V2

DIFFERENTIAL GEOMETRY AND TOPOLOGY

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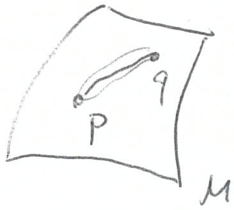
UCSC

lecture XXXVI

- The Hopf-Rinow theorem

Prerequisites for Hopf-Rinow (review)

Riemannian distance



$$d(p,q) = \inf \{ \underbrace{L(\alpha)}_{\text{length}} \;/\; \alpha \text{ piecewise smooth connecting } p \text{ and } q \}$$

Let $o \in M$, $\varepsilon > 0$

$$\mathcal{N}_\varepsilon(o) = \{ p \in M \;/\; d(o,p) < \varepsilon \}$$

ε - neighbourhood of o

- If $\mathcal{U}$ is a normal neighbourhood of o (image of exp)

  and $W \subset \mathcal{U} = \exp_o(B_\varepsilon(0))$

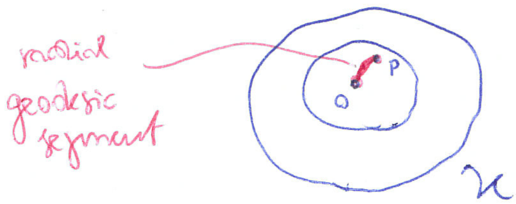
  ball in T_oM centred at 0 and with radius $\varepsilon > 0$

Then, for $\varepsilon > 0$ sufficiently small, and for $p \in W$, the radial geodesic segment

$$\sigma : [0,1] \to W$$

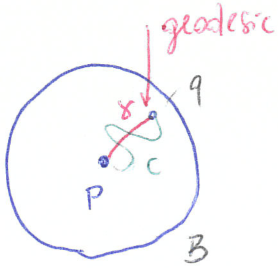
from o to p is the unique shortest curve in M (from o to p), up to reparametrization

This is a consequence of Gauss' lemma



See next page for amplification

☆ Crucial fact In a geodesic ball centred at p, the geodesics issuing from p minimise length



$\gamma : [0,1] \to B \qquad \gamma(0) = p \quad \gamma(1) = q$

$c : [0,1] \to B \qquad c(0) = p \quad c(1) = q$

Then $\ell(c) \geq \ell(\gamma)$

and "=" holds $\Leftrightarrow$ $\gamma([0,1]) = c([0,1])$

Proof. First assume that $c([0,1]) \subset B$

Then (being exp a local diffeomorphism)

$$c(t) = \exp_p(r(t)\, v(t)) \qquad t \mapsto v(t) \qquad \|v(t)\| = 1$$

$$\equiv f(\underbrace{r(t)}_{t}; \underbrace{t}_{s})$$

$r : (0,1] \to \mathbb{R}$ positive and piecewise differentiable

(assume $c(t_1) \neq p$ if $t_1 \in (0,1]$)

Therefore, up to a finite number of points

$$\boxed{\dot{c} = \frac{\partial f}{\partial r}\dot{r} + \frac{\partial f}{\partial t}}$$

Now, by Gauss $\left\langle \frac{\partial f}{\partial s}, \frac{\partial f}{\partial t} \right\rangle = 0$ and also $\left|\frac{\partial f}{\partial r}\right| = 1$ geodesics have constant speed

Therefore

① $$\|\dot{c}\|^2 = |\dot{r}|^2 + \left|\frac{\partial f}{\partial t}\right|^2 \geq |\dot{r}|^2$$

$\Downarrow$

② $$\int_\varepsilon^1 \|\dot{c}\|\, dt \geq \int_\varepsilon^1 |\dot{r}|\, dt \geq \int_\varepsilon^1 \dot{r}\, dt = \underset{\ell(\gamma)}{\overset{\|}{r(1)}} - r(\varepsilon) \searrow 0 \text{ if } \varepsilon \to 0^+$$

$\varepsilon > 0$

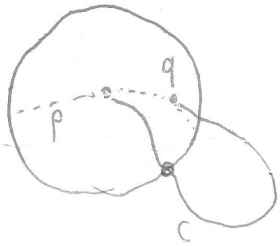
whence

$$\boxed{\ell(c) \geq \ell(\gamma)}$$

">" holds if either ① or ② is strict. "=" entails $\left|\frac{\partial f}{\partial t}\right| = 0 \Rightarrow v = v(t) = \text{const}$ and $\dot{r}(t) > 0$ (monotone reparametrization).

Subsequently, if $c([0,1]) \not\subset B$, let $t_1 \in (0,1)$ the first point such that $c(t_1) \in \partial B$. If ρ is the radius of B, we have

$$\boxed{\ell(c) \geq \ell_{[0,t_1]}(c) \geq \rho > \ell(\gamma)}$$



- A stronger statement is true :

Let $o \in M$

1. For $\varepsilon > 0$ small enough, $N_\varepsilon(o)$ is normal

2. For a normal ε-neighbourhood, the radial geodesic from o to $p \in N_\varepsilon(o)$ is the unique shortest curve from o to p

whence

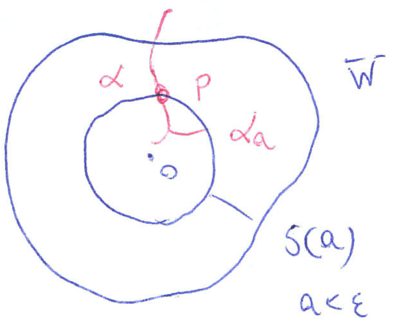
3. $\underset{r(p)}{d(o,p)} = L(\sigma)$

Ad 1 : If $U = \exp_o(\hat{\mathcal{U}})$ is a normal neighbourhood of o, then, for suitable $\varepsilon > 0$, $\hat{\mathcal{U}} \supset \underset{\text{star shaped}}{B_\varepsilon(o)}$. Then $\exp_o B_\varepsilon(o) =: W$ is normal

$\Rightarrow$ $\forall p \in W$, the radial geodesic σ from o to p is the unique shortest curve in W joining these two pts.

$L(\sigma) = \|v\|$ $\quad v = \exp_o^{-1}(p)$ and $L(\sigma) < \varepsilon$

Now, let us prove that if α starts from o and leaves W, then $L(\alpha) \geq \varepsilon$. If α leaves $\bar{W}$, then it must intersect every $S(a)$ $(a < \varepsilon)$. Let p be the intersection point. Then



$L(\alpha_a) \geq a \Rightarrow L(\alpha) \geq a \quad \forall a < \varepsilon$

$\Rightarrow L(\alpha) \geq \varepsilon$

Ad 2 : This directly follows from the above proposition

★ Consequence

$$\boxed{d(p,q)=0 \quad \Rightarrow \quad p=q}$$

Indeed, if $p \neq q$, $\exists$ $W \ni p$ normal neighbourhood of p

s.that $W \not\ni q$ $\Rightarrow$ $\exists$ a ε-normal neighbourhood $\not\ni q$

$\Rightarrow$ (by the above result) $d(p,q) \geq \varepsilon$

Symmetry $(d(p,q) = d(q,p))$ and triangle inequality

$(d(p,q) + d(q,r) \geq d(p,r))$ hold

(★★) $\Rightarrow$ d is a metric, inducing the same topology as the given one on M (Indeed every neighbourhood of $p \in M$ contains an ε-neighbourhood, and the latter is an open set)

A curve segment σ from p to q is a shortest curve (minimising curve) if and only if $L(\sigma) = d(p,q)$ (existence and uniqueness not guaranteed in general)

segment: minimising curve parametrized by a multiple of arc length

→ comes from an unbroken geodesic

The Hopf-Rinow theorem

Let (M,g) be a Riemannian manifold

The following statements are equivalent:

1. M is complete with respect to the distance d induced by g
2. There exists $p \in M$ such that $\exp_p$ is defined on all of T_pM
3. M is geodesically complete: $\exp_o$ defined on T_oM $\forall\, o \in M$
4. Every closed bounded subset of M is compact

$3 \Rightarrow 2$ is trivial; $4 \Rightarrow 1$: since a Cauchy sequence is bounded, so, being its closure compact (4), it contains a convergent subsequence, but this implies that the full sequence converges. Therefore, it is enough to show that $1 \Rightarrow 3$ and $2 \Rightarrow 4$

(then $\underbrace{1 \Rightarrow 3}_{\text{to be proven}} \underbrace{\Rightarrow 2}_{\text{trivial}} \underbrace{\Rightarrow 4}_{\text{to be proven}} \underbrace{\Rightarrow 1}_{\text{proved}}$)

$\boxed{1 \Rightarrow 3}$

Let $\gamma : [0,b)$ a geodesic (unit speed). Then γ can be extended beyond b (finite). Let $t_i \to b$ ($\Rightarrow \{t_i\}$ is Cauchy)

To be shown

Then $d(\gamma(t_i), \gamma(t_j)) \leq |t_i - t_j| \Rightarrow \{\gamma(t_j)\}$ is Cauchy

$$\int_{t_1}^{t_2} \|\dot\gamma(t)\|\,dt \qquad \|\cdot\| = \sqrt{g(\cdot,\cdot)}$$

$\Downarrow$

$\gamma(t_j) \to q \in M$

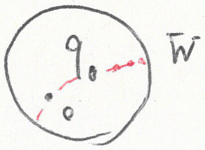
(we used 1)

Then $d(\gamma(s), q) \leq d(\gamma(s), \gamma(t_i)) + d(\gamma(t_i), q)$

$\Rightarrow s \to b \Rightarrow \gamma(s) \to q$.

normal neighbourhood of any of its pts $\equiv$ totally normal neighbourhood in Do Carmo

Let $\bar{W} \ni q$ a convex neighbourhood of q



for t close enough to b,

$o := \gamma(t)$ belongs to W

$\Rightarrow$ $\gamma\big|_{[t,b)}$ is a $\underbrace{\text{radial}}_{\text{image of } \exp_o}$ geodesic emanating from o

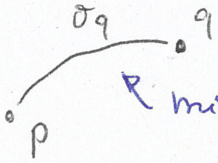
$\Rightarrow$ it can be extended past b (until it hits ∂W) $\square$

$\boxed{2 \Rightarrow 4}$

crux of the argument

($\star$) Let $p \in M$ such that $\exp_p$ is defined for all $v \in T_pM$.

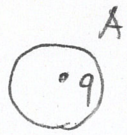
Therefore, $\forall q \in M$, $\exists$ $\sigma_q : [0,1] \to M$ minimising geodesic joining p to q.



minimising

Given ($\star$),

let q vary in a **bounded, closed** set A



Then $\{ d(p,q) = \| \sigma_q'(0) \| \}_{q \in A}$ is bounded

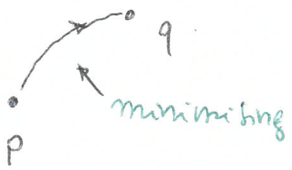
$\Rightarrow$ $\{ \sigma_q'(0) \}_{q\in A} \subset \underset{\text{closed ball}}{B} \subset T_pM$. B is compact

Then $\underset{\text{closed}}{A} \subset \underbrace{\exp_p(B)}_{\text{compact}}$ is compact (closed $\subset$ compact is compact in a Hausdorff space)

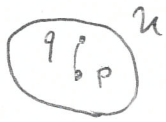
The proof of **HR** is complete $\square$

(modulo ($\star$), see next page)

Let us sketch the proof of (*) (due to G. de Rham)



If $\exp_p$ is def $\forall\, v \in T_pM$ then there exists, for any $q \in M$, a minimizing geodesic connecting p to q



If U is a normal ε-neighbourhood of p, the theorem is true (this is a consequence of Gauss' lemma)

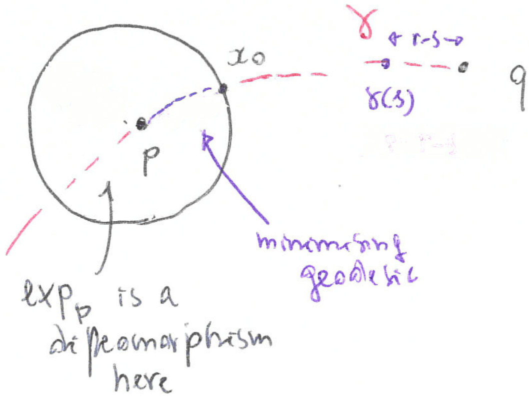
$d(p,q) = r$

Let $x_0 \in \exp_p(\delta v)$

$\|v\| = 1 \quad v \in T_pM$

chosen in such a way that $d(x_0, q)$ is minimal

[$\{\exp_p \delta v\}$ is compact]

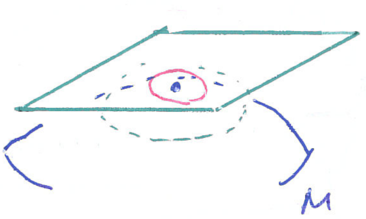


Then consider

$$\boxed{\gamma(s) := \exp_p(s v)}$$

This is a geodesic defined $\forall\, s \in \mathbb{R}$ extending the previous one

One proves that $d(\gamma(s), q) = r - s$



Thus, if $s = r$, $\gamma(r) = q$: the above geodesic connects p to q, and we are done.