Lectures on DIFFERENTIAL GEOMETRY AND TOPOLOGY V2

Prof. Mauro Spera, UCSC Brescia

Lecture XXXV

CURVATURE TENSORS

$$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$$

- Curvature operator: $R(X,Y) : Z \longmapsto R(X,Y)Z$

- Curvature tensor

$\langle \, , \, \rangle$ Riemannian metric

$$R(X,Y,Z,T) := \langle R(X,Y)Z, T \rangle$$

in coordinates R_{ijkl} $(\in T^{0,4})$

- Ricci tensor

Trace ↓ ; Ricci operator

$$\mathrm{Ric}(U,V) = \mathrm{Tr}\left(W \longmapsto R(U,W)\bar{V} \right)$$

$$= \sum_i \langle R(U,e_i)\bar{V}, e_i \rangle$$

(e_i) orthonormal bases

locally $\quad R_{ij} = g^{kl} R_{ikjl}$

- Sectional curvature : $\langle R(U,V)U, V \rangle$ = Gaussian curvature of $\exp_p(u\bar{U} + v\bar{V})$

(determines Riemann)

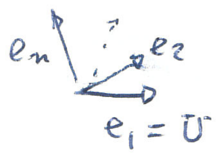
the tensor originally defined by Riemann

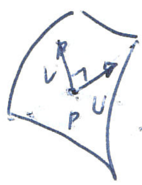
$\|U\| = \|V\| = 1$

$\mathrm{Ricci} = \sum_{i=1}^{n-1}$ sectional curvatures

$\quad \parallel$

$\mathrm{Ric}(U,U)$





$\langle U, V \rangle = 0$

embedded surface

- Scalar curvature $\quad R = \sum_{i,j} \langle R(e_i,e_j)e_i, e_j \rangle$

in coordinates: $\quad R = g^{ij} R_{ij}$

If $\dim M \geq 4$ $\quad Ric = 0 \not\Rightarrow Riem = 0$

Digression

$\star\star\star$ Einstein equations (on a pseudo-Riemannian manifold, signature (1,3))

Einstein tensor

gravitational constant

$$\boxed{\underbrace{R_{ij} - \frac{1}{2} R g_{ij}}_{\text{Einstein tensor}} = \frac{8\pi G}{c^4} T_{ij}}$$

Ricci

scalar curvature

metric

velocity of light in vacuo

energy-momentum tensor

① = ②

"marble" "straw"

"General Relativity is the triumph of the absolute differential calculus of Ricci and Levi-Civita."

(A. Einstein)

$$(\star) \quad \boxed{R_{ij} = 0}$$

vacuum

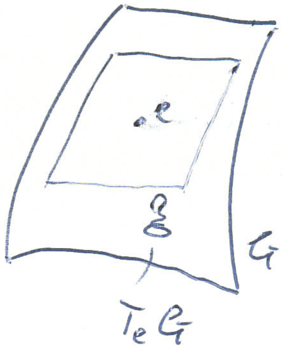
One may derive ($\star$) from the Einstein-Hilbert action $\int R$ (scalar curvature)

$$\left\{ \delta \int R = 0 \right\}$$

variational principle

★ An important example

Let G be a compact Lie group equipped with a bi-invariant metric



(manufactured from an Ad-invariant scalar product on $\mathfrak{g}$ (Lie algebra of G)

$$(*) \quad \langle \underbrace{g\,Y\,g^{-1}}_{Ad(g)Y},\ \underbrace{g\,Z\,g^{-1}}_{Ad(g)Z} \rangle = \langle Y, Z \rangle \qquad g \in G,\ Y \in \mathfrak{g}$$

$\langle\ ,\ \rangle$ is ad-invariant as well, namely

for simplicity, let G be a matrix group

$$(**) \quad \langle \underbrace{[X,Y]}_{ad_X(Y)}, Z \rangle + \langle Y, \underbrace{[X,Z]}_{ad_X(Z)} \rangle = 0$$

$(**)$ is the infinitesimal version of $(*)$: set $g = g(t) = e^{tX}$ in $(*)$ and take its derivative at $t=0$ ...

Every such metric is proportional to the $(*)$ **Killing-Cartan metric**

$$\langle X, Y \rangle_{KC} := \underset{\text{trace}}{Tr}\,(ad_X \cdot ad_Y)$$

$ad_X : \mathfrak{g} \to \mathfrak{g}$

$ad_X Z := [X, Z]$

Example If $G = SO(3)$, $\mathfrak{g} = \mathfrak{so}(3)$

$= \{ A \in M_3(\mathbb{R}) / A^T = -A \}$

antisymmetric matrices

$\dim G = \dim \mathfrak{g} = 3$

$$\underbrace{\langle X, Y \rangle_{KC}}_{\text{Killing - Cartan}} = -\operatorname{Tr}(XY)$$

Let us check bi-invariance:

$$\langle Ad(g)X, Ad(g) \rangle_{KC} = -\operatorname{Tr}(gXg^{-1} \cdot gYg^{-1})$$

$$= -\operatorname{Tr}(gXYg^{-1}) \underset{\substack{\text{cyclical}\\\text{property}\\\text{of Tr}}}{=} -\operatorname{Tr}(g^{-1}g \cdot XY) = -\operatorname{Tr}(XY) = \langle X, Y \rangle_{KC}$$

Let us work out the expression for the Levi-Civita connection

One gets $\boxed{\nabla_X Y = \frac{1}{2}[X, Y]}$ for $X, Y \in \mathfrak{g}$ left-invariant v. fields

Cartan's connection

this is enough, since at each point p, left invariant v. fields span T_pG

[Beware! The position $\tilde{\nabla}_X Y = [X, Y]$, $X, Y \in \mathfrak{X}(M)$ does not define a connection tout court: tensoriality in X does not hold $[\alpha X, Y](f) = \alpha X(Y(f)) - Y(\alpha X(f))$

$= \alpha XY(f) - Y(\alpha) X(f) - \alpha YX(f) =$

$\alpha [X, Y](f) - Y(\alpha) X(f)$ $\quad [\alpha X, Y] = \underbrace{\alpha [X, Y]}_{\text{ok}} - \underbrace{d\alpha(Y) X}_{\text{non tensorial part}}$

Let us check metricity ① and absence of torsion ②

①
$$\langle \nabla_X Y, Z \rangle + \langle Y, \nabla_X Z \rangle =$$
$$= \frac{1}{2} \langle [X,Y], Z \rangle + \frac{1}{2} \langle Y, [X,Z] \rangle = 0$$
(ad-invariance)

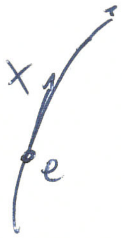
Also
$$X \langle Y, Z \rangle = \frac{d}{dt} \langle Ad(e^{tX}) Y, Ad(e^{tX}) Z \rangle \Big|_{t=0}$$
$$= \langle [X,Y], Z \rangle + \langle Y, [X,Z] \rangle = 0$$

②
$$\nabla_X Y - \nabla_Y X = \frac{1}{2} [X,Y] - \frac{1}{2} [Y,X]$$
$$= \frac{1}{2} [X,Y] + \frac{1}{2} [X,Y]$$
$$= [X,Y]$$
✓

★ **Geodesics** It is enough to determine those passing through the neutral element, and they coincide with the one-parameter subgroups

Indeed, a one-parameter subgroup is a geodesic:

X: generator
$$\nabla_X X = \frac{1}{2}[X,X] = 0$$



Conversely, one resorts to the Cauchy-Lipschitz theorem: there exists a unique geodesic passing through a given point with a given initial velocity

As an exercise, let us recover Cartan's connection from the requirement that all 1-parameter subgroups are geodesics: $\nabla_Z Z = 0$ $\quad Z \in \mathfrak{g}$

We have, for $X, Y \in \mathfrak{g}$:

$$0 = \nabla_{X+Y}(X+Y) = \underbrace{\nabla_X X}_{=0} + \underbrace{\nabla_Y Y}_{=0} + \nabla_X Y + \nabla_Y X$$

Also (absence of torsion) $\quad \nabla_X Y - \nabla_Y X = [X,Y]$

Therefore:
$$\begin{cases} \nabla_X Y - \nabla_Y X = [X,Y] \\ \nabla_X Y + \nabla_Y X = 0 \end{cases}$$

whence $\quad 2\nabla_X Y = [X,Y]$, i.e. $\boxed{\nabla_X Y = \frac{1}{2}[X,Y]}$

i.e. we get Cartan's connection □

Curvature of Cartan's connection

Let us prove that

$$\boxed{\begin{array}{ll} ① & R(X,Y)Z = \frac{1}{4}[[X,Y],Z] \quad \left(= \frac{1}{4}\,\overset{\overset{[ad_X,\, ad_Y]Z}{\shortparallel}}{ad_{[X,Y]}Z}\right) \\ ② & \langle R(X,Y)Z, W\rangle = \frac{1}{4}\langle [X,Y],[Z,W]\rangle \end{array}}$$

recall: $R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$

Proof. Let us check ① (② directly follows from it)

$$R(X,Y)Z = \frac{1}{2}\underbrace{[[X,Y],Z]}_{\circledast} - \frac{1}{4}\Big\{[X[Y,Z]] - [Y[X,Z]]\Big\}$$

$$= \frac{1}{4}\circledast + \underbrace{\frac{1}{4}\circledast - \frac{1}{4}[X[Y,Z]] + \frac{1}{4}[Y,[X,Z]]}_{\text{regroup}}$$

$$= \frac{1}{4}[[X,Y],Z] + \frac{1}{4}\Big\{\underbrace{[[X,Y],Z] + \overset{!}{[[Y,Z],X]} + \overset{!}{[[Z,X],Y]}}_{\overset{\shortparallel}{0} \ \text{(Jacobi identity)}}\Big\}$$

$$= \frac{1}{4}[[X,Y],Z] \qquad \square$$

Corollary $\langle R(X,Y)X, Y\rangle = \frac{1}{4}\|[X,Y]\|^2 \geq 0$

$\Rightarrow$ the sectional curvature is non-negative

In applications, left or right invariant metrics on G are relevant. The ensuing theory turns out to be more complicated.

Let us confine ourselves to a few remarks

$G = SO(3)$
$\mathfrak{g} = so(3)$ with a left-invariant metric

$\rightsquigarrow$ geodesic equation $\equiv$ Euler equation for the rigid body ★★★

⚠ serious analytical difficulties

$G = s\mathrm{Diff}(\mathbb{R}^3) = \{$ volume preserving diffeomorphisms of $\mathbb{R}^3$ (rapidly approaching the identity at infinity)$\}$

"Lie group"

$\mathfrak{g} = \{ X \in \mathfrak{X}(\mathbb{R}^3) \,/\, \operatorname{div} X = 0,\ X$ rapidly vanishing at infinity $\}$

divergence

"Lie algebra of G"

On this G there exists a natural right-invariant metric

geodesics $\equiv$ solutions of Euler's equation for perfect fluids ★★★

(Locally) symmetric spaces, i.e. those satisfying $\nabla R = 0$ (covariant differential) are important as well.

... But this is another story.