Lectures on

# DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

Prof. Mauro Spera, UCSC Brescia

## Lecture XXXII

## GEODESICS ON RIEMANNIAN MANIFOLDS

- geodesics
- exponential map

### ★ Geodesics

A geodesic $\gamma$ on a Riemannian manifold $(M, g)$, $\dim M = n$ is a self-parallel curve therein, namely,

$\nabla$ is the Levi-Civita connection

$$(\diamond) \qquad \boxed{\nabla_{\dot\gamma}\dot\gamma = 0}$$

$\gamma = \gamma(t)$

$t \in \mathcal{I} = [0,1]$, say

In local coordinates: $(\gamma : x^i = x^i(t))$

$$(\diamond') \qquad \boxed{\ddot x^r + \Gamma^r_{ij}\dot x^i \dot x^j = 0} \qquad r = 1, 2 \dots n$$

Notice that geodesics come equipped with a natural parameter $s$: $ds = \sqrt{g_{ij}\dot x^i \dot x^j}\, dt$ (arc-length element)

($\diamond$ is not invariant under a parameter change) [$\nabla_X Y$ is tensorial only in $X$]

Actually ($\diamond$) or ($\diamond'$) can be deduced from two different variational principles, associated with Lagrangians

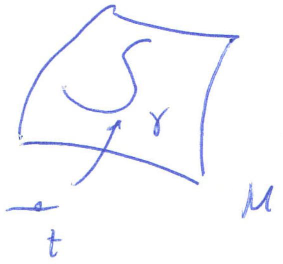
$$\mathcal{L}_1 := \frac{1}{2} g_{ij}\dot x^i \dot x^j \equiv \mathcal{L}_1(x, \dot x)$$

~ kinetic energy of a unit mass particle freely moving on $M$.

with action

$$S_1(\gamma) = \int_0^1 \mathcal{L}_1(x, \dot x)\, dt$$

"energy of $\gamma$"



and $\mathcal{L}_2 = \sqrt{g_{ij}\dot{x}^i\dot{x}^j}$

with action $\underset{\text{length of } \gamma}{\underset{\parallel}{S_2(\gamma)}} = \int_0^1 \sqrt{g_{ij}\dot{x}^i\dot{x}^j}\,dt$

(★) Precisely: (◇) (actually (◇')) can be viewed as the Euler-Lagrange equations attached to $\mathcal{L}_1$ or $\mathcal{L}_2$, provided $t \propto s$
proportional

$$\boxed{\begin{gathered}\mathcal{L} = \mathcal{L}(q^i, \dot{q}^i) \\ \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^j} - \frac{\partial \mathcal{L}}{\partial q^j} = 0 \\ j = 1 \ldots n\end{gathered}}$$

The E-L equation for $\mathcal{L}_2$ becomes, indeed

$$\frac{d}{dt}\,\frac{g_{kj}\dot{x}^j}{\sqrt{g_{ij}\dot{x}^i\dot{x}^j}} = \frac{1}{2}\,\frac{\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j}{\sqrt{g_{ij}\dot{x}^i\dot{x}^j}}$$

constant

"Newton's law"
$\underline{F} = m\underline{a}$

$$\Rightarrow \left[\frac{d}{dt}\,\underbrace{g_{kj}\dot{x}^j}_{p_k} = \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\dot{x}^i\dot{x}^j\right]$$

momentum

which is (◇'), after a short computation

notice: it is a covector, equal to $(\text{velocity})^\flat$

Let us check (★)

depend only on $(x^i)$

$$\boxed{\mathcal{L} = \frac{1}{2} g_{ij}\, \dot{x}^i \dot{x}^j}$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^k} = \frac{1}{2} g_{ij} \underbrace{\frac{\partial \dot{x}^i}{\partial \dot{x}^k}}_{\delta_{ik}} \dot{x}^j + \frac{1}{2} g_{ij}\, \dot{x}^i \underbrace{\frac{\partial \dot{x}^j}{\partial \dot{x}^k}}_{\delta_{jk}}$$

$k$ fixed

$$= \frac{1}{2} g_{kj}\, \dot{x}^j + \frac{1}{2} g_{ik}\, \dot{x}^i = g_{kj}\, \dot{x}^j$$

equal

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^k}\right) = \frac{d}{dt}\left(g_{kj}\, \dot{x}^j\right) = \dot{g}_{kj}\, \dot{x}^j + g_{kj}\, \ddot{x}^j$$

chain rule

$$= \frac{\partial g_{kj}}{\partial x^i} \dot{x}^i \dot{x}^j + g_{kj}\, \ddot{x}^j .$$

Compute

$$\frac{\partial \mathcal{L}}{\partial x^k} = \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k} \dot{x}^i \dot{x}^j$$

The expression $\boxed{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}^k} - \frac{\partial \mathcal{L}}{\partial x^k} = 0}$ becomes

$$g_{kj}\, \ddot{x}^j + \left(\frac{\partial g_{ki}}{\partial x^i} - \frac{1}{2} \frac{\partial g_{ij}}{\partial x^k}\right) \dot{x}^i \dot{x}^j = 0$$

Let us multiply both sides by the inverse matrix $(g^{km})$,

$$g^{km} g_{jk} = g^{mk} g_{kj} = \delta^m_j = \delta_{mj}$$

We obtain: $\longrightarrow$

$$\ddot{x}^m + g^{km}\left(\frac{\partial g_{jk}}{\partial x^i} - \frac{1}{2}\frac{\partial g_{ij}}{\partial x^k}\right)\dot{x}^i\dot{x}^j = 0$$

m is fixed

Now (crucial step!) (the sums over $k, i, j$. are equal)

$$g^{km}\frac{\partial g_{jk}}{\partial x^i}\dot{x}^i\dot{x}^j = \frac{1}{2}\dot{x}^i\dot{x}^j g^{km}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j}\right)$$

=>

$$\ddot{x}^m + \underbrace{\frac{1}{2}g^{km}\left(\frac{\partial g_{jk}}{\partial x^i} + \frac{\partial g_{ik}}{\partial x^j} - \frac{\partial g_{ij}}{\partial x^k}\right)}_{\Gamma^m_{ij}}\dot{x}^i\dot{x}^j = 0$$

whence

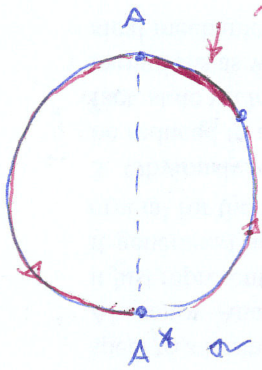
$$\boxed{\ddot{x}^m + \Gamma^m_{ij}\dot{x}^i\dot{x}^j = 0} \qquad m = 1, 2 \ldots n$$

which is $(\diamond')$.

Thus the geodesics, parametrized by arc length, are critical points of the length functional

They are not necessarily minima!

Sphere:



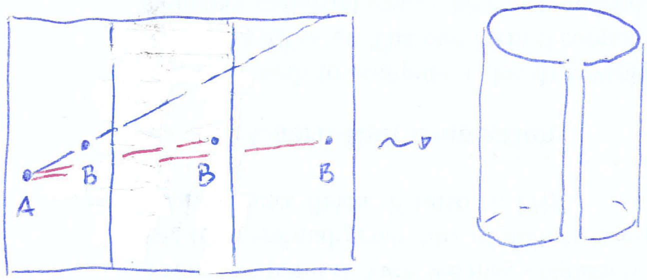

minimal geodesic $\widehat{AB}$

$\widehat{AA^*B}$ is not a minimal geodesic

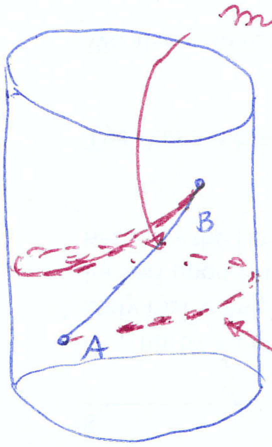
$A^*$ = antipodal point $\leadsto$ $A^*$ is a conjugate point (to A)

geodesics = great circles
(cerchi massimi)

Another example (cylinder)




upon wrapping, the straight lines become helices (geodesics on a cylinder)




minimal geodesic

not minimal

... infinitely many!

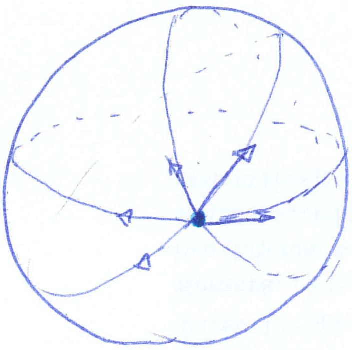
The present discussion would naturally lead us to "calculus of variations in the large" (global analysis)

a really fascinating topic which will not be touched upon in the present elementary course.

★ (Riemannian) exponential mapping

(or map)

precursor: Al-Biruni, XI century A.C.



Let $(M, g)$ be a Riemannian manifold. Define, for $p \in M$

$$\mathrm{Exp}_p : T_pM \longrightarrow M$$

$$\begin{array}{c} \cup \\ X = X_p \longmapsto \gamma^{u}(\|X\|) \end{array}$$

length of $X$

$X = \|X\| \cdot u$

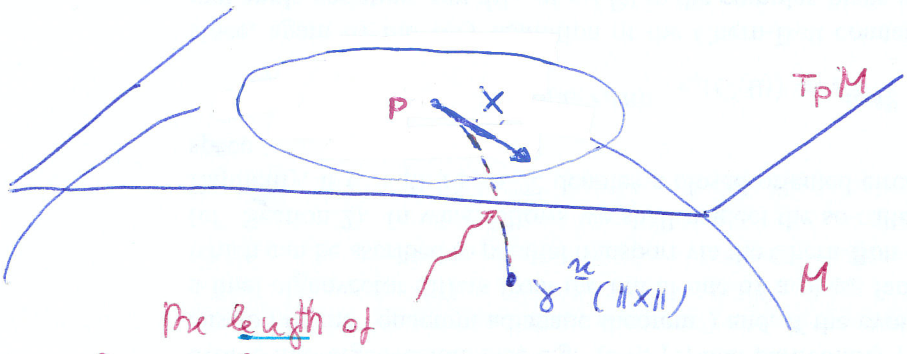
unit vector

geodesic emanating from $p$, with unit velocity - upon employment of arc-length - along $X$

In words: given $X$, traverse the geodesic path leaving from $p$ in the direction of $X$ for a stretch of length $\|X\|$.

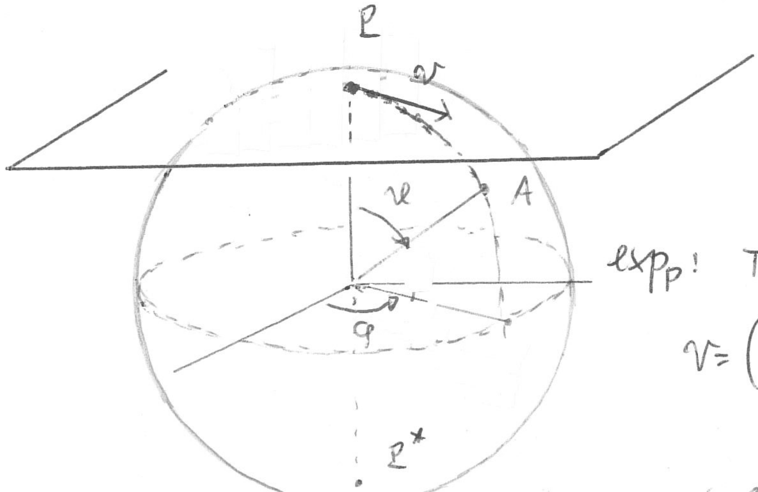
Locally, $\mathrm{Exp}_p$ is a diffeomorphism (since its differential at $0$ is the identity, but, in general, it is not a global one (see below)

$0 \in T_pM$




the length of this geodesic arc is $\|X\|$

# Example : The exponential map on $S^2$




$p : (0,0,1)$

$T_p S^2 \cong \{ z=1 \} \cong \mathbb{R}^2$

$\exp_p : T_p S^2 \to S^2$

$$v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \longmapsto \begin{pmatrix} x = \sin\sqrt{v_1^2+v_2^2}\,\dfrac{v_1}{\sqrt{v_1^2+v_2^2}} \\ y = \sin\sqrt{\ }\cdot\dfrac{v_2}{\sqrt{v_1^2+v_2^2}} \\ z = \cos\sqrt{\ } \end{pmatrix}$$

$\vartheta = \theta = \|v\| = (v_1^2+v_2^2)^{\frac{1}{2}}$

$$\begin{cases} x = \sin\vartheta\cos\varphi \\ y = \sin\vartheta\sin\varphi \\ z = \cos\vartheta \end{cases},$$

$\vartheta \in [0,\pi]$ "colatitude"

$\varphi \in [0,2\pi)$ "longitude"

At first order, for $v \to 0$

$\dfrac{\sin\xi}{\xi} \to 1$ if $\xi \to 0$

$$\begin{cases} x = v_1 + \dots \\ y = v_2 + \dots \\ z = 1 + \dots \end{cases}$$

i.e. $\boxed{(\exp_p)_*\big|_0 = \mathrm{Id}}$

Notice that <u>$\exp_p$ is not injective</u> : all $v$ s.t. $\|v\| = \pi$ yield <u>the same point $p^*$</u> $= (0,0,-1)$ (<u>conjugate</u> to $p$)

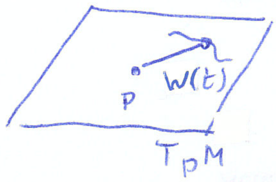
for $\|v\| < \pi$ $\exp_p$ is <u>injective</u> ($\pi$ = <u>injectivity radius</u>)

## * The Gauss lemma

In a Riemannian manifold $(M, g)$, at a given $p \in M$, consider the immersed parametric surfaces $\Sigma$:

$$\gamma(s,t) = \exp_p s\,\bar{w}(t), \quad \text{with } \|\bar{w}(t)\| = 1$$

$t \in \mathcal{I}$ some interval





the map

$$t \longmapsto \gamma(\ell, t)$$

is called an arc of geodesic circle on $\Sigma$, with centre $p$ and radius $\ell$

One has then the following, important – intuitively clear –

* Gauss lemma: geodesic circles are perpendicular to the geodesic emanating from their centres.

Proof. We have to show that

$$\left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle = 0$$

We compute $\displaystyle \frac{d}{ds}\left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \right\rangle =$

$$\Big\langle \underbrace{\frac{\nabla}{ds}\frac{\partial\gamma}{\partial s}}_{=0}, \frac{\partial\gamma}{\partial t} \Big\rangle + \left\langle \frac{\partial\gamma}{\partial s}, \frac{\nabla}{ds}\frac{\partial\gamma}{\partial t} \right\rangle$$

($\gamma(\cdot, t)$ is a geodesic)

$$= \left\langle \frac{\partial\gamma}{\partial s}, \frac{\nabla}{dt}\frac{\partial\gamma}{\partial s} \right\rangle$$

symmetry property of $\nabla$

[we shall check this in a moment]

$$= \frac{1}{2}\frac{d}{dt}\underbrace{\left\langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial s} \right\rangle}_{=1}$$

$$= 0$$

Therefore $\frac{d}{ds} \langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \rangle = 0$

$\Rightarrow \langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \rangle$ does not depend on $s$.

But, since $\gamma(0,t) = p \quad \forall t$, we have

$$\frac{\partial\gamma}{\partial t}(0,t) = 0 \Rightarrow \langle \frac{\partial\gamma}{\partial s}, \frac{\partial\gamma}{\partial t} \rangle = 0$$

as asserted.

We complete the proof as follows

$$\boxed{\frac{\nabla}{\partial s}} \frac{\partial\gamma}{\partial t} = \frac{\nabla}{\partial s}\Big(\underbrace{\frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}}_{\text{local coord.}}\Big) = \frac{\partial^2 x^i}{\partial s\partial t}\frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial t}\frac{\nabla}{\partial s}\Big(\frac{\partial}{\partial x^i}\Big) =$$

$$= \frac{\partial^2 x^i}{\partial s\partial t}\frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\underbrace{\nabla_{\frac{\partial}{\partial x^j}}\Big(\frac{\partial}{\partial x^i}\Big)}_{=\Gamma^k_{ji}\partial_k}$$

$$= \frac{\partial^2 x^i}{\partial s\partial t}\frac{\partial}{\partial x^i} + \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s}\Big(\nabla_{\frac{\partial}{\partial x^i}}\frac{\partial}{\partial x^j}\Big) = \boxed{\frac{\nabla}{\partial t}\frac{\partial\gamma}{\partial s}}$$

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$$\frac{\nabla}{ds}\Big(\frac{\partial}{\partial x^i}\Big) = \nabla_{\frac{\partial x^j}{\partial s}\frac{\partial}{\partial x^j}}\frac{\partial}{\partial x^i} = \frac{\partial x^j}{\partial s}\nabla_{\partial_j}\partial_i$$

tensoriality

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