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Fractional Caffarelli–Kohn–Nirenberg inequalities ☆

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ABSTRACT

We establish a full range of Caffarelli–Kohn–Nirenberg inequalities and their variants for fractional Sobolev spaces.

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1. Introduction

Let $d \geq 1$, $p \geq 1$, $q \geq 1$, $\tau > 0$, $0 \leq a \leq 1$, α, β, $\gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d}, \quad \frac{1}{p} + \frac{\alpha}{d}, \quad \frac{1}{q} + \frac{\beta}{d} > 0$$

and

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - 1}{d}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big).$$

In the case $a > 0$, assume in addition that, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \leq \alpha - \sigma$$

and

$$\alpha - \sigma \leq 1 \quad \text{if} \quad \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - 1}{d}.$$

Caffarelli, Kohn, and Nirenberg [5] (see also [4]) proved the following well-known inequality

$$\| |x|^\gamma u \|_{L^\tau(\mathbb{R}^d)} \leq C \| |x|^\alpha \nabla u \|^a_{L^p(\mathbb{R}^d)} \| |x|^\beta u \|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d). \tag{1.1}$$

In this paper, we extend this family of inequalities to fractional Sobolev spaces $W^{s,p}$. In the case $a = 1$, $\tau = p$, the corresponding inequality was obtained for $\alpha = 0$ and $\gamma = -s$ in [7,6] and for $\tau = pd/(d - sp)$, $-(d - sp)/p < \alpha = \gamma < 0$, and $1 < p < d/s$ in [1]. To our knowledge, a general version of such inequalities in the framework of fractional Sobolev spaces was not available.

For $p > 1$, $0 < s < 1$, $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ with $\alpha_1 + \alpha_2 = \alpha$, and Ω a measurable subset of $\mathbb{R}^d$, set

$$|u|^p_{W^{s,p,\alpha}(\Omega)} = \int_\Omega \int_\Omega \frac{|x|^{\alpha_1 p} |y|^{\alpha_2 p} |u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy \leq +\infty \quad \text{for } u \in L^1(\Omega).$$

In the case $\alpha_1 = \alpha_2 = \alpha = 0$, we simply denote $|u|_{W^{s,p,0}(\Omega)}$ by $|u|_{W^{s,p}(\Omega)}$.

Let $d \geq 1$, $p > 1$, $q \geq 1$, $\tau > 0$, $0 \leq a \leq 1$, α, β, $\gamma \in \mathbb{R}$ be such that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - s}{d}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big). \tag{1.2}$$

In the case $a > 0$, assume in addition that, with $\gamma = a\sigma + (1-a)\beta$,

$$0 \le \alpha - \sigma \tag{1.3}$$

and

$$\alpha - \sigma \le s \quad \text{if} \quad \frac{1}{\tau} + \frac{\gamma}{d} = \frac{1}{p} + \frac{\alpha - s}{d}. \tag{1.4}$$

Then, we have

Theorem 1.1. *Let $d \ge 1$, $p > 1$, $0 < s < 1$, $q \ge 1$, $\tau > 0$, $0 < a \le 1$, α_1, α_2, α, β, $\gamma \in \mathbb{R}$ be such that $\alpha = \alpha_1 + \alpha_2$, and (1.2), (1.3), and (1.4) hold. We have*

i) if $1/\tau + \gamma/d > 0$, then

$$\| |x|^\gamma u \|_{L^\tau(\mathbb{R}^d)} \le C |u|^a_{W^{s,p,\alpha}(\mathbb{R}^d)} \| |x|^\beta u \|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d),$$

ii) if $1/\tau + \gamma/d < 0$, then

$$\| |x|^\gamma u \|_{L^\tau(\mathbb{R}^d)} \le C |u|^a_{W^{s,p,\alpha}(\mathbb{R}^d)} \| |x|^\beta u \|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d \setminus \{0\}).$$

Assertion *ii*) was established in [6] for $a = 1$, $\tau = p$, $\alpha_1 = \alpha_2 = 0$, and $\gamma = -s$.

The proof of Theorem 1.1 is given in Section 2. Note that the conditions

$$\frac{1}{p} + \frac{\alpha}{d}, \quad \frac{1}{q} + \frac{\beta}{d} > 0$$

are *not* required in Theorem 1.1. Without these conditions, the RHSs in the estimates of Theorem 1.1 are finite for $u \in C^1_c(\mathbb{R}^d)$. The case $1/\tau + \gamma/d = 0$ will be considered in Section 3. In contrast with the mentioned results on fractional Sobolev spaces where the condition $\alpha_1 = \alpha_2 = \alpha/2$ is used, this is *not* necessary in our work.

The idea of the proof is quite elementary and inspired by the work [5]. In the case $0 \le \alpha - \sigma \le s$, the proof uses a variant of Gagliardo–Nirenberg's interpolation inequality for fractional Sobolev spaces (Lemma 2.2) and is as follows. We decompose $\mathbb{R}^d$ into annuli $\mathscr{A}_k$ defined by

$$\mathscr{A}_k := \left\{ x \in \mathbb{R}^d : 2^k \le |x| < 2^{k+1} \right\},$$

and apply the interpolation inequality to have

$$\left(\fint_{\mathscr{A}_k} \left| u - \fint_{\mathscr{A}_k} u \right|^\tau dx \right)^{1/\tau} \le C \left(2^{-(d-sp)k} |u|_{W^{s,p}(\mathscr{A}_k)} \right)^{a/p} \left(\fint_{\mathscr{A}_k} |u|^q \right)^{(1-a)/q}.$$

Here and in what follows, we denote

$$\fint_D v = \frac{1}{|D|}\int_D v\,dx$$

for a measurable subset D of $\mathbb{R}^d$ and for $v \in L^1(D)$. Using again the interpolation inequality in a slightly different way, we can obtain appropriate estimates for the averages and derive the desired conclusion. This is the novelty in our approach. The proof in the case $\alpha - \sigma > s$ is via interpolation and has its roots in [5]. Similar ideas in this paper are used in [8] to obtain several improvements of (1.1) in the classical setting. In the case $1 < p < d$, $\alpha = 0$, and $\sigma > -1$, one can derive (1.1) using the results in [2], [3] and [7] (see Remark 2.3).

The paper is organized as follows. In Section 2, we present the proof of Theorem 1.1. In Section 3, we discuss the case $1/\tau + \gamma/d = 0$.

2. Proof of the main result

We first state a variant of Gagliardo–Nirenberg inequality for fractional Sobolev spaces.

Lemma 2.1. *Let $d \geq 1$, $0 < s < 1$, $p > 1$, $q \geq 1$, $\tau > 0$, and $0 < a \leq 1$ be such that*

$$\frac{1}{\tau} = a\left(\frac{1}{p} - \frac{s}{d}\right) + \frac{1-a}{q}. \tag{2.1}$$

We have

$$\|u\|_{L^\tau(\mathbb{R}^d)} \leq C|u|^a_{W^{s,p}(\mathbb{R}^d)}\|u\|^{1-a}_{L^q(\mathbb{R}^d)} \qquad \text{for } u \in C^1_c(\mathbb{R}^d),$$

for some positive constant C independent of u.

Proof. The result is essentially known. Here is a short proof of it. We first consider the case $1/p - s/d > 0$. Set $p^* := pd/(d - sp)$. We have, by Sobolev's inequality for fractional Sobolev spaces,

$$\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq C|u|_{W^{s,p}(\mathbb{R}^d)}.$$

In this proof, C denotes a positive constant independent of u. Inequality (2.2) is now a consequence of Hölder's inequality. We next consider the case $1/p - s/d \leq 0$. Since

$$1/p - s/d \neq 1/q,$$

by a change of variables, one can assume that

$$|u|_{W^{s,p}(\mathbb{R}^d)} = \|u\|_{L^q(\mathbb{R}^d)} = 1.$$

Since $\tau > q \geq 1$ by (2.1), it follows from John–Nirenberg's inequality that

$$\|u\|_{L^\tau(\mathbb{R}^d)} \leq C.$$

The proof is complete. $\square$

The following result is a consequence of Lemma 2.1 and is used in the proof of Theorem 1.1.

Lemma 2.2. *Let $d \geq 1$, $p > 1$, $0 < s < 1$, $q \geq 1$, $\tau > 0$, and $0 < a \leq 1$ be such that*

$$\frac{1}{\tau} \geq a\left(\frac{1}{p} - \frac{s}{d}\right) + \frac{1-a}{q}.$$

Let $\lambda > 0$ and $0 < r < R$ and set

$$D := \left\{x \in \mathbb{R}^d : \lambda r < |x| < \lambda R\right\}.$$

Then, for $u \in C^1(\bar{D})$,

$$\left(\fint_D \left|u - \fint_D u\right|^\tau dx\right)^{1/\tau} \leq C\left(\lambda^{sp-d}|u|^p_{W^{s,p}(D)}\right)^{a/p} \left(\fint_D |u|^q\, dx\right)^{(1-a)/q} \tag{2.2}$$

for some positive constant C independent of u and λ.

Proof. By scaling, one can assume that $\lambda = 1$. Let $0 < s' \leq s$ and $\tau' \geq \tau$ be such that

$$\frac{1}{\tau'} = a\Big(\frac{1}{p} - \frac{s'}{d}\Big) + \frac{1-a}{q}.$$

From Lemma 2.1, we derive that

$$\left\|u - \fint_D u\right\|_{L^{\tau'}(D)} \leq C\,|u|^a_{W^{s',p}(D)}\,\|u\|^{1-a}_{L^q(D)}.$$

The conclusion now follows from Jensen's inequality and the fact $|u|_{W^{s',p}(D)} \leq C\,|u|_{W^{s,p}(D)}$. $\square$

We are ready to give

• **Proof of Theorem 1.1 in the case $\alpha - \sigma \leq s$.** By Lemma 2.2, we have, for $k \in \mathbb{Z}$,

$$\Big(\fint_{\mathscr{A}_k} \left|u - \fint_{\mathscr{A}_k} u\right|^\tau dx\Big)^{1/\tau} \leq C\Big(2^{-(d-sp)k}|u|^p_{W^{s,p}(\mathscr{A}_k)}\Big)^{a/p}\Big(\fint_{\mathscr{A}_k} |u|^q\, dx\Big)^{(1-a)/q}. \tag{2.3}$$

Using (1.2), we derive from (2.3) that

$$\int_{\mathscr{A}_k} |x|^{\gamma\tau}|u|^\tau \, dx \leq C2^{(\gamma\tau+d)k}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau + C|u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k)}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k)}. \tag{2.4}$$

Let $m, n \in \mathbb{Z}$ be such that $m \leq n-2$. Summing (2.4) with respect to k from m to n, we obtain

$$\int_{\{2^m<|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau \, dx \leq C\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau + C\sum_{k=m}^{n} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k)}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k)}. \tag{2.5}$$

Step 1: Proof of i). Choose n such that

$$\operatorname{supp} u \subset B_{2^n}.$$

We have

$$\Big|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\Big|^\tau \leq C\Big(2^{-(d-sp)k}|u|^p_{W^{s,p}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\Big)^{a\tau/p}\Big(\fint_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} |u|^q \, dx\Big)^{(1-a)\tau/q}.$$

It follows that, with $c = [(1+2^{\gamma\tau+d})/2]^{-1} < 1$,

$$2^{(\gamma\tau+d)k}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau \leq c2^{(\gamma\tau+d)(k+1)}\Big|\fint_{\mathscr{A}_{k+1}} u\Big|^\tau + C|u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}.$$

We derive that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau \leq C\sum_{k=m}^{n} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}. \tag{2.6}$$

Combining (2.5) and (2.6) yields

$$\int_{\{|x|>2^m\}} |x|^{\gamma\tau}|u|^\tau \, dx \leq C\sum_{k=m}^{n} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}.$$

One has, for $s \geq 0$, $t \geq 0$ with $s+t \geq 1$, and for $x_k \geq 0$ and $y_k \geq 0$,

$$\sum_{k=m}^{n} x_k^s y_k^t \leq \Big(\sum_{k=m}^{n} x_k\Big)^s\Big(\sum_{k=m}^{n} y_k\Big)^t. \tag{2.7}$$

Applying this inequality with $s = a\tau/p$ and $t = (1-a)\tau/q$, we obtain that

$$\int_{\{|x|>2^m\}} |x|^{\gamma\tau}|u|^\tau \, dx \le C|u|^{a\tau}_{W^{s,p,\alpha}(\bigcup_{k=m}^\infty \mathscr{A}_k)} \| |x|^\beta u\|^{(1-a)\tau}_{L^q(\bigcup_{k=m}^\infty \mathscr{A}_k)}, \tag{2.8}$$

since $a/p + (1-a)/q \ge 1/\tau$ thanks to the fact $\alpha - \sigma - s \le 0$.

Step 2: Proof of *ii*). Choose m such that

$$\operatorname{supp} u \cap B_{2^m} = \emptyset.$$

We have

$$\Big| \fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u \Big|^\tau \le C \Big(2^{-(d-sp)k} |u|^p_{W^{s,p}(\mathscr{A}_k \cup \mathscr{A}_{k+1})} \Big)^{a\tau/p} \Big(\fint_{\mathscr{A}_k \cup \mathscr{A}_{k+1}} |u|^q \Big)^{(1-a)\tau/q}.$$

It follows that, with $c = (1 + 2^{\gamma\tau+d})/2 < 1$,

$$2^{(\gamma\tau+d)(k+1)} \Big| \fint_{\mathscr{A}_{k+1}} u \Big|^\tau \le c 2^{(\gamma\tau+d)k} \Big| \fint_{\mathscr{A}_k} u \Big|^\tau + C|u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k \cup \mathscr{A}_{k+1})} \| |x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}.$$

We derive that

$$\sum_{k=m}^{n} 2^{(\gamma\tau+d)k} \Big| \fint_{\mathscr{A}_k} u \Big|^\tau \le C \sum_{k=m-1}^{n-1} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k \cup \mathscr{A}_{k+1})} \| |x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}. \tag{2.9}$$

Combining (2.5) and (2.9) yields

$$\int_{\{|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau \, dx \le C \sum_{k=m-1}^{n-1} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k \cup \mathscr{A}_{k+1})} \| |x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k \cup \mathscr{A}_{k+1})}.$$

As in Step 1, we derive from (2.7) that

$$\int_{\{|x|<2^{n+1}\}} |x|^{\gamma\tau}|u|^\tau \, dx \le C|u|^{a\tau}_{W^{s,p,\alpha}(\bigcup_{k=-\infty}^n \mathscr{A}_k)} \| |x|^\beta u\|^{(1-a)\tau}_{L^q(\bigcup_{k=-\infty}^n \mathscr{A}_k)}.$$

The proof is complete in the case $\alpha - \sigma \le s$. □

We next turn to

- **Proof of Theorem 1.1 in the case $\alpha - \sigma > s$.** We follows the strategy in [5]. Since

$$\frac{1}{p} + \frac{\alpha - s}{d} \ne \frac{1}{q} + \frac{\beta}{d},$$

by scaling, one might assume that

$$|u|_{W^{s,p,\alpha}(\mathbb{R}^d)} = 1 \quad \text{and} \quad \|u\|_{L^q(\mathbb{R}^d)} = 1.$$

It is necessary from (1.4) that $0 < a < 1$. Let $0 < a_1, a_2 < 1$ (a_1, a_2 are close to a and are chosen later) and $\tau_1, \tau_2 > 0$ be such that

$$\begin{aligned} \frac{1}{\tau_1} &= \frac{a_1}{p} - \frac{a_1 s}{d} + \frac{1-a_1}{q} \quad \text{if} \quad \frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} > 0, \\ \frac{1}{\tau} > \frac{1}{\tau_1} &\geq \frac{a_1}{p} - \frac{a_1 s}{d} + \frac{1-a_1}{q} \quad \text{if} \quad \frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} \leq 0, \end{aligned} \tag{2.10}$$

and

$$\frac{1}{\tau_2} = \frac{a_2}{p} + \frac{1-a_2}{q}.$$

Set

$$\gamma_1 = a_1\alpha + (1-a_1)\beta \quad \text{and} \quad \gamma_2 = a_2(\alpha - s) + (1-a_2)\beta.$$

We have

$$\frac{1}{\tau_1} + \frac{\gamma_1}{d} \geq a_1\Big(\frac{1}{p} + \frac{\alpha - s}{d}\Big) + (1-a_1)\Big(\frac{1}{q} + \frac{\beta}{d}\Big) \tag{2.11}$$

and

$$\frac{1}{\tau_2} + \frac{\gamma_2}{d} = a_2\Big(\frac{1}{p} + \frac{\alpha - s}{d}\Big) + (1-a_2)\Big(\frac{1}{q} + \frac{\beta}{d}\Big). \tag{2.12}$$

Recall that

$$\frac{1}{\tau} + \frac{\gamma}{d} = a\Big(\frac{1}{p} + \frac{\alpha - s}{d}\Big) + (1-a)\Big(\frac{1}{q} + \frac{\beta}{d}\Big). \tag{2.13}$$

We now assume that

$$|a_1 - a| \text{ and } |a_2 - a| \text{ are small enough}, \tag{2.14}$$

$$a_1 < a < a_2 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - s}{d} < \frac{1}{q} + \frac{\beta}{d}, \tag{2.15}$$

$$a_2 < a < a_1 \quad \text{if} \quad \frac{1}{p} + \frac{\alpha - s}{d} > \frac{1}{q} + \frac{\beta}{d}. \tag{2.16}$$

Using (2.14), (2.15) and (2.16), we derive from (2.11), (2.12), and (2.13) that

$$0 < \frac{1}{\tau_2} + \frac{\gamma_2}{d} < \frac{1}{\tau} + \frac{\gamma}{d} < \frac{1}{\tau_1} + \frac{\gamma_1}{d}. \tag{2.17}$$

Since $a > 0$ and $\alpha - \sigma > s$, it follows from (2.14) that

$$\frac{1}{\tau} - \frac{1}{\tau_2} = (a - a_2)\Big(\frac{1}{p} - \frac{1}{q}\Big) + \frac{a}{d}(\alpha - \sigma - s) > 0 \tag{2.18}$$

and, if $\frac{a}{p} - \frac{as}{d} + \frac{1-a}{q} > 0$,

$$\frac{1}{\tau} - \frac{1}{\tau_1} = (a - a_1)\Big(\frac{1}{p} - \frac{s}{d} - \frac{1}{q}\Big) + \frac{a}{d}(\alpha - \sigma) > 0. \tag{2.19}$$

Since, by (2.10), (2.18), and (2.19),

$$1/\tau > 1/\tau_1 \text{ and } 1/\tau > 1/\tau_2,$$

it follows from (2.17) and Hölder's inequality that

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d \setminus B_1)} \le C\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{R}^d)} \quad \text{and} \quad \||x|^\gamma u\|_{L^\tau(B_1)} \le C\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)}.$$

Applying the previous case, we have

$$\||x|^{\gamma_1} u\|_{L^{\tau_1}(\mathbb{R}^d)} \le C|u|^{a_1}_{W^{s,p,\alpha}(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a_1)}_{L^q(\mathbb{R}^d)} \le C$$

and

$$\||x|^{\gamma_2} u\|_{L^{\tau_2}(\mathbb{R}^d)} \le C|u|^{a_2}_{W^{s,p,\alpha}(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a_2)}_{L^q(\mathbb{R}^d)} \le C.$$

The conclusion follows. □

Remark 2.3. In the case $0 < p < d$, one has, for $1/2 < s < 1$ (see [7]),

$$\Big\| u - \fint_D u \Big\|_{L^{p^*}(D)} \le C(1-s)^{1/p}|u|_{W^{s,p}(D)}.$$

The same proof yields, with $\alpha_1 = \alpha_2 = \alpha = 0$, $\sigma > -s$, and $1/\tau + \gamma/d > 0$,

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \le C(1-s)^{a/p}|u|^a_{W^{s,p}(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d).$$

Using the results in [2,3], one knows that

$$\lim_{s \to 1}(1-s)^{1/p}|u|_{W^{s,p}(\mathbb{R}^d)} = C_{d,p}\|\nabla u\|_{L^p(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d).$$

We then derive that

$$\||x|^\gamma u\|_{L^\tau(\mathbb{R}^d)} \le C\|\nabla u\|^a_{L^p(\mathbb{R}^d)}\||x|^\beta u\|^{(1-a)}_{L^q(\mathbb{R}^d)} \quad \text{for } u \in C^1_c(\mathbb{R}^d).$$

Remark 2.4. In the case $\alpha - \sigma \le s$, the proof also shows that if $1/\tau + \gamma/d > 0$, then

$$\| |x|^{\gamma} u \|_{L^{\tau}(\mathbb{R}^d \setminus B_r)} \le C |u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d \setminus B_r)} \| |x|^{\beta} u \|^{(1-a)}_{L^q(\mathbb{R}^d \setminus B_r)} \quad \text{for } u \in C^1_c(\mathbb{R}^d),$$

and if $1/\tau + \gamma/d < 0$, then

$$\| |x|^{\gamma} u \|_{L^{\tau}(B_r)} \le C |u|^{a}_{W^{s,p,\alpha}(B_r)} \| |x|^{\beta} u \|^{(1-a)}_{L^q(B_r)} \quad \text{for } u \in C^1_c(\mathbb{R}^d \setminus \{0\}),$$

for any $r > 0$. In fact, the proof gives the result with $r = 2^j$ with $j = m$ in the first case and $j = n + 1$ in the second case. However, a change of variables yields the result mentioned here.

3. On the limiting case $1/\tau + \gamma/d = 0$

The main result in this section is

Theorem 3.1. *Let $d \ge 1$, $p > 1$, $0 < s < 1$, $q \ge 1$, $\tau > 1$, $0 < a \le 1$, α_1, α_2, α, β, $\gamma \in \mathbb{R}$ be such that $\alpha = \alpha_1 + \alpha_2$, (1.2) holds, and*

$$0 \le a - \sigma \le s.$$

Let $u \in C^1_c(\mathbb{R}^d)$, and $0 < r < R$. We have

i) if $1/\tau + \gamma/d = 0$ and $\operatorname{supp} u \subset B_R$, then

$$\Big(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2R/|x|)} |u|^{\tau}\, dx \Big)^{1/\tau} \le C |u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d)} \| |x|^{\beta} u \|^{(1-a)}_{L^q(\mathbb{R}^d)},$$

ii) if $1/\tau + \gamma/d = 0$ and $\operatorname{supp} u \cap B_r = \emptyset$, then

$$\Big(\int_{\mathbb{R}^d} \frac{|x|^{\gamma\tau}}{\ln^{\tau}(2|x|/r)} |u|^{\tau}\, dx \Big)^{1/\tau} \le C |u|^{a}_{W^{s,p,\alpha}(\mathbb{R}^d)} \| |x|^{\beta} u \|^{(1-a)}_{L^q(\mathbb{R}^d)}.$$

Proof. In this proof, we use the notations in the proof of Theorem 1.1. We only prove the first assertion. The second assertion follows similarly as in the spirit of the proof of Theorem 1.1. Fix $\xi > 0$. Summing (2.4) with respect to k from m to n, we obtain

$$\int_{\{|x| > 2^m\}} \frac{1}{\ln^{1+\xi}(\tau/|x|)} |x|^{\gamma\tau} |u|^{\tau}\, dx$$

$$\le C \sum_{k=m}^{n} \frac{1}{(n-k+1)^{1+\xi}} \Big| \fint_{\mathscr{A}_k} u \Big|^{\tau} + C \sum_{k=m}^{n} |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k)} \| |x|^{\beta} u \|^{(1-a)\tau}_{L^q(\mathscr{A}_k)}. \quad (3.1)$$

By Lemma 2.2, we have

$$\Big|\fint_{\mathscr{A}_k} u - \fint_{\mathscr{A}_{k+1}} u\Big| \le C\Big(2^{-(d-sp)k}|u|^p_{W^{s,p}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\Big)^{a/p}\Big(\fint_{\mathscr{A}_k\cup\mathscr{A}_{k+1}} |u|^q\Big)^{(1-a)/q}.$$

Applying Lemma 3.2 below with $c = (n-k+1)^\xi/(n-k+1/2)^\xi$, we deduce that

$$\frac{1}{(n-k+1)^\xi}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau \le \frac{1}{(n-k+1/2)^\xi}\Big|\fint_{\mathscr{A}_{k+1}} u\Big|^\tau$$
$$+ C(n-k+1)^{\tau-1-\xi}|u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}. \tag{3.2}$$

We have, for $\xi > 0$ and $k \le n$,

$$\frac{1}{(n-k+1)^\xi} - \frac{1}{(n-k+3/2)^\xi} \sim \frac{1}{(n-k+1)^{\xi+1}}. \tag{3.3}$$

Taking $\xi = \tau - 1 > 0$, we derive from (3.2) and (3.3) that

$$\sum_{k=m}^n \frac{1}{(n-k+1)^{1+\xi}}\Big|\fint_{\mathscr{A}_k} u\Big|^\tau \le C\sum_{k=m}^n |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}. \tag{3.4}$$

Combining (3.1) and (3.4), as in (2.8), we obtain

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\xi}(2^{n+1}/|x|)}|u|^\tau\,dx \le C\sum_{k=m}^n |u|^{a\tau}_{W^{s,p,\alpha}(\mathscr{A}_k\cup\mathscr{A}_{k+1})}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\mathscr{A}_k\cup\mathscr{A}_{k+1})}.$$

Applying inequality (2.7) with $s = a\tau/p$ and $t = (1-a)\tau/q$, we derive that

$$\int_{\{|x|>2^m\}} \frac{|x|^{\gamma\tau}}{\ln^{1+\xi}(2^{n+1}/|x|)}|u|^\tau\,dx \le C|u|^{a\tau}_{W^{s,p,\alpha}(\bigcup_{k=m}^\infty \mathscr{A}_k)}\||x|^\beta u\|^{(1-a)\tau}_{L^q(\bigcup_{k=m}^\infty \mathscr{A}_k)}.$$

This yields the conclusion. □

In the proof of Theorem 3.1, we used the following elementary lemma:

Lemma 3.2. *Let $\Lambda > 1$ and $\tau > 1$. There exists $C = C(\Lambda, \tau) > 0$, depending only on Λ and τ such that, for all $1 < c < \Lambda$,*

$$(|a| + |b|)^\tau \le c|a|^\tau + \frac{C}{(c-1)^{\tau-1}}|b|^\tau \text{ for all } a, b \in \mathbb{R}.$$

Remark 3.3. In Theorem 3.1, we only deal with the case $\tau > 1$ (recall that Theorem 1.1 holds for $\tau > 0$). Similar proof as in the one of Theorem 3.1 holds for the case $\tau > 0$ under the condition that the constant τ for the power log is replaced by any positive constant (strictly) greater than 1.

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