Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

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Lecture XXXIII

CURVATURE

★ Riemann curvature tensors

Let (M, g) be a Riemannian manifold, with Levi-Civita connection ∇. $\mathfrak{X} \equiv \mathfrak{X}(M)$

The Riemann curvature operator $R : \mathfrak{X} \times \mathfrak{X} \to End(\mathfrak{X})$ (endomorphisms of $\mathfrak{X}$, viewed as a vector space) is defined as

$$\boxed{R(X,Y) = \underbrace{\nabla_{[X,Y]}}_{\text{Lie bracket}} - \underbrace{[\nabla_X, \nabla_Y]}_{\text{operator commutator}}} = \nabla_{[X,Y]} - \nabla_X \nabla_Y + \nabla_Y \nabla_X$$

This yields indeed a linear operator on $\mathfrak{X}$:

$$Z \longmapsto R(X,Y)Z$$

Setting again $g = \langle \, , \, \rangle$, the Riemann curvature tensor is the $(0,4)$ tensor defined via:

$$\boxed{R(X,Y,Z,W) := \langle R(X,Y)Z, W \rangle}$$

in coordinates ($X = \partial_i$ etc.) $\{ R_{ijkl} \}$

Algebraically, $-R(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$ represents the obstruction against the linear map $\mathfrak{X} \ni X \longmapsto \nabla_X \in End(\mathfrak{X})$ being a Lie algebra representation as well.

$R(\cdot,\cdot,\cdot,\cdot)$ is indeed a tensor,

one checks, in fact, via direct computation

$$(\diamond) \qquad \boxed{R(\alpha X, \beta Y, \gamma Z, \delta W) = \alpha\beta\gamma\delta\, R(X,Y,Z,W)}$$

The map $(X, Y, Z) \longmapsto R(X,Y)Z$

yields, in turn, a $(1,3)$ tensor (also called Riemann curvature tensor) input: 3 vectors, output: 1 vector

(its tensor character being assessed "en route" to $(\diamond)$).

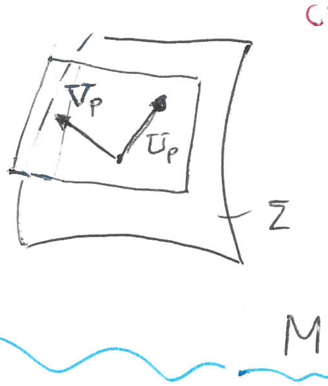
Let $p \in M$, $U_p, V_p \in T_pM$, $\langle U_p, V_p\rangle_p = 0$, $\|U_p\| = \|V_p\| = 1$

The number $\boxed{\star\ \langle R(U_p, V_p)\, U_p, V_p\rangle_p}$ is called

sectional curvature of M at p along the 2-plane with "position" $= \mathrm{span}(U_p, V_p)$.

We shall check that $\star$ is the Gaussian curvature, at p of the parametric surface $Z : (u,v) \longmapsto \mathrm{Exp}_p(uU + vV)$

! $\star$ Sectional curvatures allow reconstruction of the full curvature tensor



This will follow from the following identities satisfied by the curvature tensor

(i) $R(X,Y,Z,W) = -R(Y,X,Z,W) = -R(X,Y,W,Z)$ clear

cyclical permutations..

(ii) $R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$ $\star$ Bianchi identity

(iii) $R(X,Y,Z,W) = R(Z,W,X,Y)$

Let us provide some details:

First, proving $R(X,Y,Z,W) = -R(X,Y,W,Z)$

is equivalent to proving

$$R(X,Y,Z,Z) = 0$$

As for the latter, we compute

$$R(X,Y,Z,Z) = \langle \nabla_Y \nabla_X Z - \nabla_X \nabla_Y Z + \nabla_{[X,Y]} Z, Z \rangle$$

Now $\langle \nabla_Y \nabla_X Z, Z \rangle = Y \langle \nabla_X Z, Z \rangle - \langle \nabla_X Z, \nabla_Y Z \rangle$

(by ①)

and $\langle \nabla_{[X,Y]} Z, Z \rangle = \frac{1}{2} [X,Y] \langle Z, Z \rangle$

(again by ①: $X \langle Y,Y \rangle = \langle \nabla_X Y, Y \rangle + \langle Y, \nabla_X Y \rangle$
$= 2 \langle \nabla_X Y, Y \rangle$)

Therefore:

$$R(X,Y,Z,Z) = Y \langle \nabla_X Z, Z \rangle - X \langle \nabla_Y Z, Z \rangle + \frac{1}{2} [X,Y] \langle Z,Z \rangle$$

$$= \underbrace{\frac{1}{2} YX \langle Z,Z \rangle - \frac{1}{2} XY \langle Z,Z \rangle}_{-\frac{1}{2}[X,Y]\langle Z,Z\rangle} + \frac{1}{2} [X,Y] \langle Z,Z \rangle = 0.$$

In order to prove $R(X,Y,Z,W) = R(Z,W,X,Y)$

one starts from Bianchi (which can be checked directly)

$$R(X,Y,Z,W) + R(Y,Z,X,W) + R(Z,X,Y,W) = 0$$

and cyclically permutes all arguments, obtaining four equations, which, after summation, lead to

$$2\,R(Z,X,Y,W) + 2\,R(W,Y,Z,X) = 0$$

$$\Rightarrow \quad R(Z,X,Y,W) = R(Y,W,Z,X)$$, equivalent to the desired assertion

$\square$

★★ Theorem: The sectional curvature determines the Riemann curvature.

Proof (sketch) The Bianchi identity shows that

$$\boxed{6R(X,Y,Z,W) = \frac{\partial^2}{\partial\alpha\,\partial\beta}\Big(R(X+\alpha Z,\ Y+\beta W,\ X+\alpha Z,\ Y+\beta W) - R(X+\alpha W,\ Y+\beta Z,\ X+\alpha W,\ Y+\beta Z)\Big)\Big|_{\substack{\alpha=0\\ \beta=0}}}$$

yielding the conclusion. □

Let us compute the Riemann curvature tensor in local coordinates $\nabla_k \equiv \nabla_{\partial_k}$ etc.

$$\Gamma^k_{ij} = \Gamma^k_{ji}$$

$$\underbrace{(\nabla_k \nabla_\ell - \nabla_\ell \nabla_k)}_{(1,2)} \underset{(1,0)}{Z^i} = -\underset{(1,3)}{\overbrace{R^i_{qk\ell}}^{(1,2)}} Z^\ell$$

$$\nabla_k \nabla_\ell Z^i = \nabla_k\left(\frac{\partial Z^i}{\partial x^\ell} + \Gamma^i_{q\ell} Z^q\right) = \frac{\partial}{\partial x^k}\left(\frac{\partial Z^i}{\partial x^\ell} + \Gamma^i_{q\ell} Z^q\right) +$$

$$+ \Gamma^i_{pk}\left(\frac{\partial Z^p}{\partial x^\ell} + \Gamma^p_{q\ell} Z^q\right) - \Gamma^p_{\ell k}\left(\frac{\partial Z^i}{\partial x^p} + \Gamma^i_{q,p} Z^q\right) \quad ⚠$$

with a similar expression for $\nabla_\ell \nabla_k Z^i$. Thus we obtain

$$(\nabla_k \nabla_\ell - \nabla_\ell \nabla_k) Z^i = \left(\frac{\partial \Gamma^i_{q\ell}}{\partial x^k} - \frac{\partial \Gamma^i_{qk}}{\partial x^\ell}\right) Z^q + \left(\Gamma^i_{pk}\Gamma^p_{q\ell} - \Gamma^i_{p\ell}\Gamma^p_{qk}\right) Z^q$$

$$+ \underbrace{\left(\Gamma^p_{\ell k} - \Gamma^p_{k\ell}\right)}_{\substack{\parallel \\ 0}} \frac{\partial Z^i}{\partial x^p}$$

$$\Rightarrow \boxed{-R^{i}_{qkl} = \frac{\partial \Gamma^{i}_{ql}}{\partial x^{k}} - \frac{\partial \Gamma^{i}_{qk}}{\partial x^{l}} + \Gamma^{i}_{pk}\Gamma^{p}_{ql} - \Gamma^{i}_{pl}\Gamma^{p}_{qk}}$$

The Riemann tensor R_{ijkl} is obtained via

$$\boxed{R_{ijkl} = g_{im} R^{m}_{jlk}}$$

★ Geometric meaning of curvature

up to now we just performed algebra ; we must understand the geometric nature of curvature lying beneath the formalism. Actually, we follow Riemann himself. Let us tackle the following problem : is it possible, on a Riemannian manifold, to find a local coordinate system such that

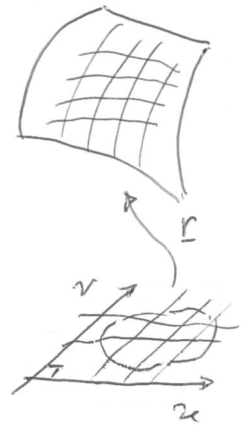
$$\underset{g}{ds^2} = \sum_{i=1}^{n} dy^{i^2} = \delta_{ij}\, dy^i dy^j \qquad \text{(Euclidean metric)}$$

The answer is, in general, NO. This happens precisely when $R_{ijkl} = 0$, i.e. when all Christoffel symbols vanish. [Notice that even if all Christoffel symbols can be made to vanish at a point, their partial derivatives do not vanish thereat, in general]. So, the Riemann curvature tensor measures the obstruction against finding local coordinate system having a Euclidean character. We shall now try and understand this.

Since the Riemann tensor is determined by the sectional curvatures, we resort to a 2-dimensional situation, and resume the formula we used (see the geometry course) in proving the Gauss Theorema Egregium (on a surface Σ)

Gaussian curvature

$$[\nabla_u, \nabla_v]\, \underline{r}_u = K \cdot F\, \underline{r}_u - K \cdot E\, \underline{r}_v$$



Levi Civita connection, defined geometrically (†)

Set $F = 0$ for simplicity

$E = \langle r_u, r_u \rangle = g_{uu}$

$$[\nabla_u, \nabla_v]\, \underline{r}_u = -K\, g_{uu}\, \underline{r}_v$$

Then

$$\langle [\nabla_u, \nabla_v]\, \underline{r}_u, \underline{r}_v \rangle = -K\, g_{uu}\, g_{vv}$$

|| abstract interpretation

$$+\langle R(\partial_u, \partial_v)\partial_u, \partial_v \rangle = +K\, g_{uu}\, g_{vv}$$

Choose, at $p \in \Sigma$, an orthonormal basis $(\underset{u}{e_1}, \underset{v}{e_2})$ of $T_p\Sigma$

Then

$$\langle R(e_1, e_2)e_1, e_2 \rangle = K\, \underset{1}{g_{11}}\, \underset{1}{g_{22}}$$

$$\Rightarrow \quad \boxed{R_{1212} = K}$$

in general

$$R_{1212} = K(g_{11}g_{22} - g_{12}^2) = eg - f^2$$

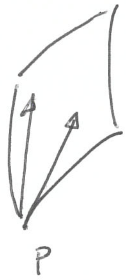
coeff. of the second fund. form

This achieves the geometric significance of the sectional curvature we hinted at above.

(†) $(\nabla_u \underline{V})(p) = \dfrac{\partial V}{\partial u} - \langle \underline{N}, \underline{V} \rangle \underline{N} = \Pi_p \dfrac{\partial V}{\partial u}$

i.e. compute $\dfrac{\partial V}{\partial u}$ and project onto $T_p\Sigma$ (at each p)

Also, we see that the curvature tensor detects parallel transport along an "infinitesimal rectangle"



$$\boxed{[\nabla_i, \nabla_j]\, w^k = -\underset{\text{a matrix}}{R^k_{\ ijl}}\, w^l}$$

[One can devise "non abelian generalizations of Stokes' theorem]

★ This is the original Levi-Civita interpretation.

Remember that for surfaces we had the formula

$$\alpha = \iint_{\partial D} K\, d\sigma$$

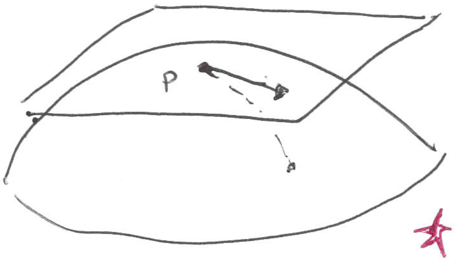
angle of parallelism

Therefore, much can be obtained by reducing to a 2-dimensional context.

Let us introduce geodesic polar coordinates, and their cartesian counterparts (called normal coordinates)

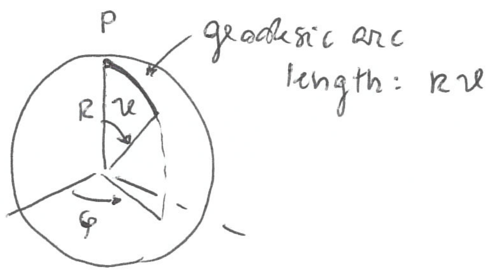
Since $K = K(E, \dots E_u, \dots E_{uu} \dots)$

partial derivatives of the Ist fundamental form (metric) up to order 2



★ one can work, up to order 2, and locally, on a (pseudo-) sphere of radius $R = \frac{1}{\sqrt{K}}$ $K = K(P)$

(assume $K > 0$ for definiteness)



$$ds^2 = R\,d\vartheta^2 + R^2 \sin^2\vartheta \, d\varphi^2$$

* geodesic polar coordinates

$$\begin{cases} x = R\vartheta \cos\varphi \\ y = \underbrace{R\vartheta}_{\rho} \sin\varphi \end{cases} \qquad \begin{cases} \varphi = \arctan \dfrac{y}{x} \\ \vartheta = \dfrac{1}{R}\sqrt{x^2+y^2} \end{cases}$$

normal coordinates

Now (crucial remark)

$$\sin^2\vartheta = \left(\vartheta - \frac{\vartheta^3}{6} + \dots\right)^2 = \vartheta^2 - \frac{1}{3}\vartheta^4 + \dots$$

Therefore

$$ds^2 = \underbrace{R^2(d\vartheta^2 + \vartheta^2 d\varphi^2)}_{\substack{\text{Euclidean polar coordinates} \\ \| \\ dx^2+dy^2}} \overbrace{\underbrace{- \frac{1}{3} R^2 \vartheta^4 d\varphi^2}_{\text{compute this!}}}^{(\diamond)}$$

$$d\varphi = \frac{x\,dy - y\,dx}{x^2+y^2} \qquad \vartheta = \frac{1}{R}\sqrt{x^2+y^2}$$

we get:

$$(\diamond) = -\frac{1}{3} \cdot R^2 \cdot \frac{1}{R^4}(x^2+y^2)^2 \frac{(x\,dy - y\,dx)^2}{(x^2+y^2)^2}$$

$$= -\frac{1}{3} \underbrace{\frac{1}{R^2}}_{K} \left(y^2 dx^2 - 2xy\,dx\,dy + x^2 dy^2 \right)$$

$$ds^2 = g_{ij}\,dx^i dx^j$$

namely

$$\begin{cases} g_{11} = E = 1 - \dfrac{K}{3} y^2 \\ g_{12} = g_{21} = F = \dfrac{2}{3} K xy \\ g_{22} = G = 1 - \dfrac{K}{3} x^2 \end{cases}$$

← off-diagonal terms

Notice that the presence of $K = K(p) > 0$ (even at a single point), prevents us from defining a Cartesian coordinate system in a neighborhood of p, due to the presence of non-diagonal terms in the coefficients of the metric.

This argument is general:

$$\boxed{g_{ij} = \delta_{ij} - \frac{1}{3} \overset{\circ}{R}_{ikje} \, x^k x^e + \dots}$$

normal coordinates ↗

This achieves Riemann's interpretation of his curvature tensor

★ Also notice that, from the above expressions, in a normal coordinate system (centred at p) Christoffel symbols vanish at p (but not their derivatives, in general...)

(since $\frac{\partial g_{ij}}{\partial x^k}(p) = 0$).

Another sketchy argument runs as follows:
In dimension 2 everything is clear in view of the existence of isothermal coordinates, whereby one finds $g = \underset{>0}{\lambda(u,v)}(du^2 + dv^2)$

so $K = 0$ is tantamount to $\Delta \log \lambda = 0$

$\Rightarrow \lambda =$ constant (under suitable conditions).

The above reasoning can be carried out for all sectional curvatures: they all vanish if and only if the embedded exp-surfaces are flat (locally, 2-planes). Therefore, since sectional curvatures determine the full curvature tensor, their vanishing implies the vanishing of the latter.