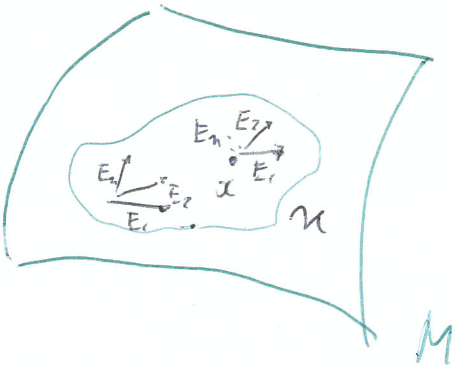
☆ Riemannian geometry à la Cartan

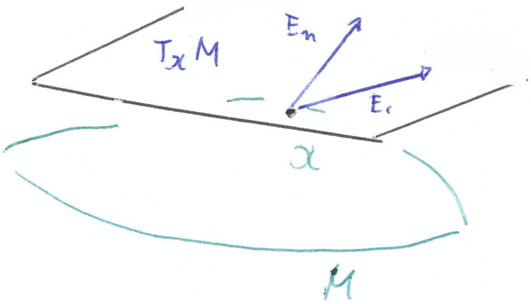
LECTURE XXXIV

• Cartan calculus & Riemannian geometry



frame field $E = (E_1 \dots E_n)$

E_i, $i = 1 \dots n$ vector fields on M (defined on a certain open set $\mathcal{U} \subset M$, yielding, at each point $x \in \mathcal{U}$ for $T_x M$.



$$\Theta = \begin{pmatrix} \Theta^1 \\ \vdots \\ \Theta^n \end{pmatrix}$$ dual collection of differential forms

$$\boxed{\Theta^i(E_j) = \delta^i_j}$$

example: $E = (\partial_1 \dots \partial_n)$ $\Theta = \begin{pmatrix} dx^1 \\ \vdots \\ dx^n \end{pmatrix}$ (local coordinates)

If $(M, \underset{\langle , \rangle}{g})$ is a Riemannian manifold, then one can define orthonormal frames : $\langle E_i, E_j \rangle = \delta_{ij}$

Consequently if $v \in T_x M$, $v = \Theta^i(v) E_i$

and $g(v,v) \equiv \langle v, v \rangle = \sum_{i=1}^{n} [\Theta^i(v)]^2$

or, in short

$$(*) \qquad \boxed{ds^2 = \sum_{i=1}^{n} (\Theta^i)^2} \qquad g = \Theta^T \Theta$$

Conversely, given $*$, one gets an orthonormal frame field.

Tautological tensor field of type (1,1)

$$I(v) = v \qquad \forall\, v \in TM_p$$

concretely, in terms of a frame field

$$I = E_j \otimes \theta^j \qquad \text{(Einstein)}$$

$$\equiv E \otimes \theta$$

connection & curvature forms

$$\boxed{\omega^i{}_j(\underset{\text{v. field}}{\xi}) := \theta^i(\nabla_\xi E_j)}$$ Levi-Civita

$$\nabla_\xi E_j = \omega^m{}_j(\xi) E_m$$

Setting $\omega = (\omega^i{}_j)$, one can prove

Cartan's first structural equation

$$(\star) \quad \boxed{d\theta + \omega \wedge \theta = 0}$$

i.e. in full $\quad d\theta^i + \omega^i{}_m \wedge \theta^m = 0$

Proof. Start from

$$d\theta(X,Y) = X(\theta(Y)) - Y(\theta(X)) - \theta([X,Y])$$

Then we have

$$d\theta^i(E_a,E_b) = \underbrace{E_a(\theta^i(E_b))}_{=0} - \underbrace{E_b(\theta^i(E_a))}_{=0} - \theta^i([E_a,E_b])$$

constants = 0, 1

$$\Rightarrow \quad d\theta^i(E_a,E_b) = -\theta^i([E_a,E_b])$$

Then compute

$$(\underbrace{\omega^i_m \wedge \theta^m}_{\alpha})(E_a, E_b) = (i_{E_a}\alpha)(E_b)$$

$\theta^m(E_b) = \delta^m_b$

$$\Big[\omega^i_m(E_a)\,\theta^m - \underbrace{\theta^m(E_a)}_{\delta^m_a}\,\omega^i_m\Big](E_b)$$

$$= \omega^i_b(E_a) - \omega^i_a(E_b) =$$

see

$\omega^i_j(\xi) = \theta^i(\nabla_\xi E_j)$

$$= \theta^i(\nabla_{E_a}(E_b)) - \theta^i \nabla_b(E_a) =$$

$$= \theta^i\big(\nabla_{E_a}(E_b) - \nabla_b(E_a)\big)$$

$$= \theta^i([E_a, E_b])$$

The Levi-Civita connection is torsion free:

$T(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y] = 0$

The result follows. $\square$

One also has

$$\omega^i_j(\xi) = \theta^i(\nabla_\xi(E_j)) = \langle \nabla_\xi E_j, E_i \rangle$$

Then, from $d\langle E_i, E_j\rangle = 0$ we have

metricity

$$\boxed{0 = \xi(\langle E_i, E_j\rangle)} = \langle \nabla_\xi E_i, E_j\rangle + \langle E_i, \nabla_\xi E_j\rangle$$

$$= \omega^i_j(\xi) + \boxed{\omega^j_i(\xi)},$$

namely

$$\omega^i_j = -\omega^j_i$$

i.e ω is antisymmetric

The clue to a generic Cartan-like computation is to get ω from $d\theta + \omega \wedge \theta = 0$ (see below)

curvature form

Let $\Omega = (\Omega^i_j)$ with Ω^i_j defined so that

→ matrix of 2-forms, see below

(★) $\boxed{R_{\xi\eta}(E_j)(p) = \Omega^i_j(\xi,\eta) E_i(p)}$

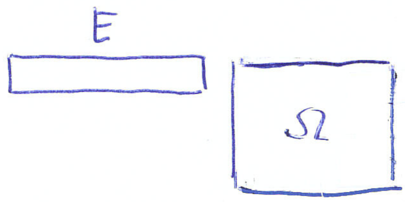
curvature operator

Since $R_{\xi\eta} = -R_{\eta\xi}$, $\Omega^i_j(\xi,\eta) = -\Omega^i_j(\eta,\xi)$

skew-symmetry

$\Rightarrow \Omega = (\Omega^i_j)$ matrix of 2-forms

Let us interpret (★) in matrix form:



right multiplication of E by Ω

We are in a position to prove

Cartan's second structural equation

(★★) $\boxed{d\omega + \omega \wedge \omega = \Omega}$

That is, in full $\boxed{\Omega^i_j = d\omega^i_j + \omega^i_m \wedge \omega^m_j}$

! different convention

$$R(X,Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$$

Proof

$$R_{E_a E_b}(E_j) = \overset{①}{\nabla_{E_a}(\nabla_{E_b} E_j)} - \underset{②}{\nabla_{E_b}(\nabla_{E_a} E_j)} - \underset{③}{\nabla_{[E_a,E_b]} E_j}$$

From $\nabla_{E_b}(E_j) = \omega^i_j(E_b) E_i$ we get, successively:

① $$\nabla_{E_a}(\nabla_{E_b} E_j) = E_a(\omega^i_j(E_b)) E_i + \omega^m_j(E_b) \nabla_{E_a} E_m$$

$$\nabla_{E_a}(\underbrace{\omega^i_j(E_b)}_{f} \underbrace{E_i}_{Y})$$

$$\nabla_X(fY) = (df, X) Y + f \nabla Y = X(f) Y + f \nabla Y$$

$$= E_a(\omega^i_j(E_b)) E_i + \omega^m_j(E_b)\, \omega^i_m(E_a) E_i$$

② $$\nabla_{E_b}(\nabla_{E_a} E_j) = E_b(\omega^i_j(E_a)) E_i + \omega^m_j(E_a)\, \omega^i_m(E_b) E_i$$

③ $$\nabla_{[E_a,E_b]} E_j = \omega^i_j([E_a,E_b]) E_i$$

$$\Rightarrow \quad R_{E_a E_b} \cdot E_j = \overbrace{E_a\, \omega^i_j(E_b) - E_b\, \omega^i_j(E_a) - \omega^i_j([E_a,E_b])}^{d\omega^i_j(E_a,E_b) \;=\; d\omega} E_i$$

$$+ \underbrace{[\omega^i_m(E_a)\, \omega^m_j(E_b) - \omega^i_m(E_b)\, \omega^m_j(E_a)]}_{(\omega^i_m \wedge \omega^m_j)(E_a,E_b) \;\equiv\; \omega \wedge \omega} E_i$$

$$\boxed{E} \quad \boxed{\Omega}$$

$$\boxed{d\omega + \omega \wedge \omega = \Omega}$$

(*) $d\theta + \omega \wedge \theta = 0$

(**) $d\omega + \omega \wedge \omega = \Omega$

Cartan structural equations

$(\ast)$

Solving $\{ d\theta + \omega \wedge \theta = 0 \}$ for ω $(= -\omega^T)$

<u>Lemma</u> (Cartan). Let $\{x_i\}$ $i = 1..p$ l. ind. $(x_i \in \bar{V})$

Let $\{y_i\}_{i=1..p}$ $y_i \in \bar{V}$ satisfying

$(\ast)$ $\boxed{x_1 \wedge y_1 + \dots x_p \wedge y_p = 0}$

Then $\boxed{y_j = \sum_{k=1}^{p} A_{jk} x_k \qquad \text{with} \quad A_{jk} = A_{kj}}$

<u>Proof</u>. Complete $(x_1 .. x_p)$ to a basis of $\bar{V}$: $(x_1 - x_p \, x_{p+1} - x_n)$

Write $y_i = \sum_{j=1}^{p} A_{ij} x_j + \sum_{k=p+1}^{n} B_{ik} x_k$

Substitution into $(\ast)$ yields

$$\sum_{1 < j \leq p} (A_{ij} - A_{ji}) x_i \wedge x_j + \sum_{i \leq p < k} B_{ik} x_i \wedge x_k = 0$$

But $\{x_k \wedge x_l\}_{k<l}$ yield a basis for $\Lambda^2(\bar{V})$,

whence $A_{ij} = A_{ji}$, $B_{ik} = 0$ □

** Now, the solution of $(\ast)$ is <u>unique</u>: indeed, let $\sigma = \omega - \omega'$

Then $\sigma \wedge \theta = 0 \Rightarrow \sigma^i{}_k = \sum_j A^i{}_{jk} \theta^j$ $\quad A^i{}_{jk} = A^i{}_{kj}$

we also have $A_{jk}{}^i = -A^j{}_{ki}$ (ω, ω' are skew-symmetric)

Therefore $\{A^i{}_{jk} = A^i{}_{kj} = -A_{ij}{}^k = A^j{}_{ki} = A^j{}_{ik} = -A^i{}_{jk}\}$

$\Rightarrow A^i{}_{jk} = 0$

★ Corollary: We get a new proof of Levi-Civita's theorem

An important example

☆ Hyperbolic space

$$\boxed{M = \{(x,y,z) \,/\, z>0\}}$$

upper half-space

Metric: $$\boxed{g = \frac{dx^2+dy^2+dz^2}{z^2}}$$

coframe field : $$\boxed{\theta = \frac{1}{z}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}$$ $$\boxed{g = \theta^T \otimes \theta}$$

(globally defined)

$$\boxed{d\theta = \frac{1}{z^2}\begin{pmatrix} dx \wedge dz \\ dy \wedge dz \\ 0 \end{pmatrix}}$$

Sample computation

$$d\left(\frac{dx}{z}\right) = d\left(\frac{1}{z}\right)\wedge dx$$

$$= -\frac{1}{z^2}\, dz \wedge dx$$

$$= +\frac{1}{z^2}\, dx \wedge dz$$

Let us find $\omega = (\omega^i_j)$, $\omega^i_j = -\omega^j_i$ (matrix of 1-forms) enforcing Cartan's equation $d\theta + \omega \wedge \theta = 0$

(ω is unique)

We find $$\boxed{\omega = \frac{1}{z}\begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix}}$$

Check: $$\underbrace{\frac{1}{z}\begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix}}_{\omega} \wedge \underbrace{\frac{1}{z}\begin{pmatrix} dx \\ dy \\ dz \end{pmatrix}}_{\theta} = \frac{1}{z^2}\begin{pmatrix} -dx\wedge dz \\ -dy \wedge dz \\ 0 \end{pmatrix} = -d\theta$$

Curvature form: $$\boxed{\Omega = d\omega + \omega \wedge \omega}$$

$$d\omega = \frac{1}{z^2}\begin{pmatrix} 0 & 0 & -dx\wedge dz \\ 0 & 0 & -dy\wedge dz \\ dx\wedge dz & dy \wedge dz & 0 \end{pmatrix} ; \quad \omega\wedge\omega = \frac{1}{z^2}\begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix} \wedge \begin{pmatrix} 0 & 0 & -dx \\ 0 & 0 & -dy \\ dx & dy & 0 \end{pmatrix}$$

$$= \frac{1}{z^2}\begin{pmatrix} 0 & -dx\wedge dy & 0 \\ -dy\wedge dx & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \frac{1}{z^2}\begin{pmatrix} 0 & -dx\wedge dy & 0 \\ dx \wedge dy & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Hence

$$\boxed{\Omega = d\omega + \omega \wedge \omega = \frac{1}{z^2}\begin{pmatrix} 0 & -dx\wedge dy & -dx\wedge dz \\ dx\wedge dy & 0 & -dy\wedge dz \\ dx\wedge dz & dy\wedge dz & 0 \end{pmatrix}}$$

Let us compute the Riemannian curvature tensor

$$\boxed{E = (E_1, E_2, E_3) \quad \text{frame field}}$$

dual to the coframe field

$$\theta_i(E_j) = \delta_{ij} \qquad E_1 = z\frac{\partial}{\partial x},\quad E_2 = z\frac{\partial}{\partial y},\quad E_3 = z\frac{\partial}{\partial z}$$

$$\langle E_1, E_2\rangle = \langle z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}\rangle = \frac{1}{z^2} z^2 \cdot 0 = 0$$

$$\langle E_1, E_1\rangle = 1 \cdot 1 = 1$$

Recall: $\boxed{R_{E_a E_b}(E_j) = \Omega^i_j(E_a, E_b) E_i}$ (Einstein)

in matrix form $E \cdot \Omega$ — "row" vector, matrix multiplication

compute $\Omega(E_1, E_2)$

sample: $\frac{1}{z^2}(dx\wedge dy)\left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}\right) = 1$

the other terms vanish:

$$\boxed{\Omega(E_1, E_2) = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\begin{array}{c} E_1 \qquad E_2 \qquad E_3 \\ \left(z\frac{\partial}{\partial x}, z\frac{\partial}{\partial y}, z\frac{\partial}{\partial z}\right) \end{array} \begin{array}{c} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ 1 \quad 2 \quad 3 \end{array} = (E_2, -E_1, 0)$$

Therefore

$$R(E_1, E_2)E_1 = E_2$$
$$R(E_1, E_2)E_2 = -E_1 \quad \Rightarrow \quad \langle R(E_1,E_2)E_1, E_2\rangle = 1$$
$$R(E_1, E_2)E_3 = 0 \qquad\qquad \langle R(E_1,E_2), E_1, E_3\rangle = 0$$

$\Rightarrow \langle R(E_1,E_2)E_2, E_1\rangle = -1 \quad \langle R(E_1,E_2)E_3, E_j\rangle = 0 \quad j = 1, 2$

$\Rightarrow$ { M has constant sectional curvature $= -1$ }

For constant (sectional) curvature spaces one has the handy expression

$$\Omega = k \cdot \Theta \wedge \Theta^T$$

sectional curvature

(immediately checked in the previous example)