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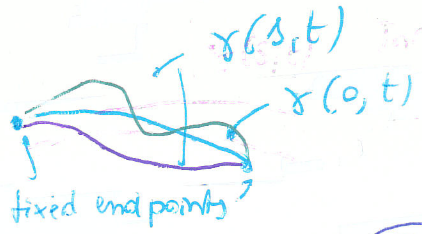
Energy variations

$$\boxed{E(s) = \int_0^1 (\dot\gamma(s,t), \dot\gamma(s,t))\, dt}$$



$\dot\gamma \sim \frac{\partial}{\partial t}$

Compute:

$\dot\gamma \sim \frac{\partial}{\partial t}$

$\frac{\partial}{\partial t} = \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}$

$\frac{\partial}{\partial s} = \frac{\partial x^j}{\partial s}\frac{\partial}{\partial x^j}$

$\left[\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right] = 0$

(independent variations. also clear by a direct computation)

(I) first variation

This is essentially the Lagrangian approach looked upon more intrinsically

$$\frac{dE}{ds}(s) = \int_0^1 \frac{\partial}{\partial s}\langle \dot\gamma(s,t), \dot\gamma(s,t)\rangle\, dt$$

$$= 2\int_0^1 \left(\frac{\nabla\dot\gamma}{ds}, \dot\gamma\right) dt$$

(compatibility with metric)

$\frac{\nabla}{ds}$ $\quad \nabla_{\frac{\partial}{\partial s}} = \nabla_{\frac{\partial x^j}{\partial s}\frac{\partial}{\partial x^j}}$

+ fixed endpoints

Compute at $s = 0$, getting

$$2\int_0^1 (\nabla_{\frac{\partial}{\partial s}}\dot\gamma, \dot\gamma)\, dt$$

$\dot\gamma = \frac{d\gamma}{dt} = \frac{\partial x^i}{\partial t}\frac{\partial}{\partial x^i}$

$\frac{\partial}{\partial s} = \frac{\partial x^j}{\partial s}\cdot\frac{\partial}{\partial x^j}$

$(,) = \frac{\partial x^i}{\partial t}\frac{\partial x^j}{\partial s} g_{ij}$

Setting (*) $\boxed{E'(0) = 0}$, we have

by symmetry

$$0 = \int_0^1 (\nabla_{\frac{\partial}{\partial s}}\dot\gamma, \dot\gamma)\, dt = \int_0^1 (\nabla_{\dot\gamma}\tfrac{\partial}{\partial s}, \dot\gamma)\, dt$$

$$= -\int_0^1 \left(\nabla_{\dot\gamma}\dot\gamma, \frac{\partial}{\partial s}\right) dt + \underbrace{\int_0^1 \frac{d}{dt}\left(\dot\gamma, \frac{\partial}{\partial s}\right) dt}_{=\ \left(\dot\gamma, \frac{\partial}{\partial s}\right)(1) - \left(\dot\gamma, \frac{\partial}{\partial s}\right)(0)}$$

They are both $= 0$

$\frac{\partial x^j}{\partial s}(p) = \frac{\partial x^j}{\partial s}(q) = 0$

Recall $(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) + (\dot\gamma, \nabla_{\dot\gamma}\dot\gamma)$

$= 2(\nabla_{\dot\gamma}\dot\gamma, \dot\gamma) = d(\underbrace{\dot\gamma\,\dot\gamma}_{1}) = 0$

Then (**) $\nabla_{\dot\gamma}\dot\gamma = 0$ (along γ) i.e. γ is a geodesic (self-parallel curve)

Conversely, starting from (**) we get (*)

(II) Second variation

$\frac{\partial}{\partial s} \sim \gamma_s$

$\frac{\partial}{\partial t} \sim \gamma_t \equiv \mathring{\gamma}$

Recall $\nabla_{\frac{\partial}{\partial s}} \frac{\partial}{\partial t} - \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s} = [\frac{\partial}{\partial s}, \frac{\partial}{\partial t}] = 0$

symmetry

γ is a geodesic

$$\frac{d^2E}{ds^2}(s) = 2\int_0^1 \frac{\partial}{\partial s}\left(\frac{\nabla}{ds}\mathring{\gamma}, \mathring{\gamma}\right) dt =$$

$R(X,Y) = \nabla_{[X,Y]} - [\nabla_X, \nabla_Y]$

$\nabla_{\frac{\partial}{\partial s}}(\frac{\partial}{\partial t})$

$$= 2\int_0^1 \Big\{ \underbrace{\left(\left(\frac{\nabla}{ds}\right)^2 \mathring{\gamma}, \mathring{\gamma}\right)} + \underbrace{\left(\frac{\nabla}{ds}\mathring{\gamma}, \frac{\nabla}{ds}\mathring{\gamma}\right)} \Big\} dt$$

$$\left(\nabla_{\frac{\partial}{\partial s}}(\nabla_{\frac{\partial}{\partial s}} \mathring{\gamma}), \mathring{\gamma}\right) \quad \text{symmetry} \qquad \left(\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}, \nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\right)$$

$$= \left(\nabla_{\frac{\partial}{\partial s}}\left[\nabla_{\frac{\partial}{\partial t}} \frac{\partial}{\partial s}\right], \mathring{\gamma}\right)$$

$$= \left(\left\{\nabla_{\frac{\partial}{\partial t}} \nabla_{\frac{\partial}{\partial s}} - R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)\right\}\frac{\partial}{\partial s}, \mathring{\gamma}\right)$$

Now, assume that the variation is geodesic: $\nabla_{\frac{\partial}{\partial s}}\left(\frac{\partial}{\partial s}\right) = 0$ (along γ)

$$= \left(-R\left(\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right)\frac{\partial}{\partial s}, \mathring{\gamma}\right)$$

$$= -\left(R\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)\frac{\partial}{\partial t}, \frac{\partial}{\partial s}\right)$$

property of the Riemann tensor

* Now, let us employ a parallel frame (repère mobile à la Cartan)

Thus $\nabla_{\mathring{\gamma}} \xi = \mathring{\xi}$, $\quad \xi \sim \frac{\partial}{\partial s}$

$\nabla f s_0 = df\, s_0 + f \nabla s_0 = df\, s_0$

general principle

$\mathcal{R}\xi$

Then, for $s=0$:

$$\boxed{E''(0) = 2\int_0^1 \left\{(\mathring{\xi}, \mathring{\xi}) - \underbrace{(R(\xi, \mathring{\gamma})\xi, \mathring{\gamma})}_{(R(\mathring{\gamma},\xi)\mathring{\gamma},\xi)}\right\} dt}$$

(integration by parts)

$$= 2\int_0^1 \left(-\ddot{\xi} - \mathcal{R}\xi, \xi\right) dt$$

Equivalently one can use Fermi's theorem: the Christoffel symbols can be made to vanish on any given curve $\mathcal{C}$ $\Gamma|_{\mathcal{C}} = 0$ (in particular, along a geodesic)

"Hessian"

index form (Morse)

In general: work with arclength and interval $[a,b]$.

Next, we are going to take $[a,b]$ sufficiently small

$$\boxed{\ddot{\xi} + \mathcal{R}\xi = 0}$$ Jacobi equation

Jacobi field

* We wish to show (Jacobi's theorem) that in this case $E''(0) > 0$ i.e. geodesics are locally energy (and length) minimising

Lemma : Poincaré inequality   Special case

Let $\xi \in C^\infty([a,b])$, $\xi(a)=\xi(b)=0$. $\xi = \xi(t)$

Then, $\exists\, C_1 > 0$ such that

$$\boxed{\int_a^b |\xi|^2 \leq C_1 \int_a^b |\xi'|^2}$$

Proof. $\int_a^b |\xi|^2 dt = \int_a^b \left| \int_a^t \xi'(\tau) d\tau \right|^2 dt \leq \int_a^b \left| \int_a^t |\xi'| d\tau \right|^2 dt$

Schwarz

But $\int_a^t |\xi'| d\tau = \int_0^t |\xi'| \cdot 1 \, d\tau \leq \left(\int_0^t |\xi'|^2\right)^{\frac{1}{2}} \left(\int_0^t d\tau\right)^{\frac{1}{2}} =$

$= \left(\int_0^t |\xi'|^2 d\tau\right)^{\frac{1}{2}} \cdot t^{\frac{1}{2}}$. Thus

$$\boxed{\int_a^b |\xi|^2 dt \leq \int_a^b \left(\int_0^t |\xi'|^2 d\tau\right) \cdot t \cdot dt \leq \left[\int_a^b |\xi'|^2 dt\right] \underbrace{\frac{(b-a)^2}{2}}_{C_1}}$$

* Therefore $E''(0) > 0$ if $b-a$ is small enough

($(\mathcal{R}\xi, \xi)$ is bounded)

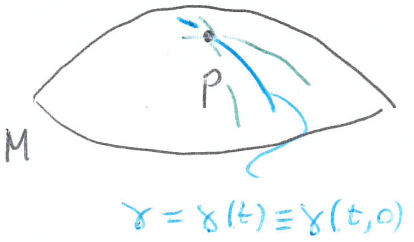
*** Jacobi fields

(M, g) Riemannian manifold, $p \in M$

Let $\boxed{\gamma(t,s) = \exp_p t\, v(s)}$ $\quad 0 \le t \le 1$, $\quad -\varepsilon \le s \le \varepsilon$

$(t,s) \mapsto \gamma(t,s)$ parametrized surface $\subset M$

a geodesic fan stemming from p ($\gamma(\cdot, s)$ is a geodesic)



Let $\gamma = \gamma(t) \equiv \gamma(t,0)$ the "central" geodesic

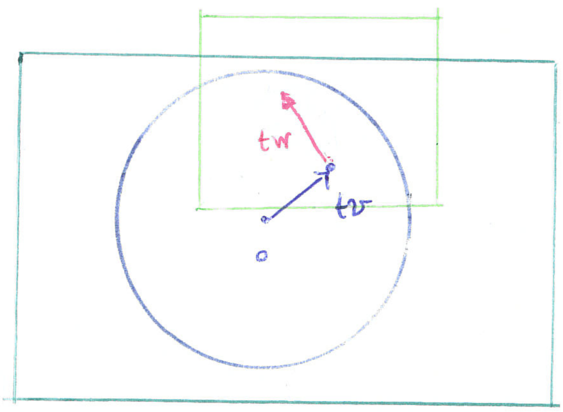
Let $v(0) = v$ $\quad \overset{\frac{d}{ds}}{v'}(0) = w$

Define $\boxed{J(t) := \frac{\partial \gamma}{\partial s}(t,0)}$ $\quad$ * Jacobi field along γ (geodesic variation field)

compute:

$$\boxed{J(t) = \frac{\partial \gamma}{\partial s}(t,0) = (d \exp_p)\Big|_{\underbrace{t\,v(0)}_{t v}} \big(\overbrace{t\, v'(0)}^{t w}\big)}$$

$$= (d \exp_p)\Big|_{tv} (t w) = \boxed{t\, (d \exp_p)_{tv} (w)}$$



J satisfies a second order linear differential equation, the Jacobi equation, derived immediately below

Start from $\frac{\nabla}{dt} \frac{\partial \gamma}{\partial t} = 0$ $\quad$ ($\gamma(\cdot, s)$ is a geodesic)

Then

$$0 = \frac{\nabla}{ds}\left(\frac{\nabla}{dt} \frac{\partial \gamma}{\partial t}\right) = \frac{\nabla}{dt}\left(\frac{\nabla}{ds} \frac{\partial \gamma}{\partial t}\right) - \underset{\text{curvature}}{R}\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial t} \qquad \left[\frac{\partial}{\partial s}, \frac{\partial}{\partial t}\right] = 0$$

$$= \frac{\nabla}{dt}\left(\frac{\nabla}{dt} \frac{\partial \gamma}{\partial s}\right) - R\left(\frac{\partial \gamma}{\partial s}, \frac{\partial \gamma}{\partial t}\right)\frac{\partial \gamma}{\partial t} \implies$$

(symmetry) $\qquad$ evaluating at $(t,0)$

$(+)$ $J=\gamma \quad \dot J = \dot\gamma = 0 \quad \ddot J = 0$
$R(\dot\gamma, \dot\gamma)\dot\gamma = 0$
$J = t\dot\gamma \quad \dot J = \dot\gamma + t\ddot\gamma = \dot\gamma$
$\ddot J = \ddot\gamma = 0 \quad R(\dot\gamma, t\dot\gamma) = tR(\dot\gamma,\dot\gamma) = 0$

$$\left(\frac{\nabla}{dt}\right)^2 J(t) + R\left(\frac{\partial\gamma}{\partial t}, J\right)\frac{\partial\gamma}{\partial t}$$

$$\equiv \boxed{\left(\frac{\nabla}{dt}\right)^2 J + R(\dot\gamma, J)\dot\gamma = 0} \quad \text{on } \gamma$$

★★ **Jacobi equation**

Solutions : Jacobi fields

in general : $2n$ l.i. solutions

using a parallel frame ("répère mobile") one gets an ordinary system (in $\mathbb{R}^n$)

$$\boxed{\ddot J + R(\dot\gamma, J)\dot\gamma = 0}$$

One may restrict to orthogonal variations ($\dot\gamma$ and $t\dot\gamma$ are Jacobi fields $(+)$) and they do not satisfy

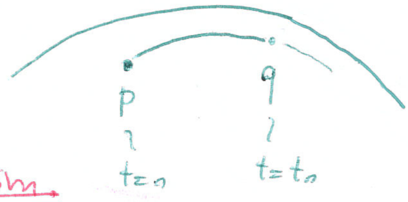
Now let $t_0 \in (0,1]$ and $J(t_0) = 0 = J(0)$

Then $0 = J(t_0) = t_0 (d\exp_p)_{t_0 v}(w) = 0$

$\Longleftrightarrow \quad w \in \ker (d\exp_p)_{t_0 v}$

$\neq$

that is, $\exp_p$ ceases to be a diffeomorphism

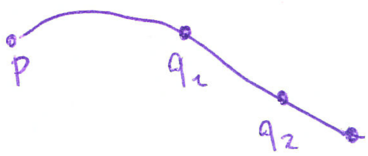


$q = \gamma(t_0) \equiv$ conjugate point to p (along γ)

$\dim \ker(d\exp) \equiv$ multiplicity $\leq n-1$

One finds (Morse)

geodesic



★★★ The index of the index form "Hessian of Energy" ($\equiv$ # negative eigenvalues) along a geodesic from p equals the sum of the multiplicities of the conjugate points thereon

★ **Example** : on S^2

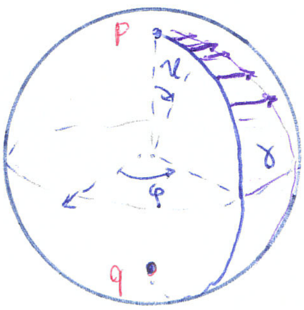
$\gamma = \vartheta \quad J(\theta) = \sin\theta \left[\frac{1}{\sin\theta}\frac{\partial}{\partial\varphi}\right] v \quad \|v\| = 1$

$J \sim \sin\theta$

$(\perp \gamma \quad \dot\gamma \equiv \frac{\partial}{\partial\vartheta} \quad ; \quad v$ is parallel along γ

$\dot{} = \frac{\partial}{\partial\vartheta}$ (clear...)

$\ddot J = -\sin\vartheta \quad K =$ (sectional) curvature of $S^2 = +1$



rotational abuse

$\ddot J + KJ = -\sin\vartheta + 1\cdot\sin\vartheta \equiv 0$