_Y(s) \right) \Big|_{\substack{t=0 \\ s=0}} = [X, Y]$$

(slightly differently, work with $s \to \sqrt{s}$, $t \to \sqrt{s}$

$$\ldots = I + s[X, Y] + \ldots \qquad \frac{\partial}{\partial s}(\quad)\Big|_{s=0} = [X, Y]$$

recall $[X, Y]_{Lie} = [X, Y]_{matrix}$