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Research Article

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Existence results for fractional p-Laplacian problems via Morse theory

Abstract: We investigate a class of quasi-linear nonlocal problems, including as a particular case semi-linear problems involving the fractional Laplacian and arising in the framework of continuum mechanics, phase transition phenomena, population dynamics and game theory. Under different growth assumptions on the reaction term, we obtain various existence as well as finite multiplicity results by means of variational and topological methods and, in particular, arguments from Morse theory.

Keywords: Fractional p-Laplacian problems, Morse theory, existence and multiplicity of weak solutions, regularity of solutions

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1 Introduction

1.1 General overview

Let Ω be a bounded domain in $\mathbb{R}^N$, $N \geq 2$, with Lipschitz boundary $\partial\Omega$. Recently, much attention has been paid to the semi-linear problem

$$\begin{cases} (-\Delta)^s u = f(x,u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{1.1}$$

from the point of view of existence, nonexistence and regularity, where f is a Carathéodory function satisfying suitable growth conditions. Several existence results via variational methods are proved in a series of papers of Servadei and Valdinoci [41–43, 45] (see also Iannizzotto and Squassina [21] for the special case $s = 1/2$, $p = 2$ and $N = 1$, with exponential nonlinearity). The issues of regularity and non-existence of solutions are examined by Caffarelli and Silvestre [7], Ros Oton and Serra [38–40]. The corresponding equation in $\mathbb{R}^N$ is studied by Cabré and Sire [4, 5]. Although the fractional Laplacian operator $(-\Delta)^s$, and more generally pseudodifferential operators, have been a classical topic in harmonic analysis and partial differential equations for a long time, the interest in such operators has constantly increased during the last few years. Nonlocal operators such as $(-\Delta)^s$ naturally arise in continuum mechanics, phase transition phenomena, population dynamics and game theory, see e.g. Caffarelli [6] and the references therein. In the works of Metzler and Klafter [29, 30], the description of anomalous diffusion via fractional dynamics is investigated and various fractional partial differential equations are derived from *Lévy random walk* models, extending *Brownian walk* models in a natural way. In particular, in the paper of Laskin [23] a fractional Schrödinger equation was obtained, which extends to a Lévy framework the classical result that path integral over Brownian trajectories leads to the Schrödinger equation. Fractional operators are also involved in financial mathematics, since Lévy processes with jumps revealed as more appropriate models of stock pricing, compared to the Brownian ones used in the celebrated Black and Scholes option pricing model (see Applebaum [1]).

Very recently, a new nonlocal and nonlinear operator was considered, namely for $p \in (1,\infty)$, $s \in (0,1)$ and u smooth enough

$$(-\Delta)_p^s\, u(x) = 2 \lim_{\varepsilon \searrow 0} \int\limits_{\mathbb{R}^N \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{N+sp}}\, dy, \quad x \in \mathbb{R}^N, \tag{1.2}$$

consistent, up to some normalization constant depending upon N and s, with the linear fractional Laplacian $(-\Delta)^s$ in the case $p = 2$. For the motivations that lead to the study of such operators, we refer the reader again to the review paper [6]. This operator, known as the *fractional p-Laplacian*, leads naturally to the study of the quasi-linear problem

$$\begin{cases} (-\Delta)_p^s\, u = f(x,u) & \text{in } \Omega, \\ \qquad u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{1.3}$$

One typical feature of the aforementioned operators is the *nonlocality*, in the sense that the value of $(-\Delta)_p^s u(x)$ at any point $x \in \Omega$ depends not only on the values of u on the whole Ω, but actually on the whole $\mathbb{R}^N$, since $u(x)$ represents the expected value of a random variable tied to a process randomly jumping arbitrarily far from the point x. While in the classical case, by the continuity properties of the Brownian motion, at the exit time from Ω one necessarily is on $\partial\Omega$, due to the jumping nature of the process, at the exit time one could end up anywhere outside Ω. In this sense, the natural non-homogeneous Dirichlet boundary condition consists in assigning the values of u in $\mathbb{R}^N \setminus \Omega$ rather than merely on $\partial\Omega$. Then, it is reasonable to search for solution in the space of functions $u \in W^{s,p}(\mathbb{R}^N)$ vanishing on the outside of Ω. It should be pointed out that, in a bounded domain, this is not the only possible way of providing a formulation of the problem.

In the works of Franzina and Palatucci [16] and of Lindgren and Linqvist [25], the eigenvalue problem associated with $(-\Delta)_p^s u$ is studied, and particularly some properties of the first eigenvalue and of the higher order (variational) eigenvalues are obtained. Then, Iannizzotto and Squassina [22] obtained some Weyl-type estimates for the asymptotic behavior of variational eigenvalues λ_j defined by a suitable cohomological index. From the point of view of regularity theory, some results can be found in [25] even though that work is most focused on the case where p is large and the solutions inherit some regularity directly from the functional embeddings themselves. More recently Di Castro, Kuusi and Palatucci [13] and Brasco and Franzina [3] obtained relevant results about the local boundedness and Hölder continuity for the solutions to the problem of finding (s,p)-harmonic functions u, that is $(-\Delta)_p^s u = 0$ in Ω with $u = g$ on $\mathbb{R}^N \setminus \Omega$, for some function g, providing an extension of results by De Giorgi–Nash–Moser to the nonlocal nonlinear framework. Finally, in the work of Bjorland, Caffarelli and Figalli [2], some higher regularity is obtained when s gets close to 1, by showing that the solutions converge to the solutions with the p-Laplace operator $\operatorname{div}(|\nabla u|^{p-2}\nabla u)$ whenever $s \to 1$.

1.2 Plan of the paper

In the present paper, we aim at establishing existence and (finite) multiplicity of the weak solutions to (1.3) by making use of advanced tools of Morse theory. The contents of the paper are as follows:

- In Section 2, we introduce some preliminary notions and notations and set the functional framework of the problem. More precisely, in Section 2.1 we establish the variational setting for problem (1.3), in Section 2.2 we recall some basic features about the variational eigenvalues of the operator $(-\Delta)_p^s$ and related topics, and in Section 2.3 we introduce critical groups and some related notions.
- In Section 3 we establish a priori L^∞-bounds for the solutions of problem (1.3) under suitable growth conditions on the nonlinearity. These regularity results are used also in the existence theorems proved in the subsequent sections. To our knowledge, L^∞-bounds were previously obtained only for the eigenvalue problem $(-\Delta)_p^s u = \lambda|u|^{p-2}u$, see [16]. The main result of this section is Theorem 3.1.

- In Section 4, we deal with the *p-superlinear case*, namely $f(x,t) = \lambda|t|^{p-2}t + g(x,t)$, with $g(x,\cdot)$ vanishing at zero, proving via Morse-theoretical methods the existence of non-zero solutions for all values of the real parameter λ. The main result of this section is Theorem 4.1.
- In Section 5, we deal with the *coercive case*, including the case when $f(x,\cdot)$ is p-sublinear at infinity, proving via truncations the existence of a positive solution u_+ and of a negative solution u_- and the computation of critical groups at zero yields the existence of a third non-zero solution. The main result of this section is Theorem 5.3.
- In Section 6, we deal with the *asymptotically p-linear case*, namely $f(x,t) = \lambda|t|^{p-2}t + g(x,t)$ with $g(x,\cdot)$ vanishing at infinity, proving some existence results via the computation of critical groups at infinity and a multiplicity result, for λ large enough, via the Mountain Pass Theorem. The main results of this section are Theorems 6.1, 6.2 and 6.4.
- In Section 7, we discuss Pohožaev identity and consequent nonexistence results in star-shaped domains (see Conjecture 7.2).

For a short introduction to fractional Sobolev spaces, we shall refer to the Hitchhiker's guide of Di Nezza, Palatucci and Valdinoci [14]. Concerning the Morse-theoretic apparatus, topological tools as well as existence and multiplicity results for the local case $s = 1$, we shall refer the reader to the monograph of Perera, Agarwal and O'Regan [35], to the classical books by Chang [8], Mawhin and Willem [28], Milnor [32] and to the references therein.

2 Preliminaries

In this preliminary section, for the reader's convenience, we collect some basic results that will be used in the forthcoming sections. In the following, for any functional Φ and any Banach space $(X, \|\cdot\|)$ we will denote

$$\begin{aligned} \Phi^c &= \{u \in X : \Phi(u) \le c\} && (c \in \mathbb{R}), \\ \overline{B}_\rho(u_0) &= \{u \in X : \|u - u_0\| \le \rho\} && (u_0 \in X,\ \rho > 0). \end{aligned}$$

Moreover, in the proofs of our results, C will denote a positive constant (whose value may change case by case).

2.1 Variational formulation of the problem

Let $\Omega \subset \mathbb{R}^N$ be a bounded domain with smooth boundary $\partial\Omega$, and for all $1 \le \nu \le \infty$ denote by $\|\cdot\|_\nu$ the norm of $L^\nu(\Omega)$. Moreover, let $0 < s < 1 < p < \infty$ be real numbers, and the fractional critical exponent be defined as

$$p_s^* = \begin{cases} \frac{Np}{N-sp} & \text{if } sp < N, \\ \infty & \text{if } sp \ge N. \end{cases}$$

First we introduce a variational setting for problem (1.3). The Gagliardo seminorm is defined for all measurable function $u : \mathbb{R}^N \to \mathbb{R}$ by

$$[u]_{s,p} = \left(\int\limits_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^p}{|x - y|^{N+sp}} \, dx \, dy \right)^{1/p}.$$

We define the fractional Sobolev space

$$W^{s,p}(\mathbb{R}^N) = \{u \in L^p(\mathbb{R}^N) : u \text{ measurable}, [u]_{s,p} < \infty\},$$

endowed with the norm

$$\|u\|_{s,p} = (\|u\|_p^p + [u]_{s,p}^p)^{\frac{1}{p}}.$$

For a detailed account on the properties of $W^{s,p}(\mathbb{R}^N)$ we refer the reader to [14]. We shall work in the closed linear subspace

$$X(\Omega) = \{u \in W^{s,p}(\mathbb{R}^N) : u(x) = 0 \text{ a.e. in } \mathbb{R}^N \setminus \Omega\},$$

which can be equivalently renormed by setting $\|\cdot\| = [\cdot]_{s,p}$ (see [14, Theorem 7.1]). It is readily seen that $(X(\Omega), \|\cdot\|)$ is a uniformly convex Banach space and that the embedding $X(\Omega) \hookrightarrow L^\nu(\Omega)$ is continuous for all $1 \le \nu \le p_s^*$, and compact for all $1 \le \nu < p_s^*$ (see [14, Theorems 6.5, 7.1]). The dual space of $(X(\Omega), \|\cdot\|)$ is denoted by $(X(\Omega)^*, \|\cdot\|_*)$.

We rephrase variationally the fractional p-Laplacian as the nonlinear operator $A : X(\Omega) \to X(\Omega)^*$ defined for all $u, v \in X(\Omega)$ by

$$\langle A(u), v\rangle = \int_{\mathbb{R}^{2N}} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(v(x) - v(y))}{|x - y|^{N+sp}} \, dx \, dy.$$

It can be seen that, if u is smooth enough, this definition coincides with that of (1.2). A *(weak) solution* of problem (1.3) is a function $u \in X(\Omega)$ such that

$$\langle A(u), v\rangle = \int_{\Omega} f(x, u)v \, dx \tag{2.1}$$

for all $v \in X(\Omega)$.

Clearly, A is odd, $(p-1)$-homogeneous, and satisfies for all $u \in X(\Omega)$

$$\langle A(u), u\rangle = \|u\|^p, \quad \|A(u)\|_* \le \|u\|^{p-1}.$$

Since $X(\Omega)$ is uniformly convex, by [35, Proposition 1.3], A satisfies the following compactness condition:

Condition (S). If (u_n) is a sequence in $X(\Omega)$ such that $u_n \rightharpoonup u$ in $X(\Omega)$ and $\langle A(u_n), u_n - u\rangle \to 0$, then $u_n \to u$ in $X(\Omega)$.

Moreover, A is a potential operator, precisely A is the Gâteaux derivative of the functional $u \mapsto \|u\|^p/p$ in $X(\Omega)$. Thus, A satisfies all the structural assumptions of [35].

Now we introduce the minimal hypotheses on the reaction term of (1.3):

Hypothesis $\mathbf{H}_2$. The mapping $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$F(x, t) = \int_0^t f(x, \tau) \, d\tau \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

and

$$|f(x, t)| \le a(1 + |t|^{r-1})$$

a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $1 < r < p_s^*$).

We set for all $u \in X(\Omega)$

$$\Phi(u) = \frac{\|u\|^p}{p} - \int_{\Omega} F(x, u) \, dx. \tag{2.2}$$

By Hypothesis $\mathbf{H}_2$, we have $\Phi \in C^1(X(\Omega))$. We denote by $K(\Phi)$ the set of all critical points of Φ. If $u \in K(\Phi)$, then (2.1) holds for all $v \in X(\Omega)$, i.e., u is a weak solution of (1.3). We recall now the Palais–Smale (PS) and the Cerami (C) compactness conditions in a set $U \subseteq X$:

Condition (PS). Every sequence (u_n) in U such that $(\Phi(u_n))$ is bounded in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in $X(\Omega)^*$ admits a convergent subsequence.

Condition (C). Every sequence (u_n) in U such that $(\Phi(u_n))$ is bounded in $\mathbb{R}$ and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ in $X(\Omega)^*$ admits a convergent subsequence.

Such conditions hold for our Φ, provided that the boundedness of the sequence is assumed:

Proposition 2.1. *If Hypothesis $\mathbf{H}_2$ holds, and every sequence (u_n) in $X(\Omega)$ such that $\Phi'(u_n) \to 0$ (respectively, $(1 + \|u_n\|)\Phi'(u_n) \to 0$) in $X(\Omega)^*$ is bounded, then Φ satisfies Condition* (PS) *(respectively, Condition* (C)*) in $X(\Omega)$.*

Proof. We deal with (PS). Passing to a relabeled subsequence, we have $u_n \rightharpoonup u$ in $X(\Omega)$, and $u_n \to u$ in $L^r(\Omega)$. So we have for all $n \in \mathbb{N}$

$$\begin{aligned}|\langle A(u_n), u_n - u\rangle| &= \left|\langle \Phi'(u_n), u_n - u\rangle + \int_\Omega f(x, u_n)(u_n - u)\,dx\right| \\ &\le \|\Phi'(u_n)\|_* \|u_n - u\| + \int_\Omega (1 + |u_n|^{r-1})|u_n - u|\,dx \\ &\le \|\Phi'(u_n)\|_* \|u_n - u\| + C(1 + \|u_n\|_r^{r-1})\|u_n - u\|_r,\end{aligned}$$

and the latter tends to 0 as $n \to \infty$. So, by the (S)-property of A, we have $u_n \to u$ in $X(\Omega)$. □

The following strong maximum principle (see [3, Theorem A.1], a consequence of [13, Lemma 1.3]) will be useful in the proof of some of our results:

Proposition 2.2. *If $u \in X(\Omega) \setminus \{0\}$ is such that $u(x) \ge 0$ a.e. in Ω and*

$$\langle A(u), v\rangle \ge 0$$

for all $v \in X(\Omega)$, $v(x) \ge 0$ a.e. in Ω, then $u(x) > 0$ a.e. in Ω.

2.2 An eigenvalue problem

We consider the nonlinear eigenvalue problem

$$\begin{cases} (-\Delta)_p^s\, u = \lambda |u|^{p-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{2.3}$$

depending on the parameter $\lambda \in \mathbb{R}$. If (2.3) admits a weak solution $u \in X(\Omega) \setminus \{0\}$, then λ is an *eigenvalue* and u is a *λ-eigenfunction*. The set of all eigenvalues is referred to as the *spectrum* of $(-\Delta)_p^s$ in $X(\Omega)$ and denoted by $\sigma(s, p)$. As in the classical case of the p-Laplacian, the structure of $\sigma(s, p)$ is not completely known yet, but many properties have been detected by several authors, see for instance [16, 22, 25]. Here we recall only the results that we will use in the forthcoming sections.

We already know from continuous embedding that the Rayleigh quotient

$$\lambda_1 = \inf_{u \in X(\Omega)\setminus\{0\}} \frac{\|u\|^p}{\|u\|_p^p} \tag{2.4}$$

lies in $(0, \infty)$. The number λ_1 plays an important role in the study of problem (2.3). We list below some spectral properties of $(-\Delta)_p^s$:

Proposition 2.3. *The eigenvalues and eigenfunctions of* (2.3) *have the following properties:*
(i) *$\lambda_1 = \min \sigma(s, p)$ is an isolated point of $\sigma(s, p)$,*
(ii) *all λ_1-eigenfunctions are proportional, and if u is a λ_1-eigenfunction, then either $u(x) > 0$ a.e. in Ω or $u(x) < 0$ a.e. in Ω,*
(iii) *if $\lambda \in \sigma(s, p) \setminus \{\lambda_1\}$ and u is a λ-eigenfunction, then u changes sign in Ω,*
(iv) *all eigenfunctions are in $L^\infty(\Omega)$,*
(v) *$\sigma(s, p)$ is a closed set.*

We define a non-decreasing sequence (λ_k) of *variational eigenvalues* of $(-\Delta)_p^s$ by means of the cohomological index. This type of construction was introduced for the p-Laplacian by Perera [34] (see also Perera and Szulkin [37]), and it is slightly different from the traditional one, based on the Krasnoselskii genus (which does not give the additional Morse-theoretical information that we need here).

We briefly recall the definition of $\mathbb{Z}_2$-cohomological index by Fadell and Rabinowitz [15]. For any closed, symmetric subset M of a Banach space X, let $\overline{M} = M/\mathbb{Z}_2$ be the quotient space (in which u and $-u$ are identified), and let $\phi : \overline{M} \to \mathbb{R}\mathrm{P}^\infty$ be the classifying map of the space $\overline{M}$, which induces a homomorphism $\phi^* : H^*(\mathbb{R}\mathrm{P}^\infty) \to H^*(\overline{M})$ of the Alexander–Spanier cohomology rings with coefficients in $\mathbb{Z}_2$. We may identify $H^*(\mathbb{R}\mathrm{P}^\infty)$ with the polynomial ring $\mathbb{Z}_2[\omega]$. The *cohomological index* of M is then

$$i(M) = \begin{cases} \sup\{k \in \mathbb{N} : \phi^*(\omega^k) \neq 0\} & \text{if } M \neq \emptyset, \\ 0 & \text{if } M = \emptyset. \end{cases}$$

Now let us come back to our case. We set for all $u \in X(\Omega)$

$$J(u) = \frac{\|u\|_p^p}{p}, \quad I(u) = \frac{\|u\|^p}{p}, \quad \Psi(u) = \frac{1}{J(u)} \quad (u \neq 0)$$

and define a C^1-Finsler manifold by setting

$$\mathcal{M} = \{u \in X(\Omega) : I(u) = 1\}. \tag{2.5}$$

For all $k \in \mathbb{N}$, we denote by $\mathcal{F}_k$ the family of all closed, symmetric subsets M of $\mathcal{M}$ such that $i(M) \geq k$, and set

$$\lambda_k = \inf_{M \in \mathcal{F}_k} \sup_{u \in M} \Psi(u) \tag{2.6}$$

(note that, for $k = 1$, (2.4) and (2.6) agree). For all $k \in \mathbb{N}$, λ_k turns out to be a critical value of the restricted functional $\Psi|_{\mathcal{M}}$ (which is even and satisfies (PS) by [35, Lemma 4.5]), hence, by the Lagrange multiplier rule, an eigenvalue of $(-\Delta)_p^s$. These eigenvalues have the following remarkable properties (see [35, Theorem 4.6]):

Proposition 2.4. *The sequence* (λ_k) *defined by* (2.6) *is non-decreasing and* $\lambda_k \to \infty$ *as* $k \to \infty$. *Moreover, for all* $k \in \mathbb{N}$ *we have*

$$i(\{u \in \mathcal{M} : \Psi(u) \leq \lambda_k\}) = i(\{u \in \mathcal{M} : \Psi(u) < \lambda_{k+1}\}) = k.$$

Remark 2.5. In [22] a different construction of the variational eigenvalues is performed. Such a construction is equivalent to that described above, up to a point: precisely, one can easily see that, following the method of [22], we obtain exactly the same sequence (λ_k), while it is not certain whether the topological property in Proposition 2.4 holds, or not.

2.3 Critical groups

We recall the definition and some basic properties of critical groups, referring the reader to the monograph [35] for a detailed account on the subject. Let X be a Banach space, $\Phi \in C^1(X)$ be a functional satisfying (C), and denote by $K(\Phi)$ the set of all critical points of Φ. Let $u \in X$ be an *isolated* critical point of Φ, i.e., there exists a neighborhood U of u such that $K(\Phi) \cap U = \{u\}$, and $\Phi(u) = c$. For all $k \in \mathbb{N}_0$, the *k-th (cohomological) critical group* of Φ at u is defined as

$$C^k(\Phi, u) = H^k(\Phi^c \cap U, \Phi^c \cap U \setminus \{u\}),$$

where $H^*(M, N)$ denotes again the Alexander-Spanier cohomology with coefficients in $\mathbb{Z}_2$ for a topological pair (M, N).

The definition above is well posed, since cohomology groups are invariant under excision, so $C^k(\Phi, u)$ does not depend on U. Moreover, critical groups are invariant under homotopies preserving isolatedness of critical points (see Chang and Ghoussoub [9], Corvellec and Hantoute [11]).

Proposition 2.6. *Let* X *be a Banach space, let* $u \in X$, *and for all* $\tau \in [0, 1]$ *let* $\Phi_\tau \in C^1(X)$ *be a functional such that* $u \in K(\Phi_\tau)$. *If there exists a closed neighborhood* $U \subset X$ *of* u *such that*
(i) Φ_τ *satisfies* (PS) *in* U *for all* $\tau \in [0, 1]$,
(ii) $K(\Phi_\tau) \cap U = \{u\}$ *for all* $\tau \in [0, 1]$,
(iii) *the mapping* $\tau \mapsto \Phi_\tau$ *is continuous between* $[0, 1]$ *and* $C^1(U)$,
then for all $k \in \mathbb{N}_0$ *we have* $C^k(\Phi_1, u) = C^k(\Phi_0, u)$.

We recall some special cases in which the computation of critical groups is immediate ($\delta_{k,h}$ is the Kronecker symbol).

Proposition 2.7. *Let X be a Banach space with $\dim(X) = \infty$, let $\Phi \in C^1(X)$ be a functional satisfying (C), and let $u \in K(\Phi)$ be an isolated critical point of Φ. The following hold:*
(i) *if u is a local minimizer of Φ, then $C^k(\Phi, u) = \delta_{k,0}\, \mathbb{Z}_2$ for all $k \in \mathbb{N}_0$,*
(ii) *if u is a local maximizer of Φ, then $C^k(\Phi, u) = 0$ for all $k \in \mathbb{N}_0$.*

If the set of critical values of Φ is bounded below, we define for all $k \in \mathbb{N}_0$ the *k-th critical group at infinity* of Φ as

$$C^k(\Phi, \infty) = H^k(X, \Phi^\eta),$$

where $\eta < \inf_{u \in K(\Phi)} \Phi(u)$. We recall the *Morse identity*:

Proposition 2.8. *Let X be a Banach space and let $\Phi \in C^1(X)$ be a functional satisfying (C) such that $K(\Phi)$ is a finite set. Then, there exists a formal power series $Q(t) = \sum_{k=0}^{\infty} q_k t^k$ ($q_k \in \mathbb{N}_0$ for all $k \in \mathbb{N}_0$) such that for all $t \in \mathbb{R}$*

$$\sum_{k=0}^{\infty} \sum_{u \in K(\Phi)} \operatorname{rank} C^k(\Phi, u) t^k = \sum_{k=0}^{\infty} \operatorname{rank} C^k(\Phi, \infty) t^k + (1 + t) Q(t).$$

In the absence of a direct sum decomposition, one of the main technical tools that we use to compute the critical groups of Φ at zero is the notion of a *cohomological local splitting* introduced in [35], which is a variant of the homological local linking of Perera [33]. The following slightly different form of this notion was given in Degiovanni, Lancelotti and Perera [12].

Definition 2.9. A functional $\Phi \in C^1(X)$ has a cohomological local splitting near 0 in dimension $k \in \mathbb{N}$ if there exist symmetric cones $X_\pm \subset X$ with $X_+ \cap X_- = \{0\}$ and $\rho > 0$ such that
(i) $i(X_- \setminus \{0\}) = i(X \setminus X_+) = k$,
(ii) $\Phi(u) \le \Phi(0)$ for all $u \in \overline{B}_\rho(0) \cap X_-$, and $\Phi(u) \ge \Phi(0)$ for all $u \in \overline{B}_\rho(0) \cap X_+$.

In this case, we have the following result (see [12, Proposition 2.1]):

Proposition 2.10. *If X is a Banach space and $\Phi \in C^1(X)$ has a cohomological local splitting near 0 in dimension $k \in \mathbb{N}$, and 0 is an isolated critical point of Φ, then $C^k(\Phi, 0) \neq 0$.*

3 L^∞-bounds on the weak solutions

In this section we will prove some *a priori* L^∞-bounds on the weak solutions of problem (1.3). Similar bounds were obtained before in some special cases, namely for linear, inhomogeneous fractional Laplacian equation (see [44, Proposition 7]), and for the eigenvalue problem (2.3) (see [16, Theorem 3.2]). A fractional version of De Giorgi's iteration method was developed by Mingione [31]. Our hypothesis on the reaction term is the following:

Hypothesis $\mathbf{H}_3$. The mapping $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping satisfying a.e. in Ω and for all $t \in \mathbb{R}$

$$|f(x, t)| \le a(|t|^{q-1} + |t|^{r-1}),$$

for some $a > 0$, $1 \le q \le r < p_s^*$.

The main result of the section is the following:

Theorem 3.1. *If Hypothesis $\mathbf{H}_3$ holds with $q \le p \le r$ satisfying*

$$1 + \frac{q}{p} > \frac{r}{p} + \frac{r}{p_s^*},$$

then there exist $K > 0$ and $\alpha > 1$, only depending on s, p, Ω, a, q, and r, such that, for every weak solution $u \in X(\Omega)$ of (1.3), we have $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \le K(1 + \|u\|_r^\alpha).$$

Proof. Fix a weak solution $u \in X(\Omega)$ of (1.3) with $u^+ \neq 0$. We choose $\rho \geq \max\{1, \|u\|_r^{-1}\}$, set $v = (\rho\|u\|_r)^{-1}u$, so $v \in X(\Omega)$, $\|v\|_r = \rho^{-1}$, and v is a weak solution of the auxiliary problem

$$\begin{cases} (-\Delta)_p^s\, v = (\rho\|u\|_r)^{1-p} f(x, \rho\|u\|_r v) & \text{in } \Omega, \\ v = 0 & \text{on } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{3.1}$$

For all $n \in \mathbb{N}$ we set $v_n = (v - 1 + 2^{-n})^+$, so $v_n \in X(\Omega)$, $v_0 = v^+$, and for all $n \in \mathbb{N}$ we have $0 \leq v_{n+1}(x) \leq v_n(x)$ and $v_n(x) \to (v(x) - 1)^+$ a.e. in Ω as $n \to \infty$. Moreover, the following inclusion holds (up to a Lebesgue null set):

$$\{v_{n+1} > 0\} \subseteq \{0 < v < (2^{n+1} - 1)v_n\} \cap \{v_n > 2^{-n-1}\}. \tag{3.2}$$

For all $n \in \mathbb{N}$ we set $R_n = \|v_n\|_r^r$, so $R_0 = \|v^+\|_r^r \leq \rho^{-r}$, and (R_n) is a nonincreasing sequence in $[0, 1]$. We shall prove that $R_n \to 0$ as $n \to \infty$. By Hölder's inequality, the fractional Sobolev inequality (see [14, Theorem 6.5]), (3.2), and Chebyshev's inequality we have for all $n \in \mathbb{N}$

$$R_{n+1} \leq |\{v_{n+1} > 0\}|^{1-\frac{r}{p_s^*}} \|v_{n+1}\|_{p_s^*}^r \leq C|\{v_n^r > 2^{-r(n+1)}\}|^{1-\frac{r}{p_s^*}} \|v_{n+1}\|^r \leq C\, 2^{\left(r-\frac{r^2}{p_s^*}\right)(n+1)} R_n^{1-\frac{r}{p_s^*}} \|v_{n+1}\|^r.$$

So, what we need now is an estimate of $\|v_{n+1}\|$. Using the elementary inequality

$$|\xi^+ - \eta^+|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(\xi^+ - \eta^+) \quad (\xi, \eta \in \mathbb{R}),$$

testing (3.1) with v_{n+1}, and applying also (3.2), we obtain

$$\begin{aligned} \|v_{n+1}\|^p &\leq \langle A(v), v_{n+1}\rangle \\ &= \int_\Omega (\rho\|u\|_r)^{1-p} f(x, \rho\|u\|_r v) v_{n+1}\, dx \\ &\leq C \int_{\{v_{n+1}>0\}} \big((\rho\|u\|_r)^{q-p}|v|^{q-1} + (\rho\|u\|_r)^{r-p}|v|^{r-1}\big) v_{n+1}\, dx \\ &\leq C(\rho\|u\|_r)^{r-p} \int_{\{v_{n+1}>0\}} \big((2^{n+1} - 1)^{q-1} v_n^q + (2^{n+1} - 1)^{r-1} v_n^r\big)\, dx \\ &\leq C\, 2^{(r-1)(n+1)} (\rho\|u\|_r)^{r-p} R_n^{\frac{q}{r}}. \end{aligned}$$

Concatenating the inequalities above we have

$$R_{n+1} \leq C\, 2^{\left(r+\frac{r^2}{p}-\frac{r}{p}-\frac{r^2}{p_s^*}\right)(n+1)} (\rho\|u\|_r)^{\frac{r^2}{p}-r} R_n^{1+\frac{q}{p}-\frac{r}{p_s^*}},$$

which rephrases as the recursive inequality

$$R_{n+1} \leq H^n (\rho\|u\|_r)^{\frac{r^2}{p}-r} R_n^{1+\beta}, \tag{3.3}$$

where $H > 1$ and $0 < \beta < 1$ only depend on the data of (1.3). Now we set $\gamma = r\beta + r - r^2/p > 0$, and fix

$$\rho = \max\Big\{1, \|u\|_r^{-1}, \eta^{-\frac{1}{\gamma}} \|u\|_r^{\left(\frac{r^2}{p}-r\right)\frac{1}{\gamma}}\Big\}.$$

We prove that, provided ρ is big enough, for all $n \in \mathbb{N}$

$$R_n \leq \frac{\eta^n}{\rho^r}, \tag{3.4}$$

for $\eta = H^{-1/\beta} \in (0, 1)$. We argue by induction. We already know that $R_0 \leq \rho^{-r}$. Assuming that (3.4) holds for some $n \in \mathbb{N}$, by (3.3) we have

$$R_{n+1} \leq H^n (\rho\|u\|_r)^{\frac{r^2}{p}-r} \left(\frac{\eta^n}{\rho^r}\right)^{1+\beta} \leq \frac{\|u\|_r^{\frac{r^2}{p}-r} \eta^n}{\rho^{\gamma+r}} \leq \frac{\eta^{n+1}}{\rho^r}.$$

By (3.4) we have $R_n \to 0$. This, in turn, implies that $v_n(x) \to 0$ a.e. in Ω, so $v(x) \leq 1$ a.e. in Ω. An analogous argument applies to $-v$, so we have $v \in L^\infty(\Omega)$ and $\|v\|_\infty \leq 1$, hence $u \in L^\infty(\Omega)$ and

$$\|u\|_\infty \leq \rho\|u\|_r = \max\Big\{\|u\|_r, 1, \eta^{-\frac{1}{\gamma}} \|u\|_r^{1+\left(\frac{r^2}{p}-r\right)\frac{1}{\gamma}}\Big\} \leq K(1 + \|u\|_r^\alpha),$$

for some $K > 0$ and $\alpha > 1$ only depending on the data of (1.3). This concludes the proof. ∎

If, in Hypothesis $\mathbf{H}_3$, we assume $q = p$, then we can improve Theorem 3.1 in a twofold way: we may take any r below the critical exponent, and the inequality relating the L^∞-norms of solutions to the L^r-norms is of linear type.

Corollary 3.2. *If Hypothesis $\mathbf{H}_3$ holds with $q = p \le r < p_s^*$, then for all $0 < \varepsilon < 1$ there exists a constant $K > 0$, only depending on s, p, Ω, a, and r, such that, for every weak solution $u \in X(\Omega)$ of (1.3) with $\|u\|_r < K$, we have $u \in L^\infty(\Omega)$ and*

$$\|u\|_\infty \le K^{-1}\|u\|_r.$$

Proof. Fix $0 < \varepsilon < 1$. Let $u \in X(\Omega)$ be a weak solution of (1.3) with $u^+ \neq 0$ and $\|u\|_r \le \varepsilon$. We set $v = \varepsilon^{-1}u$. Then, $v \in X(\Omega)$ and $\|v\|_r \le 1$. For all $n \in \mathbb{N}$ we set $v_n = (v - 1 + 2^{-k})^+$ and $R_n = \|v_n\|_r^r$. Reasoning as in the proof of Theorem 3.1, we derive the following recursive inequality:

$$R_{n+1} \le H^N R_n^{1+\beta}, \tag{3.5}$$

for some $H > 1$, $0 < \beta < 1$ depending only on the data of (1.3). We set $\eta = H^{-1/\beta} \in (0, 1)$ and $\delta = \eta^{1/(\beta r)}\varepsilon \in (0, \varepsilon)$. If $\|u\|_r = \delta$, then for all $n \in \mathbb{N}$ we have

$$R_n \le \frac{\delta^r}{\varepsilon^r}\eta^n. \tag{3.6}$$

Indeed, clearly $R_0 \le \delta^r/\varepsilon^r$. Moreover, if (3.6) holds for some $n \in \mathbb{N}$, then by (3.5) we have

$$R_{n+1} \le H^n\Big(\frac{\delta^r}{\varepsilon^r}\eta^n\Big)^{1+\beta} = \frac{\delta^r}{\varepsilon^r}\eta^{n+1}.$$

By (3.6) we have $R_n \to 0$ as $n \to \infty$, so $v(x) \le 1$ a.e. in Ω. Reasoning in a similar way on $-v$, we get $\|v\|_\infty \le 1$, hence

$$\|u\|_\infty \le \varepsilon = \eta^{-\frac{1}{\beta r}}\|u\|_r.$$

We set $K = \eta^{1/\beta r}$. Letting ε span the interval $(0, 1)$, we see that for every weak solution $u \in X(\Omega)$ of (1.3) with $\|u\|_r < K$ we have $u \in L^\infty(\Omega)$ and $\|u\|_\infty \le K^{-1}\|u\|_r$. □

4 p-superlinear case

In this section we study problem (1.3), rephrased as

$$\begin{cases} (-\Delta)_p^s\, u = \lambda|u|^{p-2}u + g(x, u) & \text{in } \Omega, \\ u = 0 & \text{on } \mathbb{R}^N \setminus \Omega, \end{cases} \tag{4.1}$$

where $\lambda \in \mathbb{R}$ is a parameter and the hypotheses on the reaction term are the following:

Hypothesis $\mathbf{H}_4$. The mapping $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$G(x, t) = \int_0^t g(x, \tau)\, d\tau,$$

and

(i) $|g(x, t)| \le a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $p < r < p_s^*$),
(ii) $0 < \mu G(x, t) \le g(x, t)t$ a.e. in Ω and for all $|t| \ge R$ ($\mu > p$, $R > 0$),
(iii) $\lim_{t\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$ uniformly a.e. in Ω.

Since $g(x, \cdot)$ does not necessarily vanish at infinity, Hypothesis $\mathbf{H}_4$ classifies problem (4.1) as *p-superlinear*. Besides, by Hypothesis $\mathbf{H}_4$ (iii) we have $g(x, 0) = 0$ a.e. in Ω, so (4.1) admits the zero solution for all $\lambda \in \mathbb{R}$. By means of Morse theory and the spectral properties of $(-\Delta)_p^s$, we will prove the existence of a non-zero solution for all $\lambda \in \mathbb{R}$, requiring when necessary additional sign conditions on $G(x, \cdot)$ near zero. Results of this type were first proved for the p-Laplacian in [12] (see also Perera and Sim [36]).

The main result of this section is the following theorem.

Theorem 4.1. *If Hypothesis* $\mathbf{H}_4$ *and one of the following hold:*
(i) $\lambda \notin (\lambda_k)$,
(ii) $\lambda \in (\lambda_k)$ *and* $G(x,t) \geq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *(for some* $\delta > 0$*),*
(iii) $\lambda \in (\lambda_k)$ *and* $G(x,t) \leq 0$ *a.e. in* Ω *and for all* $|t| \leq \delta$ *(for some* $\delta > 0$*),*
then problem (4.1) *admits a non-zero solution.*

In the present case, the energy functional takes for all $u \in X(\Omega)$ the form

$$\Phi(u) = \frac{\|u\|^p}{p} - \frac{\lambda\|u\|_p^p}{p} - \int_\Omega G(x,u)\,dx.$$

Lemma 4.2. *The functional* $\Phi \in C^1(X(\Omega))$ *satisfies Condition* (PS). *Moreover, there exists an* $\eta < 0$ *such that* Φ^η *is contractible.*

Proof. By Hypothesis $\mathbf{H}_4$ (ii) we have a.e. in Ω and for all $t \in \mathbb{R}$

$$G(x,t) \geq C_0|t|^\mu - C_1 \quad (C_0, C_1 > 0). \tag{4.2}$$

Let (u_n) be a sequence in $X(\Omega)$ such that $(\Phi(u_n))$ is bounded in $\mathbb{R}$ and $\Phi'(u_n) \to 0$ in $X(\Omega)^*$. By (4.2) we have for all $n \in \mathbb{N}$

$$\begin{aligned}\Big(\frac{\mu}{p} - 1\Big)\frac{\|u_n\|^p}{2} &= \frac{\mu+p}{2}\Phi(u_n) - \langle \Phi'(u_n), u_n\rangle + \frac{\lambda}{2}\Big(\frac{\mu}{p} - 1\Big)\|u_n\|_p^p + \int_\Omega \Big(\frac{\mu+p}{2}G(x,u_n) - g(x,u_n)u_n\Big)\,dx \\ &\leq \|\Phi'(u_n)\|_*\|u_n\| + \frac{\lambda}{2}\Big(\frac{\mu}{p} - 1\Big)\|u_n\|_p^p - \frac{\mu-p}{2}\|u_n\|_\mu^\mu + C \\ &\leq \|\Phi'(u_n)\|_*\|u_n\| + C(1 + \|u_n\|_\mu^p - \|u_n\|_\mu^\mu),\end{aligned}$$

hence (u_n) is bounded in $X(\Omega)$. By Proposition 2.1, Φ satisfies Condition (PS).

Now, fix $u \in X(\Omega) \setminus \{0\}$. By (4.2) we have for all $\tau > 0$

$$\Phi(\tau u) \leq \frac{\tau^p\|u\|^p}{p} - \frac{\lambda\tau^p\|u\|_p^p}{p} - C(\tau^\mu\|u\|_\mu^\mu - 1),$$

and the latter tends to $-\infty$ as $\tau \to \infty$. In particular, Φ is unbounded below in $X(\Omega)$. Moreover, by Hypothesis $\mathbf{H}_4$ (ii) we have

$$\langle \Phi'(u), u\rangle = p\Phi(u) + \int_\Omega (pG(x,u) - g(x,u)u)\,dx \leq p\Phi(u),$$

so there exists an $\eta < 0$ such that for all $u \in \Phi^\eta$ we have

$$\langle \Phi'(u), u\rangle < 0. \tag{4.3}$$

By the considerations above, we see that, for all $u \in X(\Omega) \setminus \{0\}$, there exists a unique $\tau(u) \geq 1$ such that, for all $\tau \in [1, \infty)$,

$$\Phi(\tau u) \begin{cases} > \eta & \text{if } 1 \leq \tau < \tau(u), \\ = \eta & \text{if } \tau = \tau(u), \\ < \eta & \text{if } \tau > \tau(u). \end{cases}$$

Moreover, by the Implicit Function Theorem and (4.3), the mapping $\tau : X(\Omega) \setminus \{0\} \to [1, \infty)$ is continuous. We define a continuous deformation $h : [0,1] \times (X(\Omega) \setminus \{0\}) \to X(\Omega) \setminus \{0\}$ by setting for all $(t,u) \in [0,1] \times X(\Omega) \setminus \{0\}$

$$h(t,u) = (1-t)u + t\tau(u).$$

It is immediately seen that Φ^η is a strong deformation retract of $X(\Omega) \setminus \{0\}$. Similarly, by radial retraction we see that $\partial B_1(0)$ is a deformation retract of $X(\Omega) \setminus \{0\}$, and $\partial B_1(0)$ is contractible (as $\dim(X(\Omega)) = \infty$), so Φ^η is contractible. □

We need to compute the critical groups of Φ at 0. With this aim in mind, we define for all $\tau \in [0, 1]$ a functional $\Phi_\tau \in C^1(X(\Omega))$ by setting for all $u \in X(\Omega)$

$$\Phi_\tau(u) = \frac{\|u\|^p}{p} - \frac{\lambda\|u\|_p^p}{p} - \int_\Omega G(x, (1-\tau)u + \tau\theta(u))\,\mathrm{d}x,$$

where $\theta \in C^1(\mathbb{R}, [-\delta, \delta])$ ($\delta > 0$) is a non-decreasing mapping such that

$$\theta(t) = \begin{cases} t & \text{if } |t| \le \frac{\delta}{2}, \\ \pm\delta & \text{if } \pm t \ge \delta. \end{cases}$$

Clearly, $\Phi_0 = \Phi$. Critical groups of Φ and Φ_1 at 0 coincide:

Lemma 4.3. *The point 0 is an isolated critical point of Φ_τ, uniformly with respect to $\tau \in [0, 1]$, and*

$$C^k(\Phi, 0) = C^k(\Phi_1, 0)$$

for all $k \in \mathbb{N}_0$.

Proof. For $\varepsilon > 0$ small enough, we have $K(\Phi) \cap \overline{B}_\varepsilon(0) = \{0\}$. We prove now that, taking $\varepsilon > 0$ even smaller if necessary, we have

$$K(\Phi_\tau) \cap \overline{B}_\varepsilon(0) = \{0\} \quad \text{for all } \tau \in [0, 1]. \tag{4.4}$$

We argue by contradiction: assume that there exist sequences (τ_n) in $[0, 1]$ and (u_n) in $X(\Omega) \setminus \{0\}$ such that $\Phi'_{\tau_n}(u_n) = 0$ for all $n \in \mathbb{N}$, and $u_n \to 0$ in $X(\Omega)$. For all $n \in \mathbb{N}$, we set for all $(x, t) \in \Omega \times \mathbb{R}$

$$g_n(x, t) = (1 - \tau_n + \tau_n\theta'(t))g(x, (1-\tau_n)t + \tau_n\theta(t)),$$

where $\theta \in C^1(\mathbb{R}, [-\delta, \delta])$ is defined as above. By Hypothesis $\mathbf{H}_4$ (i) and (iii), for all $n \in \mathbb{N}$, $g_n : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping and satisfies a.e. in Ω and for all $t \in \mathbb{R}$

$$|\lambda|t|^{p-2}t + g_n(x, t)| \le a'(|t|^{p-1} + |t|^{r-1}),$$

for some $a' > 0$ independent of $n \in \mathbb{N}$. Besides, for all $n \in \mathbb{N}$, u_n is a weak solution of the auxiliary problem

$$\begin{cases} (-\Delta)_p^s\, u = \lambda|u|^{p-2}u + g_n(x, u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{4.5}$$

By Corollary 3.2, there exists a constant $K > 0$ (independent of $n \in \mathbb{N}$) such that, for all weak solution $u \in X(\Omega)$ of (4.5) with $\|u\|_r < K$ we have $u \in L^\infty(\Omega)$ with $\|u\|_\infty \le K^{-1}\|u\|_r$. By the continuous embedding $X(\Omega) \hookrightarrow L^r(\Omega)$, we have $u_n \to 0$ in $L^r(\Omega)$, hence the same convergence takes place in $L^\infty(\Omega)$ as well. In particular, for $n \in \mathbb{N}$ big enough we have $u_n \in \overline{B}_\varepsilon(0)$ and $\|u_n\|_\infty \le \delta/2$, hence by definition of Φ_{τ_n} it is easily seen that

$$\Phi'(u_n) = \Phi'_{\tau_n}(u_n) = 0,$$

i.e., $u_n \in K(\Phi) \cap \overline{B}_\varepsilon(0) \setminus \{0\}$, a contradiction. So (4.4) is achieved.

For all $0 \le \tau \le 1$ the functional $\Phi_\tau \in C^1(X(\Omega))$ satisfies hypotheses analogous to Hypothesis $\mathbf{H}_4$, hence by Lemma 4.2 Φ_τ satisfies (PS) in $\overline{B}_\varepsilon(0)$. Besides, clearly the mapping $\tau \mapsto \Phi_\tau$ is continuous in $[0, 1]$. So, by Proposition 2.6 we have $C^k(\Phi, 0) = C^k(\Phi_1, 0)$ for all $k \in \mathbb{N}_0$. □

We prove now that Φ has a non-trivial critical group at zero for all $\lambda \in \mathbb{R}$, under appropriate conditions. We begin with 'small' λ:

Lemma 4.4. *If one of the following holds:*
(i) $\lambda < \lambda_1$,
(ii) $\lambda = \lambda_1$, *and $G(x, t) \le 0$ a.e. in Ω and for all $|t| \le \delta$ (for some $\delta > 0$),*
then $C^k(\Phi, 0) = \delta_{k,0}\,\mathbb{Z}_2$ for all $k \in \mathbb{N}_0$.

Proof. By Hypothesis $\mathbf{H}_4$ (iii), for all $\varepsilon > 0$ there exists some $\rho > 0$ such that a.e. in Ω and for all $|t| \leq \rho$

$$|g(x,t)| \leq \varepsilon |t|^{p-1}.$$

So, for all $u \in X(\Omega)$ we have by Hypothesis $\mathbf{H}_4$ (i)

$$\Big| \int_\Omega G(x,u)\,dx \Big| \leq \int_{\{|u|\leq\rho\}} \frac{\varepsilon|u|^p}{p}\,dx + \int_{\{|u|>\rho\}} a\Big(|u| + \frac{|u|^r}{r}\Big)\,dx \leq \frac{\varepsilon\|u\|_p^p}{p} + C\|u\|_r^r,$$

which, together with the continuous embeddings $X(\Omega) \hookrightarrow L^p(\Omega)$, $L^r(\Omega)$ and by arbitrarity of $\varepsilon > 0$, yields

$$\int_\Omega G(x,u)\,dx = o(\|u\|^p) \quad \text{as } \|u\| \to 0. \tag{4.6}$$

Now we consider separately the two cases:

(i) By (4.6), we have for all $u \in X(\Omega)$

$$\Phi(u) \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big)\frac{\|u\|^p}{p} + o(\|u\|^p),$$

and the latter is positive for $\|u\| > 0$ small enough, hence 0 is a strict local minimizer of Φ. Thus, by Lemma 2.7, for all $k \in \mathbb{N}_0$ we have $C^k(\Phi,0) = \delta_{k,0}\mathbb{Z}_2$.

(ii) By Lemma 4.3, we may pass to $\Phi_1 \in C^1(X(\Omega))$. For all $u \in X(\Omega)$ we have $|\theta(u(x))| \leq \delta$ a.e. in Ω, so

$$\Phi_1(u) \geq \Big(1 - \frac{\lambda}{\lambda_1}\Big)\frac{\|u\|^p}{p} - \int_\Omega G(x,\theta(u))\,dx \geq 0,$$

hence 0 is a local minimizer of Φ_1. Thus, by Lemmas 2.7 and 4.3, for all $k \in \mathbb{N}_0$ we have

$$C^k(\Phi,0) = C^k(\Phi_1,0) = \delta_{k,0}\mathbb{Z}_2.$$

This concludes the proof. □

Now we consider 'big' λ:

Lemma 4.5. *If one of the following holds for some $k \in \mathbb{N}$:*
(i) $\lambda_k < \lambda < \lambda_{k+1}$,
(ii) $\lambda_k = \lambda < \lambda_{k+1}$, and $G(x,t) \geq 0$ a.e. in Ω and for all $|t| \leq \delta$ (for some $\delta > 0$),
(iii) $\lambda_k < \lambda = \lambda_{k+1}$, and $G(x,t) \leq 0$ a.e. in Ω and for all $|t| \leq \delta$ (for some $\delta > 0$),
then $C^k(\Phi,0) \neq 0$.

Proof. First we assume (i). Again, (4.6) holds. We prove that Φ has a cohomological local splitting near 0 in dimension $k \in \mathbb{N}$ (see Definition 2.9). Set

$$X_+ = \{u \in X(\Omega) : \|u\|^p \geq \lambda_{k+1}\|u\|_p^p\}, \quad X_- = \{u \in X(\Omega) : \|u\|^p \leq \lambda_k\|u\|_p^p\}.$$

Clearly, $X_\pm$ are symmetric closed cones with $X_+ \cap X_- = \{0\}$ (as $\lambda_k < \lambda_{k+1}$). Defining the manifold $\mathcal{M}$ as in (2.5), by Proposition 2.4 we have

$$i(\mathcal{M} \cap X_-) = i(\mathcal{M} \cap (X(\Omega) \setminus X_+)) = k.$$

We define a mapping $h : [0,1] \times (X_- \setminus \{0\}) \to (X_- \setminus \{0\})$ by setting for all $(t,u) \in [0,1] \times (X_- \setminus \{0\})$

$$h(t,u) = (1-t)u + t\frac{p^{1/p}u}{\|u\|}.$$

It is easily seen that, by means of h, the set $\mathcal{M} \cap X_-$ is a deformation retract of $X_- \setminus \{0\}$, so we have

$$i(X_- \setminus \{0\}) = k.$$

Analogously we see that

$$i(X(\Omega) \setminus X_+) = k.$$

Now we prove that, for $\rho > 0$ small enough,

$$\begin{aligned} \Phi(u) &\le 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_-, \\ \Phi(u) &\ge 0 \quad \text{for all } u \in \overline{B}_\rho(0) \cap X_+. \end{aligned} \tag{4.7}$$

Indeed, for all $u \in X_- \setminus \{0\}$, we have by (4.6)

$$\Phi(u) \le \Big(1 - \frac{\lambda}{\lambda_k}\Big)\frac{\|u\|^p}{p} + o(\|u\|^p) \quad \text{as } \|u\| \to 0,$$

and the latter is negative for $\|u\| > 0$ small enough. Besides, for all $u \in X_+ \setminus \{0\}$, we have

$$\Phi(u) \ge \Big(1 - \frac{\lambda}{\lambda_{k+1}}\Big)\frac{\|u\|^p}{p} + o(\|u\|^p) \quad \text{as } \|u\| \to 0,$$

and the latter is positive for $\|u\| > 0$ small enough. So (4.7) holds.

Now we apply Proposition 2.10 and conclude that $C^k(\Phi, 0) \neq 0$.

If we assume either (ii) or (iii), we can develop the same argument for Φ_1 (replacing one of the strict inequalities $\lambda_k < \lambda < \lambda_{k+1}$ with the convenient sign condition on $G(x, \theta(u(x)))$ a.e. in Ω). Then we apply Lemma 4.3 and obtain $C^k(\Phi, 0) = C^k(\Phi_1, 0) \neq 0$. □

Now we are ready to prove our main result:

Proof of Theorem 4.1. We argue by contradiction, assuming

$$K(\Phi) = \{0\}. \tag{4.8}$$

Let $\eta < 0$ be as in Lemma 4.2. Since there is no critical value for Φ in $[\eta, 0)$ and Φ satisfies (PS) in $X(\Omega)$, by the Second Deformation Theorem the set Φ^η is a deformation retract of $\Phi^0 \setminus \{0\}$. Analogously, since there is no critical value in $(0, \infty)$, Φ^0 is a deformation retract of $X(\Omega)$. So we have for all $k \in \mathbb{N}_0$

$$C^k(\Phi, 0) = H^k(\Phi^0, \Phi^0 \setminus \{0\}) = H^k(X(\Omega), \Phi^\eta) = 0.$$

We can easily check that, in all cases (i)–(iii), one of the assumptions of either Lemma 4.4 or 4.5 holds for some $k \in \mathbb{N}_0$, a contradiction. Thus, (4.8) must be false and there exists some $u \in K(\Phi) \setminus \{0\}$, which turns out to be a non-zero solution of (4.1). □

5 Multiplicity for the coercive case

In this section, following the methods of Liu and Liu [26] (see also Liu and Li [27]), we prove a multiplicity result for problem (1.3), under assumptions which make the energy functional coercive. More precisely, by a truncation argument and minimization, we prove the existence of two constant sign solutions (one positive, the other negative), then we apply Morse theory to find a third non-zero solution.

We assume that Ω has a $C^{1,1}$ boundary. The hypotheses on the reaction term f in (1.3) are the following:

Hypothesis H_5. The mapping $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$F(x, t) = \int_0^t f(x, \tau)\, d\tau \quad \text{for all } (x, t) \in \Omega \times \mathbb{R},$$

and

(i) $|f(x, t)| \le a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $1 < r < p_s^*$),

(ii) $f(x, t)t \ge 0$ a.e. in Ω and for all $t \in \mathbb{R}$,

(iii) $\lim_{t\to 0} \frac{f(x,t) - b|t|^{q-2}t}{|t|^{p-2}t} = 0$ uniformly a.e. in Ω ($b > 0$, $1 < q < p$),

(iv) $\limsup_{|t|\to\infty} \frac{pF(x,t)}{|t|^p} < \lambda_1$ uniformly a.e. in Ω.

We define Φ as in (2.2). Since we are interested in finding minimizers of truncated versions of Φ, we shall need a nonlocal analogous of a well-known result of Garcìa Azorero, Peral Alonso and Manfredi [17] about local minimizers of functionals in Hölder and Sobolev topologies, which holds under suitable regularity assumptions. We briefly discuss such issue before introducing the main result.

For all $x \in \overline{\Omega}$ we set

$$\delta(x) := \operatorname{dist}(x, \mathbb{R}^N \setminus \Omega).$$

Accordingly, we define the weighted Hölder-type spaces ($\alpha, \gamma \in (0,1)$)

$$C^0_\delta(\overline{\Omega}) := \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\delta^\gamma} \text{ admits a continuous extension to } \overline{\Omega}\Big\},$$
$$C^{0,\alpha}_\delta(\overline{\Omega}) := \Big\{u \in C^0(\overline{\Omega}) : \frac{u}{\delta^\gamma} \text{ admits an } \alpha\text{-Hölder continuous extension to } \overline{\Omega}\Big\},$$

endowed with the norms

$$\|u\|_{C^0_\delta(\overline{\Omega})} := \Big\|\frac{u}{\delta^\gamma}\Big\|_\infty,$$
$$\|u\|_{C^{0,\alpha}_\delta(\overline{\Omega})} := \|u\|_{C^0_\delta(\overline{\Omega})} + \sup_{x,y\in\overline{\Omega},\, x\neq y} \frac{|u(x)/\delta(x)^\gamma - u(y)/\delta(y)^\gamma|}{|x-y|^\alpha},$$

respectively. Clearly, if $u \in C^0_\delta(\overline{\Omega})$, then $u = 0$ on $\partial\Omega$. In general, all functions that vanish at $\partial\Omega$ will be identified with their zero-extensions to $\mathbb{R}^N$. By the Arzelà–Ascoli Theorem, the embedding $C^{0,\alpha}_\delta(\overline{\Omega}) \hookrightarrow C^0_\delta(\overline{\Omega})$ is compact for all $0 < \alpha < 1$. Further, $C^0_\delta(\overline{\Omega})$ is an ordered Banach space with order cone

$$C_+ = \{u \in C^0_\delta(\overline{\Omega}) : u(x) \geq 0 \text{ for all } x \in \overline{\Omega}\}.$$

Lemma 5.1. *The interior of C_+, with respect to the topology of $C^0_\delta(\overline{\Omega})$, is*

$$\operatorname{int}(C_+) = \{u \in C^0_\delta(\overline{\Omega}) : u(x)d(x)^{-\gamma} > 0 \textit{ for all } x \in \overline{\Omega}\}.$$

Proof. First, we prove the direct inclusion, by contradiction. Assume that $u \in \operatorname{int}(C_+)$ and there exists an $\bar{x} \in \overline{\Omega}$ such that $u(\bar{x})\delta(\bar{x})^{-\gamma} = 0$ (recall that $u\delta^{-\gamma}$ is identified with its continuous extension to $\overline{\Omega}$). We can find $\varepsilon > 0$ and a non-negative function $v \in C^0(\mathbb{R}^N)$ such that $v(x) = 0$ in $\mathbb{R}^N \setminus B_\varepsilon(\bar{x})$ and $v(\bar{x}) > 0$. For all $n \in \mathbb{N}$ set $u_n = u - v\delta^\gamma/n$. Then, $u_n \in C^0_\delta(\overline{\Omega})$ and, as $n \to \infty$,

$$\|u_n - u\|_{C^0_\delta(\overline{\Omega})} = \Big\|\frac{u_n}{\delta^\gamma} - \frac{u}{\delta^\gamma}\Big\|_{C^0(\overline{\Omega})} = \frac{\|v\|_{C^0(B_\varepsilon(\bar{x}))}}{n} \to 0,$$

while for all $n \in \mathbb{N}$ we have

$$u_n(\bar{x})\delta(\bar{x})^{-\gamma} < 0.$$

This, in turn, implies that $u_n(x_n) < 0$ for some $x_n \in \Omega$, hence $u_n \notin C_+$, a contradiction.

We now prove the reverse inclusion, arguing again by contradiction. To this end, assume that $u \in C^0_\delta(\overline{\Omega})$ and $u(x)\delta(x)^{-\gamma} > 0$ in $\overline{\Omega}$, and that there exist sequences (u_n) in $C^0_\delta(\overline{\Omega})$, (x_n) in $\overline{\Omega}$ such that $u_n \to u$ in $C^0_\delta(\overline{\Omega})$ and $u_n(x_n) < 0$ for all $n \in \mathbb{N}$. Up to a relabeled subsequence, $x_n \to x$ for some $x \in \overline{\Omega}$, so we have

$$\frac{u_n(x_n)}{\delta(x_n)^\gamma} \to \frac{u(x)}{\delta(x)^\gamma}.$$

Hence, $u(x)\delta(x)^{-\gamma} \leq 0$, a contradiction. □

We will assume that the following regularity condition holds:

Condition (RC). Let f satisfy Hypothesis $\mathbf{H}_5$ (i)–(ii). Then, there exist $\alpha, \gamma \in (0,1)$, only depending on the data of (1.3), such that:

(i) if $u \in X(\Omega)$ is a bounded weak solution of (1.3), then $u \in C^{0,\gamma}(\overline{\Omega}) \cap C^{0,\alpha}_\delta(\overline{\Omega})$ and, if $\pm u(x) > 0$ in Ω, then $\pm u(x)\delta(x)^{-\gamma} > 0$ in $\overline{\Omega}$,

(ii) if $u \in X(\Omega)$ and, for all $0 < \varepsilon < 1$, the restriction $\Phi|_{\overline{B}_\varepsilon(u)}$ attains its infimum at $u_\varepsilon \in \overline{B}_\varepsilon(u)$, then $u_\varepsilon \in C^{0,\alpha}_\delta(\overline{\Omega})$ and

$$\sup_{0<\varepsilon<1} \|u_\varepsilon\|_{C^{0,\alpha}_\delta(\overline{\Omega})} < \infty.$$

Condition (RC) plays an essential role in the proof of the following result.

Proposition 5.2. *If $u \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$ and $\rho > 0$ are such that $\Phi(u+h) \geq \Phi(u)$ for all $h \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$ satisfying $\|h\|_{C^0_\delta(\overline{\Omega})} \leq \rho$, then there exists an $\varepsilon > 0$ such that $\Phi(u+h) \geq \Phi(u)$ for all $h \in X(\Omega)$ satisfying $\|h\| \leq \varepsilon$.*

Proof. We argue by contradiction, assuming that there exists some $\rho > 0$ such that $\Phi(u+h) \geq \Phi(u)$ for all $h \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$ satisfying $\|h\|_{C^0_\delta(\overline{\Omega})} < \rho$, while there exists a sequence (h_n) in $X(\Omega)$ such that $\|h_n\| \leq 1/n$ and $\Phi(u+h_n) < \Phi(u)$, for all $n \in \mathbb{N}$. With no loss of generality we may assume that

$$\Phi(u+h_n) = \inf_{h \in \overline{B}_{1/n}(0)} \Phi(u+h),$$

so by Condition (RC) (ii) we can find $\alpha, \gamma \in (0,1)$ such that the sequence (h_n) is bounded in $C^{0,\alpha}_\delta(\overline{\Omega})$. By the compact embedding $C^{0,\alpha}_\delta(\overline{\Omega}) \hookrightarrow C^0_\delta(\overline{\Omega})$, we have, up to a relabeled subsequence, $h_n \to 0$ in $C^0_\delta(\overline{\Omega})$ (recall that $h_n(x) \to 0$ a.e. in Ω). For $n \in \mathbb{N}$ big enough, we have

$$\|h_n\|_{C^0_\delta(\overline{\Omega})} < \rho \quad \text{and} \quad \Phi(u+h_n) < \Phi(u),$$

against our assumption. □

The main result of this section is the following:

Theorem 5.3. *If Hypothesis $\mathbf{H}_5$ and Condition* (RC) *hold, then problem* (1.3) *admits at least three non-zero solutions.*

Remark 5.4. In the linear case $p = 2$, (RC) holds with $\gamma = s$ due to the results of [39, 40]. Such a case was carefully studied by Iannizzotto, Mosconi and Squassina [19], who, through different L^∞-bounds and a non-local Hopf Lemma, prove versions of Proposition 5.2 and Theorem 5.3 with no regularity assumption. How to achieve (RC) in the nonlinear case $p \neq 2$ is still an open problem, yet a partial result in this direction (namely, C^α-regularity up to the boundary of the weak solutions) is provided in the forthcoming paper [20].

We introduce two truncated energy functionals by setting for all $u \in X(\Omega)$

$$\Phi_\pm(u) = \frac{\|u\|^p}{p} - \int_\Omega F(x, \pm u^\pm)\,dx, \tag{5.1}$$

where $t^\pm = \max\{\pm u, 0\}$. The following lemma displays some properties of $\Phi_\pm$:

Lemma 5.5. *We have $\Phi_\pm \in C^1(X(\Omega))$. Moreover,*
(i) *if $u \in X(\Omega)$ is a critical point of $\Phi_\pm$, then $\pm u(x) \geq 0$ a.e. in Ω,*
(ii) *0 is not a local minimizer of $\Phi_\pm$,*
(iii) *$\Phi_\pm$ is coercive in $X(\Omega)$.*

Proof. We consider Φ_+, the argument for Φ_- being analogous. By Hypothesis $\mathbf{H}_5$ (ii), we have $f(x,0) = 0$ a.e. in Ω, so $(x,t) \to f(x,t^+)$ is Carathéodory and satisfies a growth condition similar to Hypothesis $\mathbf{H}_5$ (i). So, $\Phi_+ \in C^1(X(\Omega))$ with derivative given for all $u, v \in X(\Omega)$ by

$$\langle \Phi_+'(u), v\rangle = \langle A(u), v\rangle - \int_\Omega f(x, u^+)v\,dx.$$

Now we prove (i). Assume $\Phi_+'(u) = 0$ in $X(\Omega)^*$. We recall the elementary inequality

$$|\xi^- - \eta^-|^p \leq |\xi - \eta|^{p-2}(\xi - \eta)(\eta^- - \xi^-),$$

holding for all $\xi, \eta \in \mathbb{R}$. Testing with $-u^- \in X(\Omega)$, we have

$$\|u^-\|^p \leq \langle A(u), -u^-\rangle = -\int_\Omega f(x, u^+)u^-\,dx = 0.$$

Hence, $u \geq 0$ a.e. in Ω.

Now we prove (ii). By Hypothesis $\mathbf{H}_5$ (i)–(ii), we have a.e. in Ω and for all $t \in \mathbb{R}$

$$F(x, t^+) \geq C_0|t|^q - C_1|t|^r \quad (C_0, C_1 > 0). \tag{5.2}$$

Consider a function $\bar{u} \in X(\Omega)$, $\bar{u}(x) > 0$ a.e. in Ω. For all $\tau > 0$ we have

$$\Phi_+(\tau\bar{u}) = \frac{\tau^p\|\bar{u}\|^p}{p} - \int_\Omega F(x,\tau\bar{u})\,dx \le \frac{\tau^p\|\bar{u}\|^p}{p} - \tau^q C_0\|\bar{u}\|^q_{L^q(\Omega)} + \tau^r C_1\|\bar{u}\|^r_r,$$

and the latter is negative for $\tau > 0$ close enough to 0. So, 0 is not a local minimizer of Φ_+.

Finally, we prove (iii). By Hypothesis $\mathbf{H}_5$ (iv), for all $\varepsilon > 0$ small enough, we have a.e. in Ω and for all $t \in \mathbb{R}$

$$F(x,t^+) \le \frac{\lambda_1 - \varepsilon}{p}|t|^p + C.$$

By the definition of λ_1, we have for all $u \in X(\Omega)$

$$\Phi_+(u) \ge \frac{\|u\|^p}{p} - \frac{\lambda_1-\varepsilon}{p}\|u\|^p_{L^p(\Omega)} - C \ge \frac{\varepsilon}{p\lambda_1}\|u\|^p - C,$$

and the latter goes to ∞ as $\|u\| \to \infty$. So, Φ_+ is coercive in $X(\Omega)$. □

Now we can prove our main result:

Proof of Theorem 5.3. The functional Φ_+ is coercive and sequentially weakly lower semi-continuous in $X(\Omega)$, so there exists $u_+ \in X(\Omega)$ such that

$$\Phi_+(u_+) = \inf_{u\in X(\Omega)} \Phi_+(u).$$

Since u_+ is a critical point of Φ_+, by Lemma 5.5 (i)–(ii) we have $u_+(x) \ge 0$ a.e. in Ω and $u_+ \ne 0$. By Hypothesis $\mathbf{H}_5$ (ii) and Proposition 2.2, we have $u_+(x) > 0$ a.e. in Ω.

Now we invoke Condition (RC) (i) and find $\alpha, \gamma \in (0,1)$ such that $u_+ \in C^{0,\alpha}_\delta(\overline{\Omega})$ and $u(x)\delta(x)^{-\gamma} > 0$ in $\overline{\Omega}$. By Lemma 5.1, then, $u_+ \in \operatorname{int}(C_+)$. Let $\rho > 0$ be such that $u_+ + h \in \operatorname{int}(C_+)$ for all $h \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$, $\|h\|_{C^0_\delta(\overline{\Omega})} \le \rho$. Since Φ and Φ_+ agree on $\operatorname{int}(C_+)$, for all $h \in X(\Omega) \cap C^0_\delta(\overline{\Omega})$, $\|h\|_{C^0_\delta(\overline{\Omega})} \le \rho$ we have

$$\Phi(u_+) \le \Phi(u+h).$$

By Proposition 5.2, u_+ turns out to be a local minimizer of Φ in $X(\Omega)$, hence $\Phi'(u_+) = 0$ in $X(\Omega)^*$.

Similarly, we find a local minimizer $u_- \in X(\Omega) \cap (-\operatorname{int}(C_+))$ of Φ, with $\Phi'(u_-) = 0$.

From now on we argue by contradiction, assuming that

$$K(\Phi) = \{0, u_\pm\}. \tag{5.3}$$

Note that $\Phi(u_\pm) < \Phi(0) = 0$. In particular, 0 and $u_\pm$ are isolated critical points, so we can compute the corresponding critical groups. Clearly, since $u_\pm$ are strict local minimizers of Φ, we have for all $k \in \mathbb{N}_0$

$$C^k(\Phi, u_\pm) = \delta_{k,0}\mathbb{Z}_2. \tag{5.4}$$

Now we prove that for all $k \in \mathbb{N}_0$

$$C^k(\Phi, 0) = 0. \tag{5.5}$$

By (5.2), for all $u \in X(\Omega) \setminus \{0\}$ we can find $\tau(u) \in (0,1)$ such that $\Phi(\tau u) < 0$ for all $0 < \tau < \tau(u)$. Besides, Hypothesis $\mathbf{H}_5$ (iii) implies

$$\lim_{t\to 0} \frac{qF(x,t) - f(x,t)t}{|t|^p} = 0.$$

So, for all $\varepsilon > 0$ we can find $C_\varepsilon > 0$ such that a.e. in Ω and for all $t \in \mathbb{R}$

$$\left|F(x,t) - \frac{f(x,t)t}{q}\right| \le \varepsilon|t|^p + C_\varepsilon|t|^r.$$

By the relations above we have

$$\int_\Omega \left(F(x,u) - \frac{f(x,u)u}{q}\right) dx = o(\|u\|^p) \quad \text{as } \|u\| \to 0.$$

For all $u \in X(\Omega) \setminus \{0\}$ we have

$$\frac{1}{q}\frac{d}{d\tau}\Phi(\tau u)|_{\tau=1} = \frac{\|u\|^p}{q} - \int_\Omega \frac{f(x,u)u}{q}\,dx = \Phi(u) + \left(\frac{1}{q} - \frac{1}{p}\right)\|u\|^p + o(\|u\|^p) \quad \text{as } \|u\| \to 0.$$

So we can find some $\rho > 0$ such that, for all $u \in B_\rho(0) \setminus \{0\}$ with $\Phi(u) > 0$,

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\Phi(\tau u)|_{\tau=1} > 0. \tag{5.6}$$

This assures uniqueness of $\tau(u)$ defined as above, for all $u \in B_\rho(0)$ with $\Phi(u) > 0$. We set $\tau(u) = 1$ for all $u \in B_\rho(0)$ with $\Phi(u) \le 0$, so we have defined a mapping $\tau : B_\rho(0) \to (0, 1]$. By (5.6) and the Implicit Function Theorem, τ turns out to be continuous. We set for all $(t, u) \in [0, 1] \times B_\rho(0)$

$$h(t, u) = (1 - t)u + t\tau(u)u,$$

so $h : [0, 1]\times B_\rho(0) \to B_\rho(0)$ is a continuous deformation and the set $B_\rho(0)\cap\Phi^0$ is a deformation retract of $B_\rho(0)$. Similarly we deduce that the set $B_\rho(0) \cap \Phi^0 \setminus \{0\}$ is a deformation retract of $B_\rho(0) \setminus \{0\}$. So, by recalling that $\dim(X(\Omega)) = \infty$, we have

$$C^k(\Phi, 0) = H^k(B_\rho(0) \cap \Phi^0, B_\rho(0) \cap \Phi^0 \setminus \{0\}) = H^k(B_\rho(0), B_\rho(0) \setminus \{0\}) = 0,$$

the last passage following from contractibility of $B_\rho(0) \setminus \{0\}$.

Now we compute the critical groups at infinity. Reasoning as in Lemma 5.2, we see that Φ is coercive. So, being also sequentially weakly lower semi-continuous, Φ is bounded below in $X(\Omega)$. Take

$$\eta < \inf_{u\in X(\Omega)} \Phi(u),$$

then we have for all $k \in \mathbb{N}_0$

$$C^k(\Phi, \infty) = H^k(X(\Omega), \Phi^\eta) = \delta_{k,0}\mathbb{Z}_2. \tag{5.7}$$

We recall Proposition 2.8. In our case, by (5.4), (5.5), and (5.7), the Morse identity reads as

$$\sum_{k=0}^{\infty} 2\delta_{k,0}t^k = \sum_{k=0}^{\infty} \delta_{k,0}t^k + (1 + t)Q(t),$$

where Q is a formal power series with coefficients in $\mathbb{N}_0$. Choosing $t = -1$, the relation above leads to a contradiction, hence (5.3) cannot hold. So there exists a further critical point $\tilde{u} \in K(\Phi) \setminus \{0, u_\pm\}$ of Φ. Thus, u_+, u_-, and $\tilde{u}$ are pairwise distinct, non-zero weak solutions of (1.3). □

Remark 5.6. A careful look at the proof of Theorem 5.3 reveals the following situation: either (1.3) admits infinitely many non-zero weak solutions (if $u_\pm$ is *not* a strict local minimizer), or it admits at least three non-zero weak solutions, one of which, denoted $\tilde{u}$, is of mountain pass type, i.e. $C^1(\Phi, \tilde{u}) \neq 0$ (recall Proposition 2.7). This can be seen directly, by constructing a path joining u_+ and u_-, or by contradiction. Assume that $C^1(\Phi, \tilde{u}) = 0$. Then, from the Morse identity we would have

$$h = 1 + q_0 + (q_0 + q_1)t + t^2Q_1(t),$$

where $h \in \mathbb{N}$, $h \ge 2$, $Q(t) = q_0 + q_1t + \cdots$ ($q_k \in \mathbb{N}$ for all $k \in \mathbb{N}_0$). This implies $q_0 \ge 1$, hence a first-order term appears on the right-hand side, a contradiction.

Combining ingeniously the techniques seen above and in Section 4, we can prove a multiplicity result for problem (4.1). Such a result requires modified hypotheses (involving the *second* variational eigenvalue defined in (2.6)).

Hypothesis H_5'. The mapping $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$G(x, t) = \int_0^t g(x, \tau)\,\mathrm{d}\tau,$$

and

(i) $|g(x, t)| \le a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $p < r < p_s^*$),

(ii) $\lim_{t\to 0} \frac{g(x,t)}{|t|^{p-1}} = 0$ uniformly a.e. in Ω,

(iii) $\lambda_2|t|^p + g(x, t)t \ge 0$ a.e. in Ω and for all $t \in \mathbb{R}$,

(iv) $\lim_{|t|\to\infty} \frac{\lambda|t|^p+pG(x,t)}{|t|^p} < \lambda_1$ uniformly a.e. in Ω.

Note that, by Hypothesis $\mathbf{H}_5'$ (iv), we have in particular

$$\lim_{|t|\to\infty} \frac{G(x,t)}{|t|^p} = -\infty,$$

thus we places ourselves again in the coercive case. Our multiplicity result is the following:

Theorem 5.7. *If Hypothesis* $\mathbf{H}_5'$, *Condition* (RC), *and one of the following hold:*
(i) $\lambda > \lambda_2$, $\lambda \notin (\lambda_k)$,
(ii) $\lambda \geq \lambda_2$ *and* $G(x,t) \geq 0$ *for a.e. in* Ω *and for all* $|t| \leq \delta$ *(for some* $\delta > 0$*),*
(iii) $\lambda \geq \lambda_3$ *and* $G(x,t) \leq 0$ *for a.e. in* Ω *and for all* $|t| \leq \delta$ *(for some* $\delta > 0$*),*
then problem (4.1) *admits at least three non-zero solutions.*

Proof. Clearly, we have $0 \in K(\Phi)$. Reasoning as in the proof of Theorem 5.3, we find some $u_\pm \in K(\Phi) \setminus \{0\}$ with $C^k(\Phi, u_\pm) = \delta_{k,0}\mathbb{Z}_2$ and see that $C^k(\Phi, \infty) = \delta_{k,0}\mathbb{Z}_2$ for all $k \in \mathbb{N}_0$. Besides, in all cases (i)–(iii), we argue as in Lemma 4.5 and find $k \geq 2$ such that $C^k(\Phi, 0) \neq 0$. Then we apply [35, Proposition 3.28 (ii)] and deduce that there exists some $\tilde{u} \in K(\Phi)$ such that either $\Phi(\tilde{u}) < 0$ and $C^{k-1}(\Phi, \tilde{u}) \neq 0$, or $\Phi(\tilde{u}) > 0$ and $C^{k+1}(\Phi, \tilde{u}) \neq 0$. Clearly, $\tilde{u} \neq 0$. Moreover, since $k \geq 2$, it follows at once that $\tilde{u} \neq u_\pm$. Thus, u_+, u_-, and $\tilde{u}$ are pairwise distinct, non-zero weak solutions of (4.1). □

6 Asymptotically p-linear case

In this section we deal with problem (1.3), in the case when $f(x,\cdot)$ is *asymptotically* p*-linear* at infinity, i.e.

$$\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda$$

uniformly a.e. in Ω, for some $\lambda \in (0,\infty)$. The problem is said to be of *resonant* type if $\lambda \in \sigma(s,p)$, of *non-resonant* type otherwise. The two cases require different techniques to prove the existence of a non-zero solution (analogous results in the non-resonant case for the p-Laplacian were proved by Liu and Li [27], on the basis of Perera [34]). If $f(x,\cdot)$ has a p-linear behavior at zero as well, but with a different slope, then we can prove the existence of two non-zero solutions, one non-negative, the other non-positive, both in the resonant and non-resonant case, by employing a truncation method (see Zhang, Li, Liu and Feng [46] and Li and Zhou [24] for the p-Laplacian case).

We state here our first set of hypotheses:

Hypothesis $\mathbf{H}_6$. The mapping $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

and
(i) $|f(x,t)| \leq a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $1 < r < p_s^*$),
(ii) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda$ uniformly a.e. in Ω ($\lambda > 0$),
(iii) $\lim_{t\to 0} \frac{f(x,t) - b|t|^{q-2}t}{|t|^{p-2}t} = 0$ uniformly a.e. in Ω ($b > 0$, $1 < q < p$).

Clearly, Hypothesis $\mathbf{H}_6$ (iii) implies that

$$f(x,0) = 0 \quad \text{a.e. in } \Omega,$$

so (1.3) admits the zero solution. We seek non-zero solutions, so with no loss of generality we may assume that *all* critical points of the energy functional $\Phi \in C^1(X(\Omega))$ (defined as in (2.2)) are isolated.

First we introduce our existence result for the non-resonant case:

Theorem 6.1. *If Hypothesis* $\mathbf{H}_6$ *holds with* $\lambda \notin \sigma(s,p)$, *then problem* (1.3) *admits at least a non-zero solution.*

Proof. We first consider the case $0 < \lambda < \lambda_1$. In such a case, Φ is coercive and sequentially weakly lower semi-continuous, so it has a global minimizer $u \in K(\Phi)$. By Proposition 2.7 (i), we have $C^k(\Phi, u) = \delta_{k,0}\mathbb{Z}_2$ for all $k \in \mathbb{N}_0$. If $\lambda > \lambda_1$, then we can find $k \in \mathbb{N}$ such that $\lambda_k < \lambda < \lambda_{k+1}$. By [35, Theorem 5.7], there exists some $u \in K(\Phi)$ such that $C^k(\Phi, u) \neq 0$. In either case, we have found $u \in K(\Phi)$ with a non-trivial critical group.

By Hypothesis $\mathbf{H}_6$ (iii), reasoning as in the proof of Theorem 5.3, we can see that $C^k(\Phi, 0) = 0$ for all $k \in \mathbb{N}_0$, so $u \neq 0$. □

In the study of the resonant case, we meet a significant difficulty: the energy functional Φ need not satisfy Condition (PS). So, we need to introduce additional conditions in order to ensure compactness of critical sequences. We set for all $(x, t) \in \Omega \times \mathbb{R}$

$$H(x,t) = pF(x,t) - f(x,t)t.$$

We have the following existence result:

Theorem 6.2. *If Hypothesis $\mathbf{H}_6$ holds with $\lambda \in \sigma(s,p)$, and there exist $k \in \mathbb{N}$, $h_0 \in L^1(\Omega)$ such that one of the following holds:*
(i) $\lambda_k < \lambda \le \lambda_{k+1}$, $H(x,t) \le -h_0(x)$ *a.e. in Ω and for all $t \in \mathbb{R}$, and*

$$\lim_{|t|\to\infty} H(x,t) = -\infty$$

uniformly a.e. in Ω,
(ii) $\lambda_k \le \lambda < \lambda_{k+1}$, $H(x,t) \ge h_0(x)$ *a.e. in Ω and for all $t \in \mathbb{R}$, and*

$$\lim_{|t|\to\infty} H(x,t) = \infty$$

uniformly a.e. in Ω,
then problem (1.3) *admits at least a non-zero solution.*

Proof. Since $\lambda \in \sigma(s,p)$, by Proposition 2.3 (i) there exists some $k \in \mathbb{N}$ such that $\lambda \in [\lambda_k, \lambda_{k+1}]$, and the latter is a non-degenerate interval. We assume (i). We aim at applying [35, Theorem 5.9], but first we need to verify some technical conditions. Set for all $u \in X(\Omega)$

$$\Psi(u) = \Phi(u) - \frac{1}{p}\langle \Phi'(u), u\rangle = -\frac{1}{p}\int_\Omega H(x,u)\,\mathrm{d}x.$$

Then, for all $u \in X(\Omega)$ we have

$$\Psi(u) \ge \frac{1}{p}\|h\|_1,$$

hence Ψ is bounded below in $X(\Omega)$. Moreover, if (u_n) is a sequence in $X(\Omega)$ such that $\|u_n\| \to \infty$ and $v_n = \|u_n\|^{-1}u_n \to v$ in $X(\Omega)$, then in particular we have $v_n(x) \to v(x)$ a.e. in Ω. So, by the Fatou Lemma we have for all $n \in \mathbb{N}$, $\tau \ge 1$

$$\Psi(\tau u_n) = -\frac{1}{p}\int_\Omega H(x, \|u_n\|\tau v_n)\,\mathrm{d}x,$$

and the latter tends to ∞ as $n \to \infty$. We conclude that Condition (H_+) holds (see [35, p. 82]). So, by [35, Theorem 5.9], Φ satisfies Condition (C) and there exists some $u \in K(\Phi)$ such that $C^k(\Phi, u) \neq 0$. Reasoning as in the proof of Theorem 6.1 we see that $u \neq 0$. Thus, (1.3) has a non-zero solution.

The argument for the case (ii) is analogous. □

Remark 6.3. We note that, if we only assume Hypothesis $\mathbf{H}_6$ (i)–(ii), by the same arguments used in Theorems 6.1 and 6.2 we can prove the existence of a (possibly zero) solution. This is still a valuable information, since we have no condition on $f(\cdot, 0)$.

In the remaining part of the section we deal with the case of a reaction term f which behaves p-linearly both at infinity and at zero, but with different slopes. Our hypotheses are the following.

Hypothesis $\mathbf{H}_6'$. The mapping $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory mapping,

$$F(x,t) = \int_0^t f(x,\tau)\,d\tau \quad \text{for all } (x,t) \in \Omega \times \mathbb{R},$$

and
(i) $|f(x,t)| \leq a(1 + |t|^{r-1})$ a.e. in Ω and for all $t \in \mathbb{R}$ ($a > 0$, $1 < r < p_s^*$),
(ii) $\lim_{|t|\to\infty} \frac{f(x,t)}{|t|^{p-2}t} = \lambda$ uniformly a.e. in Ω ($\lambda > \lambda_1$),
(iii) $\lim_{t\to 0} \frac{f(x,t)}{|t|^{p-2}t} = \mu$ uniformly a.e. in Ω ($0 < \mu < \lambda_1$).

By Hypothesis $\mathbf{H}_6'$ (iii), we have $f(x,0) = 0$ a.e. in Ω, hence problem (1.3) admits the zero solution. For non-zero solutions, we have the following multiplicity result:

Theorem 6.4. *If Hypothesis $\mathbf{H}_6'$ holds, then problem* (1.3) *admits at least two non-zero solutions, one non-negative, the other non-positive.*

Remark 6.5. If, beside Hypothesis $\mathbf{H}_6'$, we also assume a sign condition of the type $f(x,t)t \geq 0$ a.e. in Ω and for all $t \in \mathbb{R}$, then by applying Proposition 2.2 we can prove the existence of a strictly positive and of a strictly negative solution.

Since $f(\,\cdot\,,0) = 0$, we can define truncated energy functionals $\Phi_\pm \in C^1(X(\Omega))$ as in (5.1). We have

$$\langle \Phi_\pm'(u), v\rangle = \langle A(u) \mp \lambda B_\pm(u), v\rangle - \int_\Omega g_\pm(x,u)v\,dx \quad \text{for all } u,v \in X(\Omega),$$

where we set for all $(x,t) \in \Omega \times \mathbb{R}$

$$g_\pm(x,t) = f(x, \pm t^\pm) \mp \lambda (t^\pm)^{p-1}$$

and for all $u, v \in X(\Omega)$

$$\langle B_\pm(u), v\rangle = \int_\Omega (u^\pm)^{p-1} v\,dx.$$

By the compact embedding $X(\Omega) \hookrightarrow L^p(\Omega)$, $B_\pm : X(\Omega) \to X(\Omega)^*$ is a completely continuous operator.

Lemma 6.6. *There exists some $\rho > 0$ such that $\|A(u) \mp \lambda B_\pm(u)\|_* \geq \rho\|u\|^{p-1}$ for all $u \in X(\Omega)$.*

Proof. We deal with $A - \lambda B_+$ (the argument for $A + \lambda B_-$ is analogous). We argue by contradiction: let (u_n), (ε_n) be sequences in $X(\Omega)$ and in $(0,\infty)$, respectively, such that $\varepsilon_n \to 0$ as $n \to \infty$, and for all $n \in \mathbb{N}$

$$\|A(u_n) - \lambda B_+(u_n)\|_* = \varepsilon_n \|u_n\|^{p-1}.$$

Since $A - \lambda B_+$ is $(p-1)$-homogeneous, we may assume $\|u_n\| = 1$ for all $n \in \mathbb{N}$. So (u_n) is bounded, and passing to a relabeled subsequence we have $u_n \rightharpoonup u$ in $X(\Omega)$, $u_n \to u$ in $L^p(\Omega)$ and $(u_n^+)^{p-1} \to (u^+)^{p-1}$ in $L^{p'}(\Omega)$. For all $n \in \mathbb{N}$ we have

$$\begin{aligned} |\langle A(u_n), u_n - u\rangle| &\leq |\langle A(u_n) - \lambda B_+(u_n), u_n - u\rangle| + \lambda|\langle B_+(u_n), u_n - u\rangle| \\ &\leq \varepsilon_n\|u_n - u\| + \lambda\|u_n^+\|_p^{p-1}\|u_n - u\|_p, \end{aligned}$$

and the latter tends to 0 as $n \to \infty$. By the (S)-property of the operator A, we deduce $u_n \to u$ in $X(\Omega)$. So, $\|u\| = 1$ and for all $v \in X(\Omega)$

$$\langle A(u), v\rangle = \lambda \int_\Omega (u^+)^{p-1} v\,dx.$$

Reasoning as in Lemma 5.5 (i), we see that $u(x) \geq 0$ a.e. in Ω. By Proposition 2.2, then, we have $u(x) > 0$ a.e. in Ω. Thus, u turns out to be a positive λ-eigenfunction with $\lambda > \lambda_1$, against Proposition 2.3 (iii). This concludes the proof. □

We point out the following technical lemma:

Lemma 6.7. *The functionals $\Phi_\pm \in C^1(X(\Omega))$ satisfy Condition* (PS) *in* $X(\Omega)$.

Proof. We deal with Φ_+ (the argument for Φ_- is analogous). Let (u_n) be a sequence in $X(\Omega)$ such that $(\Phi_+(u_n))$ is bounded in $\mathbb{R}$ and $\Phi_+'(u_n) \to 0$ in $X(\Omega)^*$. We prove that (u_n) is bounded, arguing by contradiction: assume that (passing if necessary to a subsequence) $\|u_n\| \to \infty$ as $n \to \infty$. Let $\rho > 0$ be as in Lemma 6.6. By Hypothesis $\mathbf{H}_6'$ (i) and (iii), we have $g_+(x,t) = o(t^{p-1})$ as $t \to \infty$, so there exists a constant $C_\rho > 0$ such that a.e. in Ω and for all $t \in \mathbb{R}$

$$|g_+(x,t)| \leq \frac{\rho\lambda_1}{2}(t^+)^{p-1} + C_\rho.$$

For all $n \in \mathbb{N}$, $v \in X(\Omega)$ we have

$$\begin{aligned}|\langle A(u_n) - \lambda B_+(u_n), v\rangle| &\leq |\langle \Phi_+'(u_n), v\rangle| + \int_\Omega |g_+(x,u)v|\,dx \\ &\leq \|\Phi_+'(u_n)\|_*\|v\| + \frac{\rho\lambda_1}{2}\|u_n^+\|_p^{p-1}\|v\|_p + C_\rho\|v\|_1 \\ &\leq \|\Phi_+'(u_n)\|_*\|v\| + \frac{\rho}{2}\|u_n\|^{p-1}\|v\| + C\|v\|.\end{aligned}$$

So, using also Lemma 6.6, we have for all $n \in \mathbb{N}$

$$\rho\|u_n\|^{p-1} \leq \|A(u_n) - \lambda B_+(u_n)\|_* \leq \frac{\rho}{2}\|u_n\|^{p-1} + o(\|u_n\|^{p-1}),$$

a contradiction as $n \to \infty$. Thus, (u_n) is bounded, and as in the proof of Proposition 2.1 we conclude that (u_n) has a convergent subsequence. □

Now we are ready to prove our main result:

Proof of Theorem 6.4. In Hypothesis $\mathbf{H}_6'$ (i) we can always set $p < r < p_s^*$. Choose real numbers $\mu < \alpha < \lambda_1 < \beta < \lambda$. By Hypothesis $\mathbf{H}_6'$ (i) and (iii), there exists a constant $C_\alpha > 0$ such that a.e. in Ω and for all $t \in \Omega$

$$|F(x,t^+)| \leq \frac{\alpha}{p}|t|^p + C_\alpha|t|^r.$$

For all $u \in X(\Omega)$ we have

$$\Phi_+(u) \geq \frac{\|u\|^p}{p} - \frac{\alpha}{p}\|u\|_p^p - C_\alpha\|u\|_r^r \geq \left(1 - \frac{\alpha}{\lambda_1}\right)\frac{\|u\|^p}{p} - C\|u\|^r.$$

So, we can find $R, c > 0$ such that

$$\inf_{u\in\partial B_R(0)} \Phi_+(u) = c. \tag{6.1}$$

By Hypothesis $\mathbf{H}_6'$ (i)–(ii), there exists a constant $C_\beta > 0$ such that a.e. in Ω and for all $t \in \Omega$

$$F(x,t^+) \geq \frac{\beta}{p}(t^+)^p - C_\beta.$$

Let $u_1 \in X(\Omega)$ be a positive λ_1-eigenfunction (recall Proposition 2.3 (ii)). Then for all $\tau > 0$ we have

$$\Phi_+(\tau u_1) = \frac{\tau^p\|u_1\|^p}{p} - \int_\Omega F(x,\tau u_1)\,dx \leq \frac{\tau^p\|u_1\|^p}{p} - \frac{\beta\tau^p}{p}\|u_1\|_p^p + C \leq \tau^p\left(1 - \frac{\beta}{\lambda_1}\right)\frac{\|u_1\|^p}{p} + C,$$

and the latter tends to $-\infty$ as $\tau \to \infty$. So, Φ_+ exhibits the 'mountain pass geometry'. By Lemma 6.7, the functional Φ_+ satisfies Condition (PS) in $X(\Omega)$. Hence, by the Mountain Pass Theorem, there exists some $u_+ \in K(\Phi_+)$ such that $\Phi_+(u_+) \geq c$, with c as in (6.7). In particular, then, $u_+ \neq 0$. Reasoning as in the proof of Lemma 5.5 (i) we see that $u_+(x) \geq 0$ a.e. in Ω, hence $u \in K(\Phi)$ turns out to be a non-negative, non-zero solution of problem (1.3).

In a similar way, working on Φ_-, we produce a non-positive, non-zero solution u_- of problem (1.3) (in particular, $u_+ \neq u_-$). □

Remark 6.8. We could have denoted $f(x,t) = \lambda|t|^{p-2}t + g(x,t)$ for all $(x,t) \in \Omega \times \mathbb{R}$ as in Section 4, with $g(x,t) = o(|t|^{p-1})$ at infinity. But in Theorem 6.4, this would have lead to unnatural condition on the behavior of $g(x,\cdot)$ at zero.

7 Pohožaev identity and nonexistence

In this section we discuss possible non-existence results for problems involving the operator $(-\Delta)_p^s$ via a convenient Pohožaev identity. We focus first on the autonomous equation

$$(-\Delta)_p^s u = f(u) \quad \text{in } \mathbb{R}^N, \tag{7.1}$$

where $0 < s < 1 < p < N$, $f \in C(\mathbb{R})$, and we set for all $(x,t) \in \Omega \times \mathbb{R}$

$$F(t) = \int_0^t f(\tau)\,d\tau.$$

A *weak solution* of (7.1) is a function $u \in W^{s,p}(\mathbb{R}^N)$ such that for all $v \in W^{s,p}(\mathbb{R}^N)$

$$\langle A(u), v\rangle = \int_{\mathbb{R}^N} f(u)v\,dx.$$

As usual, weak solutions of (7.1) can be detected as the critical points of an energy functional $\Phi \in C^1(W^{s,p}(\mathbb{R}^N))$ defined by setting for all $u \in W^{s,p}(\mathbb{R}^N)$

$$\Phi(u) = \frac{[u]_{s,p}^p}{p} - \int_{\mathbb{R}^N} F(u)\,dx,$$

by assuming convenient growth conditions on f.

Let $u \in W^{s,p}(\mathbb{R}^N)$ be a weak solution of (7.1). We define a continuous path $\gamma_u :]0,1] \to W^{s,p}(\mathbb{R}^N)$ by setting for all $\theta \in]0,1]$ and $x \in \mathbb{R}^N$

$$\gamma_u(\theta)(x) := u(\theta x).$$

A simple scaling argument shows that, for all $\theta \in]0,1]$,

$$\Phi \circ \gamma_u(\theta) = \frac{\theta^{sp-N}}{p}[u]_{s,p}^p - \theta^{-N}\int_{\mathbb{R}^N} F(u)\,dx$$

and

$$\frac{d}{d\theta}\Phi \circ \gamma_u(\theta)|_{\theta=1} = \frac{sp-N}{p}[u]_{s,p}^p + N\int_{\mathbb{R}^N} F(u)\,dx.$$

The general form of the *Pohožaev identity* for problem (7.1) is

$$\frac{d}{d\theta}\Phi \circ \gamma_u(\theta)|_{\theta=1} = 0,$$

which, in our case, is easily seen to be equivalent to the following formula:

$$\int_{\mathbb{R}^N}\left(\frac{N-sp}{Np}f(u)u - F(u)\right)dx = 0. \tag{7.2}$$

Identity (7.2) is a major tool to prove non-existence results for problem (7.1). Nevertheless, it requires a more sophisticated machinery, as we need to deduce that

$$\langle \Phi'(u), (x\cdot\nabla u)\rangle = 0,$$

and hence we need good regularity results in order to justify that $v = x\cdot\nabla u$ is an admissible test function for problem (7.1). Such a regularity theory is not available yet.

Remark 7.1. In the semi-linear case $p = 2$, for which the regularity theory is well established, a version of (7.2) has recently be proved by Chang and Wang [10, Proposition 4.1]. Namely, for any weak solution $u \in H^s(\mathbb{R}^N)$ of

$$(-\Delta)^s u = f(u) \quad \text{in } \mathbb{R}^N,$$

we have

$$\int_{\mathbb{R}^N}\left(\frac{N-2s}{2N}f(u)u - F(u)\right)dx = 0.$$

Now we introduce a bounded, smooth domain Ω and couple (7.1) with zero Dirichlet conditions outside Ω, i.e., we consider the problem

$$\begin{cases} (-\Delta)_p^s u = f(u) & \text{in } \Omega, \\ u = 0 & \text{in } \mathbb{R}^N \setminus \Omega. \end{cases} \tag{7.3}$$

Obviously, a *weak solution* of (7.3) is understood as $u \in X(\Omega)$ such that, for all $v \in X(\Omega)$,

$$\langle A(u), v \rangle = \int_\Omega f(u) v \, dx.$$

In this framework, things become even more involved due to the presence of a boundary contribution in the identity. A reasonable candidate to play the role of (7.2), for a weak solution $u \in X(\Omega)$ of (7.3), is the following formula:

$$\int_\Omega \Big(\frac{N - sp}{Np} f(u)u - F(u) \Big) \, dx = -M \int_{\partial\Omega} \Big| \frac{u}{d(x)^\gamma} \Big|^p (x \cdot \nu) \, d\sigma, \tag{7.4}$$

where $M > 0$, $\gamma \in (0, 1)$ depend on s, p, and N, ν denotes the outward normal unit vector to $\partial\Omega$ (see (RC) and the related discussion in Section 5). If Ω is star-shaped, by means of (7.4) one should be able to prove some non-existence results for problem (7.3) of the following type:

Conjecture 7.2. If $f \in C(\mathbb{R})$ satisfies for all $t \in \mathbb{R}$

$$\frac{N - sp}{Np} f(t)t - F(t) \geq 0,$$

then problem (7.3) does not admit any positive bounded weak solution. Moreover, if the inequality above is strict for all $t \in \mathbb{R} \setminus \{0\}$, then problem (7.3) does not admit any non-zero bounded solution.

If we reduce ourselves to the pure power-type reaction terms $f(t) = |t|^{r-2} t$ ($r > 0$), then the assumption of Conjecture 7.2 becomes $r \geq p_s^*$, so non-zero solutions are ruled out for $r > p_s^*$ (as expected).

Remark 7.3. A comparison with some well-known special cases is in order. In the local, nonlinear case ($s = 1$, $p > 1$), our (7.4) yields the classical Pohožaev identity for the p-Laplacian (see Guedda and Veron [18]), provided we set $M = (p-1)/(Np)$, $\gamma = s = 1$, and recall that for all $x \in \partial\Omega$

$$\lim_{h \to 0^+} \Big| \frac{u(x - h\nu)}{h} \Big| = |\nabla u(x) \cdot \nu|.$$

In the semi-linear case ($s \in (0, 1)$, $p = 2$), (7.4) with $\gamma = s$ and $M = \Gamma(1+s)^2/(2N)$ becomes

$$\int_\Omega \Big(\frac{N - 2s}{2N} f(u)u - F(u) \Big) \, dx = -\frac{\Gamma(1+s)^2}{2N} \int_{\partial\Omega} \Big(\frac{u}{d(x)^s} \Big)^2 (x \cdot \nu) \, d\sigma,$$

namely the Pohožaev identity obtained for the fractional Laplacian by Ros Oton and Serra [40, Theorem 1.1]. Such an identity has been applied to prove non-existence results of the type discussed above (see [40, Corollaries 1.2 and 1.3]). Note that for $s = 1$, $p = 2$, the values of M agree as $\Gamma(2) = 1$.

Remark 7.4. In the non-linear case $p \neq 2$, other approaches may lead to non-existence results. For instance, again Ros Oton and Serra [38] have obtained the following result for problem (1.3): if $f \in C^{0,1}_{\mathrm{loc}}(\overline{\Omega} \times \mathbb{R})$ is of supercritical type, i.e., if

$$(N - sp) f(x, t)t - NpF(x, t) - px \cdot F_x(x, t) > 0$$

holds for all $(x, t) \in \overline{\Omega} \times \mathbb{R}$, then (1.3) does not admit any non-zero bounded solution which belongs to $C^{1,\alpha}(\Omega)$ ($0 < \alpha < 1$).

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