Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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SU(2) & SO(3)

Lecture XXX

A digression on SU(2) and SO(3)

Recall that $SU(2) = \{ U \in U(2) \,/\, \det U = 1 \}$

$$U = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} \qquad \alpha, \beta \in \mathbb{C}, \quad |\alpha|^2 + |\beta|^2 = 1$$

α, β: Cayley-Klein parameters

Therefore, $SU(2)$ is diffeomorphic to $S^3 \subset \mathbb{R}^4 \cong \mathbb{C}^2$

$SU(2)$ acts naturally on $\mathbb{C}^2$ (spinor space, in quantum mechanics)

$$\mathfrak{su}(2) = \{ X \in M_2(\mathbb{C}) \,/\, X + X^* = 0 \text{ and } \operatorname{Tr} X = 0 \}$$

$\mathfrak{su}(2)$: Lie algebra of SU(2)

Let us write down a basis of $\mathfrak{su}(2)$:

$$\begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} =: i\sigma_1$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} =: i\sigma_2$$

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} =: i\sigma_3$$

The σ_i are called Pauli matrices (or spin matrices)

Also:

$$\begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix} = \alpha_0 I + \sum_{j=1}^{3} \alpha_j \, i\sigma_j = \rightarrow$$

$\alpha_i \in \mathbb{R}$

$$\Rrightarrow \quad = \alpha_0 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \alpha_1 \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \alpha_3 \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}$$

$$= \begin{pmatrix} \alpha_0 + i\alpha_3 & \alpha_2 + i\alpha_1 \\ -\alpha_2 + i\alpha_1 & \alpha_0 - i\alpha_3 \end{pmatrix}, \quad \text{so} \quad \begin{aligned} \alpha &= \alpha_0 + i\alpha_3 \\ \beta &= \alpha_2 + i\alpha_1 \end{aligned}$$

The condition that $U = 1$ translates into

$$\alpha_0^2 + \alpha_1^2 + \alpha_2^2 + \alpha_3^2 = 1,$$

that is, we again get the sphere S^3.

Now, in $\mathbb{R}^3$ (actually, the geometric vector space, equipped with an orthonormal basis $(\underline{i}, \underline{j}, \underline{k})$, take

$$\underline{a} = a_1 \underline{i} + a_2 \underline{j} + a_3 \underline{k}$$

Set: $\underline{\sigma} \cdot \underline{a} := \sum_{i=1}^{3} \sigma_i a_i$ and, for $\underline{n}$, $\|\underline{n}\| = 1$,

vector of Pauli matrices

$$\boxed{U_{\underline{n}}(\varphi) := e^{i\frac{\varphi}{2}\underline{\sigma}\cdot\underline{n}}}$$

$\varphi \in \mathbb{R}$

notice this

a 1-parameter group in $SU(2)$

$$= \boxed{\cos\frac{\varphi}{2} I_2 + \sin\frac{\varphi}{2}\, i\,\underline{\sigma}\cdot\underline{n}}$$

simple calculation

One verifies that

$$(\underline{\sigma}\cdot\underline{a})(\underline{\sigma}\cdot\underline{b}) = \underbrace{\underline{a}\cdot\underline{b}}_{\text{scalar product}} I_2 + i\,\underline{\sigma}\cdot\underbrace{(\underline{a}\times\underline{b})}_{\text{vector product}}$$

Let us now consider the vector space of hermitian, traceless 2×2 complex matrices: as a vector space this is just $\mathbb{R}^3$, and, with abuse of language, it will be denoted again by $\mathfrak{su}(2)$ — $X = X^*$ if and only if $(iX)^* = -iX$, i.e. iX is skew-hermitian

Thus, we write down an explicit vector space isomorphism

$$\mathbb{R}^3 \ni \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longmapsto \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \in \mathfrak{su}(2)$$

hermitian, traceless

Consider now the adjoint representation of $SU(2)$ on $\mathfrak{su}(2)$, and, specifically:

$$\begin{tikzcd}
\underline{v} \sim \begin{pmatrix} x \\ y \\ z \end{pmatrix} \equiv \underline{x} \arrow[r, mapsto] \arrow[d, "?\ \text{what is this?}"] & \begin{pmatrix} z & x-iy \\ x+iy & -z \end{pmatrix} \equiv X \arrow[d, "\mathrm{Ad}\, U_{\underline{n}}(\varphi)"] \\
\underline{v} \sim \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \equiv \underline{x}' \arrow[r, mapsto, shift left] & U_{\underline{n}}(\varphi)\, X\, U_{\underline{n}}(\varphi)^{-1} \arrow[l, shift left]
\end{tikzcd}$$

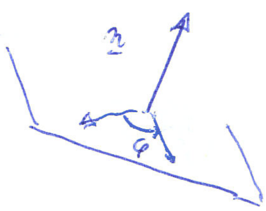
$$U_{\underline{n}}(\varphi)^{-1} = U_{\underline{n}}(\varphi)^{*} = U_{\underline{n}}(-\varphi)$$

coordinates of the same vector $\underline{v}$

commutative diagram

Answer: $\underline{x}' = R_{\underline{n}}(\varphi)\, \underline{x}$

(rotation by φ around $\underline{n}$)



That is: ordinary rotations (elements of $SO(3)$, by Euler) are described by complex 2×2 matrices, instead of real 3×3 matrices. This formalism turns out to be extremely useful in mechanics (classical and quantum), robotics, aeronautics, space navigation...

Let us check this only in a specific example

$$x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \underline{n} = \underline{\mathbb{k}}$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$U_{\underline{\mathbb{k}}}(\varphi) = \cos\frac{\varphi}{2} I_2 + i \sin\frac{\varphi}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

Then

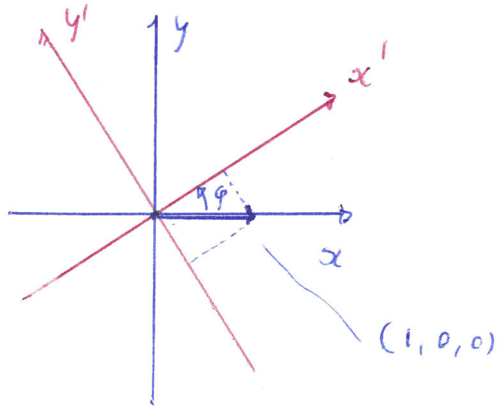
$$U_{\underline{\mathbb{k}}}(\varphi) X U_{\underline{\mathbb{k}}}(-\varphi) = \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & \\ 0 & e^{i\varphi/2} \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & e^{i\varphi/2} \\ e^{-i\varphi/2} & 0 \end{pmatrix}\begin{pmatrix} e^{-i\varphi/2} & 0 \\ 0 & e^{i\varphi/2} \end{pmatrix} = \begin{pmatrix} 0 & e^{i\varphi} \\ e^{-i\varphi} & 0 \end{pmatrix} = X'$$

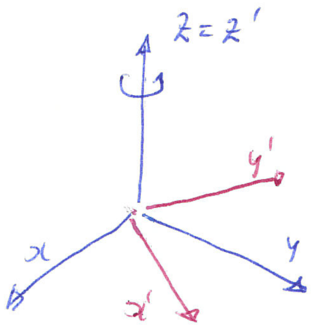
Therefore

$$X' = \begin{pmatrix} 0 & \cos\varphi + i\sin\varphi \\ \cos\varphi - i\sin\varphi & 0 \end{pmatrix} \Rightarrow x' = \begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \quad \text{with}$$

$$\begin{cases} x' = \cos\varphi \\ y' = -\sin\varphi \\ z' = 0 \end{cases} \rightsquigarrow$$



new coordinate of $\underline{v} \sim x = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ with respect to the rotated axes



* <u>Topological</u> <u>intermezzo</u>

We already know that $SU(2) \simeq S^3$, $SO(2) = \mathbb{R}P^3$
$\equiv S^3/\sim$

Let us investigate this feature in more detail.

$\sim$: identification of antipodal points

First notice that if $U_{\underline{n}} \longmapsto -U_{\underline{n}}$ (still in $SU(2)$)

one obtains the <u>same</u> rotation $R_{\underline{n}}$

Then observe that if $\varphi = 0$, $U_n(0) = I_2$,

but $\varphi = 2\pi \Rightarrow U_{\underline{n}}(2\pi) = \overset{-1}{\overset{\shortparallel}{\cos\pi}}\, I_2 + i\underline{\sigma}\cdot\underline{n}\,\overset{0}{\overset{\shortparallel}{\sin\pi}}$

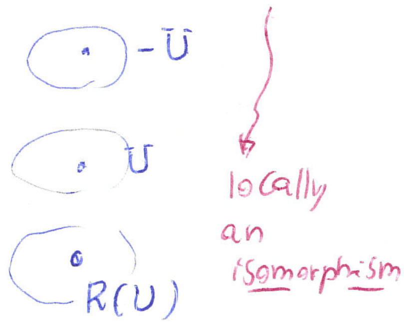
$= -I_2$

However $R_{\underline{n}}(0) = R_{\underline{n}}(2\pi) = I_3$

The upshot is the following: the map (a group homomorphism)

$$SU(2) \longrightarrow SO(3)$$
$$U \longmapsto R(U)$$

is a $2:1$ <u>covering</u> <u>map</u>



The <u>loop</u> $[0,2\pi] \ni \varphi \longmapsto R_{\underline{n}}(\varphi)$ $(\star)$

yields a <u>non-trivial</u> element in $\pi_1(SO(3))$ $(\simeq \mathbb{Z}_2)$.

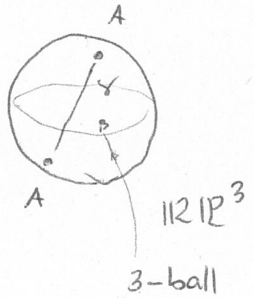
In fact, upon lifting to $[0,2\pi] \ni \varphi \longmapsto U_{\underline{n}}(\varphi)$,

we <u>do not</u> get a loop.

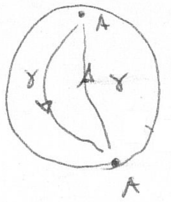
But, if we trace the first loop <u>twice</u> $(\varphi \in [0, 4\pi])$

we indeed get a loop as a lift, whose homotopy class is <u>trivial</u>, being S^3 simply connected.

The loop (◇) yields the generator $a \in \mathbb{Z}_2$ $(a^2 = 1)$

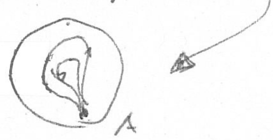


γ is a loop based on A and its class $[\gamma]$ is non trivial : indeed $[\gamma]^2 = id$



γ^2 = trivial loop at A

$\rightsquigarrow$

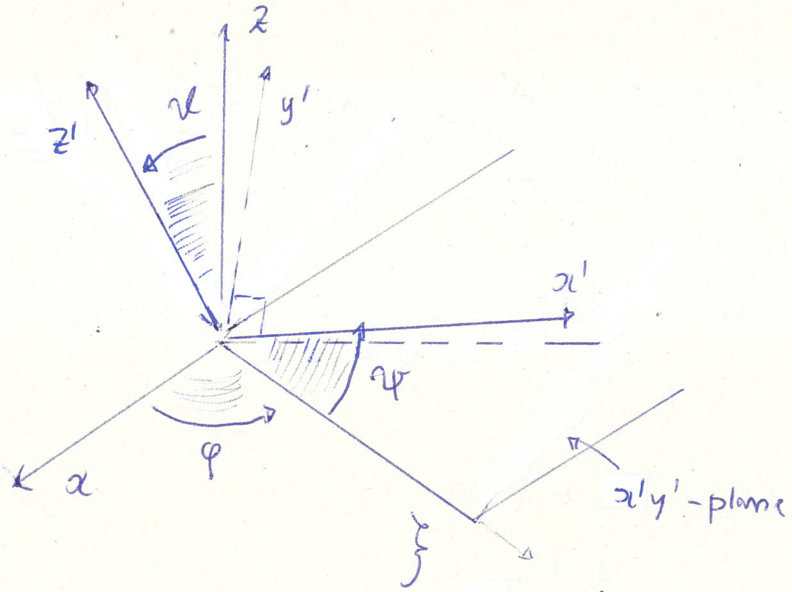


☆ SU(2) is the universal covering group of SO(3)

being simply connected

$SO(3) = SU(2)/\mathbb{Z}_2$ as a homogeneous space (still a group, in this case)

★ Connection with <u>Euler angles</u>



nodal line
||
xy-plane ∩ x'y'-plane

φ : <u>precession</u> $\varphi \in [0, 2\pi)$

ν : <u>nutation</u> $\nu \in [0, \pi]$

ψ : <u>proper</u> <u>rotation</u> $\psi \in [0, 2\pi)$

"yaw - pitch - roll"

straorzata / imbardata — inclinazione / pendenza — dondolio / rollio

(φ, ψ, ν) become coordinates on $S^3 = SU(2)$

$$\begin{cases} x_1 = \cos\varphi \cos\psi \cos\nu \\ x_2 = \sin\varphi \cos\psi \cos\nu \\ x_3 = \sin\psi \cos\nu \\ x_4 = \sin\nu \end{cases}$$

cf S^2:

$$\begin{cases} x = \cos\varphi \cos\nu \\ y = \sin\varphi \cos\nu \\ z = \sin\nu \end{cases}$$

Referring to the figure above:

A generic $O \in SO(3)$ (a rotation around a certain axis, by Euler's theorem) can be described via its action on a coordinate frame (O, x, y, z) ← origin

aside:
[this is a principal bundle viewpoint]

The $x'y'$-plane intersects the (old) xy-plane along the <u>nodal</u> <u>line</u> (with "longitude" φ, measured from the x-axis). The z'-axis is perpendicular to the $x'y'$-plane, and forms a (convex) angle ν with the z-axis. The x'-axis forms an angle ψ with the nodal line ξ. Conversely, φ, ν, ψ, subject to the above limitations, determine a rotation.

Let us work with Cayley-Klein parameters:

$$\underset{\in SU(2)}{Q} = Q_\psi \, Q_\vartheta \, Q_\varphi$$

giving rise to O

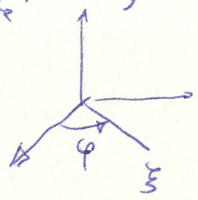
$$\begin{pmatrix} e^{i\psi/2} & 0 \\ 0 & e^{-i\psi/2} \end{pmatrix} \begin{pmatrix} \cos\frac{\vartheta}{2} & i\sin\frac{\vartheta}{2} \\ i\sin\frac{\vartheta}{2} & \cos\frac{\vartheta}{2} \end{pmatrix} \begin{pmatrix} e^{i\varphi/2} & 0 \\ 0 & e^{-i\varphi/2} \end{pmatrix}$$

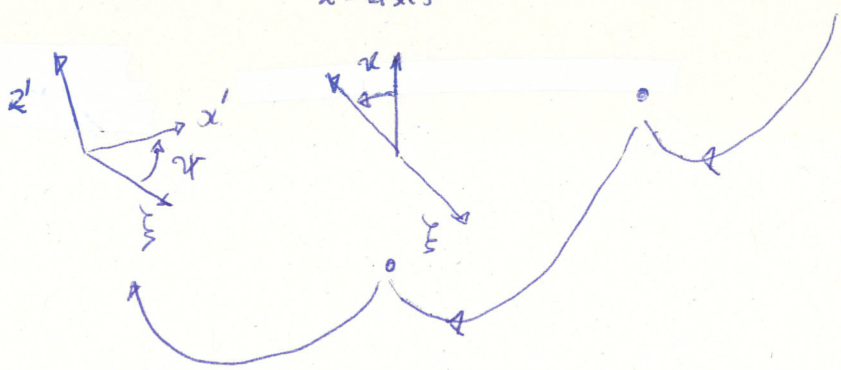
rotation around the z'-axis: yields the final position of the x'-axis and the y'-axis

rotation around the intermediate "x-axis", i.e. ξ: yields the new z-axis

rotation around the z-axis: yields ξ

work in this order: ←





one finds

$$Q = \begin{pmatrix} e^{i\frac{\psi+\varphi}{2}}\cos\frac{\vartheta}{2} & i\,e^{i\frac{\psi-\varphi}{2}}\sin\frac{\vartheta}{2} \\ i\,e^{i\frac{\psi-\varphi}{2}}\sin\frac{\vartheta}{2} & e^{-i\frac{\psi+\varphi}{2}}\cos\frac{\vartheta}{2} \end{pmatrix}$$

wherefrom one gets $O = R_{\underline{n}}(\omega)$

via a straightforward, possibly tedious calculation:

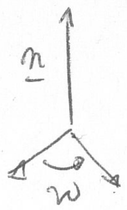
$$\cos\frac{\omega}{2} = \mathrm{Re}\left(e^{i\frac{\psi+\varphi}{2}}\cos\frac{\vartheta}{2}\right) = \cos\frac{\psi+\varphi}{2}\cos\frac{\vartheta}{2}$$

$$\underline{n} = \frac{1}{\underset{\substack{\| \\ +\sqrt{1-\cos^2\frac{\omega}{2}}}}{\sin\frac{\omega}{2}}}\left[\cos\frac{\psi-\varphi}{2}\sin\frac{\vartheta}{2}\,\underline{i} + \left(-\sin\frac{\psi-\varphi}{2}\sin\frac{\vartheta}{2}\right)\underline{j} + \sin\frac{\psi+\varphi}{2}\cos\frac{\vartheta}{2}\,\underline{k}\right]$$

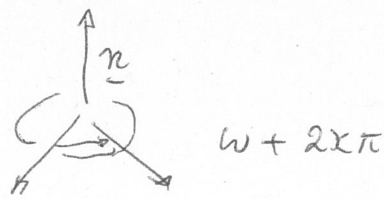
(indeed one has $\|\underline{n}\| = 1$)

Notice however the following identifications (leading to the fact that actually φ, ψ, ϑ are coordinates for the universal covering group $SU(2)$)

$(\omega, \underline{n})$



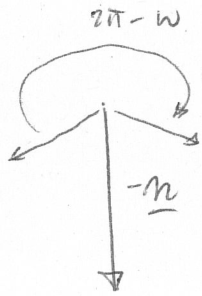
$\sim$



$(\omega + 2k\pi, \underline{n})$

$\wr$

$(2\pi - \omega, -\underline{n})$



this is counterclockwise when seen from below

4 Connection with quaternions

$$I := i\sigma_1 \qquad J := i\sigma_2 \qquad K := i\sigma_3$$

One has

$$I \circ J := i\sigma_2 \circ i\sigma_1 = \ldots \; i\sigma_3 = K$$

matrix product

notice the reverse order

and also $J \cdot K = I$, $K \cdot I = J$

together with $I^2 = J^2 = K^2 = -Id$

$\Rightarrow$ one gets the imaginary quaternions.

quaternions: $\mathbb{H} = \{ a\,Id + b\,I + c\,J + d\,K \}_{a,b,c,d \in \mathbb{R}}$

they yield a skew-field with respect to $+$ and $\cdot$

One has $SU(2) = \mathbb{H}_1$ ← unit quaternions (norm = 1)

(cf with $S^1 = U(1) =$ complex numbers with norm one)

Notice that, in view of $(\underline{\sigma}\cdot\underline{a})(\underline{\sigma}\cdot\underline{b}) = i\,\underline{\sigma}\cdot(\underline{a}\times\underline{b}) + \underline{a}\cdot\underline{b}\,I$

we see that the operations on quaternions embody the scalar and the vector products of geometric vectors.