Lectures on

DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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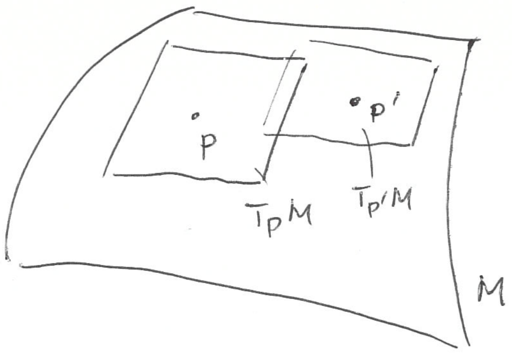
Lecture XXXI

AFFINE AND RIEMANNIAN CONNECTIONS

★ Affine connections

We use a "mélange" of classical and modern ideas: free use is made of "infinitesimals", within a modern framework. This is pedagogically instructive and avoids harshness of a strictly axiomatic treatment (cf. H. Amari)



work in local coordinates (ξ^i)

There is no a priori way of identifying T_pM and $T_{p'}M$ (though they are obviously isomorphic).

set $d\xi^i := \xi^i(p') - \xi^i(p)$

"infinitely close points" ⚠

We want to set up a linear map (infinitesimal parallel transport ≡ affine connection)

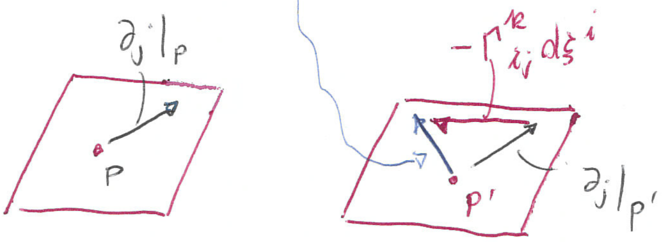
$$\Pi_{p,p'} : T_pM \longrightarrow T_{p'}M$$

(no canonical way of doing this) allowing us to consistently identify the two spaces. We want

infinitesimal   generalized Christoffel symbols

$$\Pi_{p,p'}(\partial_j|_p) = \partial_j|_{p'} - \underbrace{d\xi^i\, \Gamma^k_{ij}(p)\, \partial_k|_{p'}}$$

sum over i and k



This is a "correction at first order" of $\partial_j|_{p'}$ via an "infinitesimal vector"

Of course, we want everything to be intrinsically defined, namely, independent of the choice of a coordinate system.

So let

$$\tilde{\partial}_r = \frac{\partial}{\partial \eta^r} = \frac{\partial}{\partial \xi^i} \cdot \frac{\partial \xi^i}{\partial \eta^r} \qquad \left(= \frac{\partial \xi^i}{\partial \eta^r} \partial_i \right)$$

$\eta = (\eta^1 \cdots \eta^n)$

new coord. system

By virtue of linearity:

$$\Pi_{P,P'} \left(\tilde{\partial}_s \big|_P \right) = \frac{\partial \xi^j}{\partial \eta^s}(P) \left\{ \partial_j \big|_{P'} - d\xi^i \, \Gamma^k_{ij}(P) \, \partial_k \big|_{P'} \right\}$$

Now, from

$$\frac{\partial \xi^j}{\partial \eta^s}(P') = \frac{\partial \xi^j}{\partial \eta^s}(P) + \frac{\partial^2 \xi^j}{\partial \eta^s \partial \eta^r}(P) \, \underbrace{d\eta^r}_{\eta^r(P') - \eta^r(P)} + \cdots$$

and $d\xi^i = \frac{\partial \xi^i}{\partial \eta^r}(P) \, d\eta^r$, we find, after substitution in

$$\Pi_{P,P'} \left(\tilde{\partial}_s \big|_P \right) = \tilde{\partial}_s \big|_{P'} - d\eta^r \, \tilde{\Gamma}^t_{rs}(P) \, \tilde{\partial}_t \big|_{P'} \,,$$

which has the same form as the previous one, upon setting

$$(*) \quad \tilde{\Gamma}^t_{rs}(P) = \left\{ \Gamma^k_{ij}(P) \frac{\partial \xi^i}{\partial \eta^r}(P) \frac{\partial \xi^j}{\partial \eta^s}(P) + \underbrace{\frac{\partial^2 \xi^k}{\partial \eta^r \partial \eta^s}(P)}_{\text{non tensorial part}} \right\} \frac{\partial \eta^t}{\partial \xi^k}(P)$$

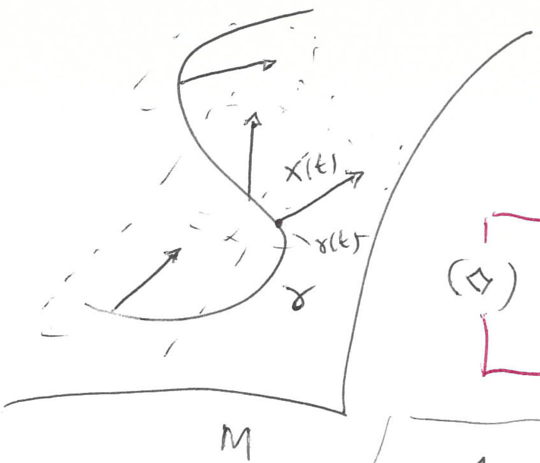
The Γ^k_{ij} do NOT define a tensor

(the correct transformation law (of a (1,2) tensor) is not respected)

The upshot is the following:

An affine connection is specified by giving $\{\Gamma^k_{ij}\}$ subject to (∗) (under a coordinate change).

It defines an infinitesimal parallel transport, allowing identification of nearby tangent spaces.

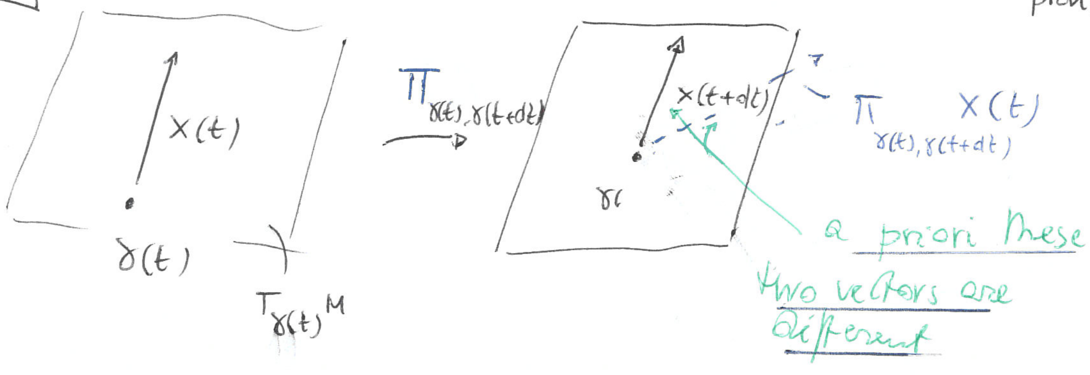


Now let $X = X(t)$ be a vector field along a curve $\gamma : [a,b] \to M$.

It is called parallel along γ if

$$(\diamond) \qquad \boxed{X(t+dt) = \Pi_{\gamma(t),\gamma(t+dt)} X(t)}$$

provisional



That is, the value of X at a nearby point of the curve is the one specified by the connection.

Let us now work in coordinates, also in order to achieve a rigorous formulation of the above concept.

In view of linearity, one has:

$$\Pi_{\gamma(t),\gamma(t+dt)} X(t) = \left\{ X^k(t) - dt\, \dot\gamma^i(t) X^j(t) \Gamma^k_{ij}(\gamma(t)) \right\} \partial_k \Big|_{\gamma(t+dt)}$$

However: $\quad X(t+dt) = X^i(t+dt)\, \partial_i \Big|_{\gamma(t+dt)}$

$\Rightarrow$ (from (◇)) $\quad$ ("crossing fingers and dividing by dt" (T. Willmore)) !

$$\dot X^k(t) + \dot\gamma^i(t) X^j(t) \Gamma^k_{ij}(\gamma(t)) = 0$$

or, more concisely:

$$\boxed{\dot X^k + \Gamma^k_{ij} \dot\gamma^i X^j = 0} \qquad k = 1..n$$

"parallel transport equation"

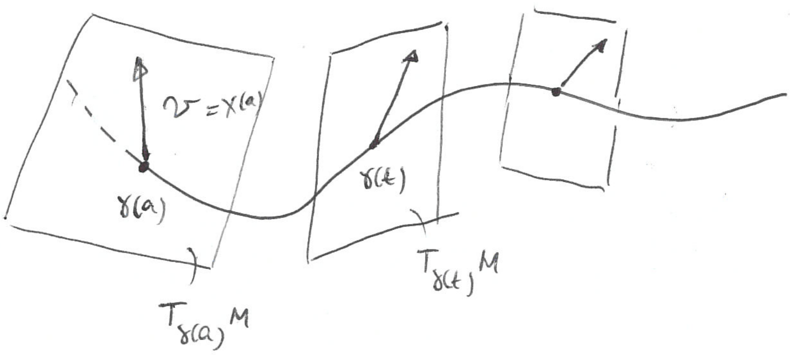
a system of linear diff. equation of 1st-order

$\Rightarrow$ by Cauchy-Lipschitz, given

$v = X(a)$, initial condition, $\quad \exists!\ X = X(t)$ parallel along γ, with $X(a) = v$ (given)

★ This is the parallel transport of v along a curve γ

cf. with the surface theory situation: here the connection $\{\Gamma^k_{ij}\}$ is given abstractly



We now wish to define the covariant derivative $\frac{\nabla}{dt} X$ of a vector field X along a curve γ

Set, provisionally,

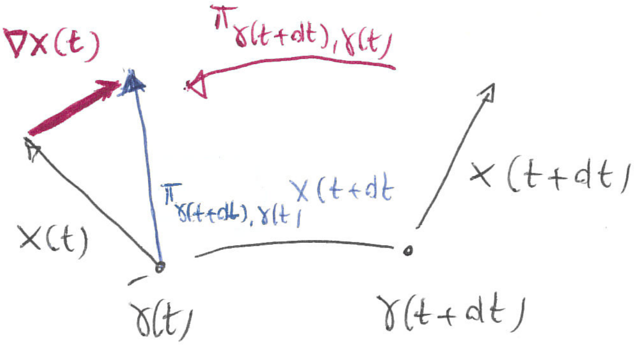
$$\nabla X(t) = \underbrace{\Pi_{\gamma(t+dt),\gamma(t)} X(t+dt)}_{\text{this is a vector at } \gamma(t) \ (\in T_{\gamma(t)}M)} - \underset{\substack{\cap \\ T_{\gamma(t)}M}}{X(t)}$$

"covariant differential"

⚠ this is a vector at $\gamma(t)$ $(\in T_{\gamma(t)}M$

.. ordinary derivative, in $\mathbb{R}^n$
$dX(t) = X(t+dt) - X(t)$..

$= \left(\pi_{\gamma(t),\gamma(t+dt)}\right)^{-1}$



Use formula for π, again
"cross your fingers and divide by dt"

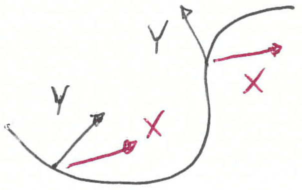
$$\boxed{\frac{\nabla X}{dt} = \left\{ \dot{X}^k(t) + \Gamma^k_{ij}(\gamma(t))\,\dot{\gamma}^i(t)\,X^j(t) \right\} \partial_k \Big|_{\gamma(t)}}$$

$$\frac{\nabla X}{dt} \equiv \frac{\nabla}{dt} X$$

covariant derivative of X (defined on γ) along γ

★ X is called parallel along γ if $\frac{\nabla}{dt} X = 0$ (along γ)

★ Extensions of ∇



one can define $(\nabla_X Y)(\gamma(t))$, covariant derivative of Y, defined on γ, along X, also defined on γ, as:

$$(\star)\quad (\nabla_X Y)(\gamma(t)) = \left\{ X^i \partial_i Y^k(t) + \Gamma^k_{ij}(\gamma(t)) X^i(t) Y^j(t) \right\} \partial_k \Big|_{\gamma(t)}$$

$$= X^i \left\{ \partial_i Y^k + \Gamma^k_{ij} Y^j \right\} \partial_k \Big|_{\gamma(t)}$$

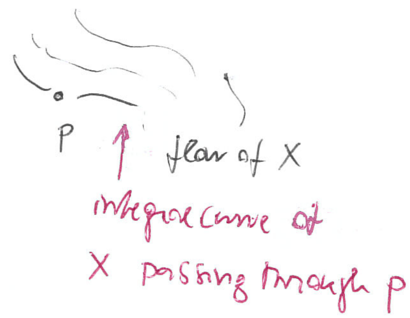
That is: just replace $\dot\gamma$ by $X = X^i \partial_i$

in coordinates:

$$\dot Y(t) = \dot Y(\gamma(t)) = \frac{\partial Y}{\partial x^i} \frac{dx^i}{dt} = \frac{dx^i}{dt} \frac{\partial}{\partial x^i} Y$$

$\dot\gamma$

Now, let $X, Y \in \mathfrak{X}(M)$.
Let $p \in M$ and let $t \mapsto F^X_t(p) =: \gamma(t)$ be the integral curve of X passing through p. Restrict X and Y to γ and use $(\star)$.
A moment's reflection then shows that the formula below, without any direct reference to γ,
defines the covariant derivative of $Y \in \mathfrak{X}(M)$ with respect to $X \in \mathfrak{X}(M)$;



"tout court":

without further specifications

everything is ultimately computed at a generic point $p \in M$

$$\boxed{\nabla_X Y = X^i \left\{ \partial_i Y^k + \Gamma^k_{ij} Y^j \right\} \partial_k}$$

If $X = \partial_i$, $Y = \partial_j$

$$\boxed{\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k}$$

further meaning of the Christoffel symbols

Abstraction: The above discussion enables us to distillate the following properties of ∇:

we have a map

$$\nabla : \mathfrak{X} \times \mathfrak{X} \longrightarrow \mathfrak{X}$$

$$(X, Y) \longmapsto \nabla_X Y$$

fulfilling

(i) $\nabla_{X+Y} Z = \nabla_X Z + \nabla_Y Z$

(ii) $\nabla_X (Y+Z) = \nabla_X Y + \nabla_X Z$

(iii) $\nabla_X (fY) = \underset{\overset{\shortparallel}{df(X)}}{X(f)} Y + f \nabla_X Y$ Leibniz

(iv) $\nabla_{fX} Y = f \nabla_X Y$ ← tensoriality in X

Does ∇ define a $((1,2))$ tensor?

No! One does not have tensoriality in Y (Leibniz!) (just in X). This reflects the non-tensorial behaviour of the Christoffel symbols!

★ Properties (i) – (iv) provide the basis for an axiomatic treatment of the notion of (affine) connection. We decided to follow a more intuitive approach, closer to the historical path, albeit not fully rigorous.

The upshot is that an affine connection on M is then a map $\nabla : \mathfrak{X}(M) \times \mathfrak{X}(M) \to \mathfrak{X}(M)$ fulfilling properties (i) – (iv)

★ The Levi-Civita connection on a Riemannian manifold (The fundamental theorem of Riemannian geometry)

Let (M, g) be a Riemannian manifold. Then

There exists a unique affine connection ∇

obeying the following conditions:

(set $g(X,Y) = \langle X, Y\rangle$)

① "metricity" $\qquad X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$

compatibility with the metric

"∇ metric"

[cf. in $\mathbb{R}^3$, the property, for $\underline{a} = \underline{a}(t)$, $\underline{b} = \underline{b}(t)$,

$$\frac{d}{dt}\langle \underline{a}, \underline{b}\rangle = \langle \frac{d\underline{a}}{dt}, \underline{b}\rangle + \langle \underline{a}, \frac{d\underline{b}}{dt}\rangle \;]$$

② "absence of torsion" or "symmetry"

"∇ torsion-free"

"∇ symmetric" $\qquad \nabla_X Y - \nabla_Y X = [X, Y]$ $\qquad \boxed{T(X,Y) = 0}$

★ torsion tensor

($T(X,Y) := \nabla_X Y - \nabla_Y X - [X,Y]$ is indeed a tensor (of type $(1,2)$), although the single summands are not tensors, as we already know.

Check this property:

[$\nabla_{\alpha X}(\beta Y) = \alpha \nabla_X(\beta Y) = \alpha(X(\beta)Y + \beta\nabla_X Y) = \alpha X(\beta) Y + \alpha\beta \nabla_X Y$

$\nabla_{\beta Y}(\alpha X) = \beta Y(\alpha) + \alpha\beta\nabla_Y X$

$[\alpha X, \beta Y] = \alpha X(\beta Y) - \beta Y(\alpha X) = \alpha X(\beta) Y + \alpha\beta XY$
$\qquad - \beta Y(\alpha) X - \alpha\beta YX$

$\Rightarrow T(\alpha X, \beta Y) = \alpha\beta T(X,Y)$ $\qquad \alpha, \beta \in \mathcal{C}^\infty(M)$]

(bilinearity is clear).

If $X = \partial_i \quad Y = \partial_j$

$$\nabla_{\partial_i} \partial_j = \Gamma^k_{ij} \partial_k$$

$T = 0$ corresponds to

$$\left[\Gamma^k_{ij} = \Gamma^k_{ji} \right]$$

Symmetry in the lower indices, whence the term "symmetric"

Important remark:

On a surface one recovers the geometrically defined Levi-Civita connection (see page XXXI - 11)

Proof just an algebraic matter:

if ∇ exists, then one has (metricity)

$$\begin{array}{r|l} + & X\langle Y,Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle \\ + & Y\langle Z,X\rangle = \langle \nabla_Y Z, X\rangle + \langle Z, \nabla_Y X\rangle \\ - & Z\langle X,Y\rangle = \langle \nabla_Z X, Y\rangle + \langle X, \nabla_Z Y\rangle \end{array}$$

Summing, with the appropriate signs we have

$$X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle$$

$$= \langle \underbrace{\nabla_X Y + \nabla_Y X}_{= 2\nabla_Y X + \underbrace{\nabla_X Y - \nabla_Y X}_{= [X,Y]}}, Z\rangle + \langle Y, \underbrace{\nabla_X Z - \nabla_Z X}_{[X,Z]}\rangle + \langle X, \underbrace{\nabla_Y Z - \nabla_Z Y}_{[Y,Z]}\rangle$$

∇ torsion-free

$$\Rightarrow X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle =$$

$$2\langle \nabla_Y X, Z\rangle + \langle Z, [X,Y]\rangle + \langle Y, [X,Z]\rangle + \langle X, [Y,Z]\rangle$$

yielding the (Koszul) six-term formula

(***)

$$\boxed{\underbrace{\langle \nabla_Y X, Z\rangle}_{\text{defined via taking scalar products with all } Z} = \frac{1}{2}\Big\{ X\langle Y,Z\rangle + Y\langle Z,X\rangle - Z\langle X,Y\rangle - \langle Z,[X,Y]\rangle - \langle Y,[X,Z]\rangle - \langle X[Y,Z]\rangle \Big\}}$$

Therefore, if ∇ exists, then it is uniquely determined.
Conversely, (***) is easily seen to define a connection, and that ① and ② hold.

In local coordinates:

$$X = \partial_j \qquad Y = \partial_i \qquad Z = \partial_\ell$$

$$\nabla_{\partial_i}\partial_j = \Gamma^k_{ij}\partial_k$$

$$[\partial_i, \partial_j] = 0 \ldots$$

$$\langle \Gamma^k_{ij}\partial_k, \partial_\ell\rangle = \Gamma^k_{ij} g_{k\ell}$$

(***) becomes:

$$\Gamma^k_{ij} g_{k\ell} = \frac{1}{2}\left\{ \frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{\ell i}}{\partial x^i} - \frac{\partial g_{ji}}{\partial x^\ell} \right\}$$

$$\Rightarrow \boxed{\Gamma^k_{ij} = \frac{1}{2} g^{k\ell}\left\{ \frac{\partial g_{i\ell}}{\partial x^j} + \frac{\partial g_{\ell i}}{\partial x^i} - \frac{\partial g_{ij}}{\partial x^\ell} \right\}}$$

inverse of g..

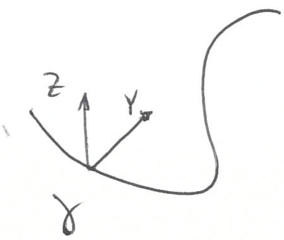
$$\big(\Gamma^k_{ij}\underbrace{g_{k\ell}\, g^{\ell m}}_{\delta^m_k} = \frac{1}{2} g^{\ell m}\{\quad\}$$

// (setting $m = k$..)

Let us comment on the metricity condition:

$$X\langle Y, Z\rangle = \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

It entails that parallel transport preserves inner products (hence angles and lengths..)



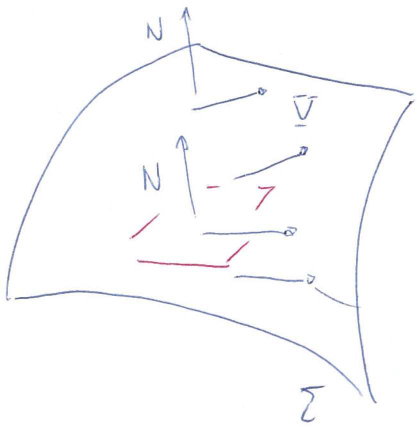
Indeed, let γ be an integral curve of X, and Y and Z parallel along γ.

Then $\nabla_X Y = \nabla_X Z = 0$

$\Rightarrow \quad X\langle Y, Z\rangle = 0 \quad \Rightarrow \quad \langle Y, Z\rangle$ is constant along γ.

◇ ◇ ◇

The Levi-Civita connection on a surface in $\mathbb{R}^3$



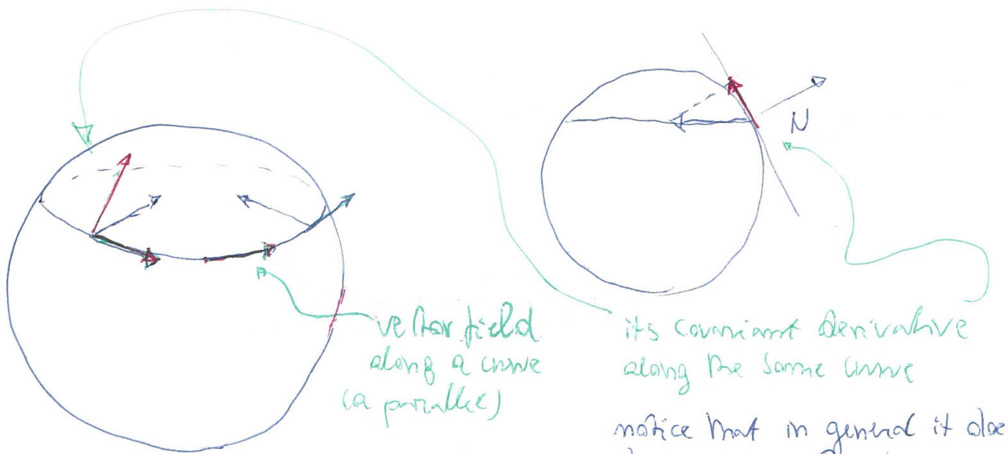
$\bar V = \bar V(u,v)$
vector field on Σ

$$\nabla_u \frac{\partial \bar V}{\partial u}(p) = \frac{\partial \bar V}{\partial u}(p) - \left\langle \underline{N}, \frac{\partial \bar V}{\partial u}\right\rangle(p)\, \underline{N}(p)$$

$$= \Pi_p \frac{\partial \bar V}{\partial u}(p)$$

orthogonal projection onto $T_p\Sigma$

Example:



vector field along a curve (a parallel)

its covariant derivative along the same curve

notice that in general it does not vanish. This happens on the equator... see further on.