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Stability of eigenvalues for variable exponent problems

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ABSTRACT

In the framework of variable exponent Sobolev spaces, we prove that the variational eigenvalues defined by inf sup procedures of Rayleigh ratios for the Luxemburg norms are all stable under uniform convergence of the exponents.

1. Introduction and main result

The differential equations and variational problems involving $p(x)$-growth conditions arise from nonlinear elasticity theory and electrorheological fluids, and have been the target of various investigations, especially in regularity theory and in nonlocal problems (see e.g. [1–3,10,16,32] and the references therein). Let $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, be a bounded domain and let $p : \bar{\Omega} \to \mathbb{R}^+$ be a continuous function such that

$$1 < p_- := \inf_{\Omega} p \leq p(x) \leq \sup_{\Omega} p =: p_+ < N \quad \text{for all } x \in \Omega. \tag{1.1}$$

We also assume that p is log-Hölder continuous, namely

$$|p(x) - p(y)| \leq -\frac{L}{\log|x-y|} \tag{1.2}$$

for some $L > 0$ and for all $x, y \in \Omega$, with $0 < |x - y| \leq 1/2$. From now on, we denote by

$$\mathscr{C} := \left\{ p \in C(\bar{\Omega}) : p \text{ satisfies (1.1) and (1.2)} \right\}$$

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the set of admissible variable exponents. The goal of this paper is to study the stability of the (variational) eigenvalues with respect to (uniform) variations of $p(\cdot)$ for the problem

$$-\operatorname{div}\Big(p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)-2}\frac{\nabla u}{K(u)}\Big)=\lambda S(u)p(x)\left|\frac{u}{k(u)}\right|^{p(x)-2}\frac{u}{k(u)},\quad u\in W_0^{1,p(x)}(\Omega),\tag{1.3}$$

where we have set

$$K(u):=\|\nabla u\|_{p(x)},\qquad k(u):=\|u\|_{p(x)},\qquad S(u):=\frac{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)}dx}{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)}dx}.$$

Following the argument contained in [21, Section 3], it is possible to derive Eq. (1.3) as the Euler–Lagrange equation corresponding to the minimization of the Rayleigh ratio

$$\frac{K(u)}{k(u)}=\frac{\|\nabla u\|_{p(x)}}{\|u\|_{p(x)}},\quad\text{among all } u\in W_0^{1,p(x)}(\Omega)\setminus\{0\},\tag{1.4}$$

where $\|\cdot\|_{p(x)}$ denotes the Luxemburg norm of the variable exponent Lebesgue space $L^{p(x)}(\Omega)$ (see Section 2). This minimization problem has been firstly introduced in [21] as an appropriate replacement for the *inhomogeneous* minimization problem

$$\frac{\displaystyle\int_\Omega|\nabla u|^{p(x)}dx}{\displaystyle\int_\Omega|u|^{p(x)}dx},\quad\text{among all } u\in W_0^{1,p(x)}(\Omega)\setminus\{0\},$$

which was previously considered in [20] to define the first eigenvalue λ_1 of the $p(x)$-Laplacian. In [20], sufficient conditions for λ_1 defined in this way to be zero or positive are provided. In particular, if $p(\cdot)$ has a strict local minimum (or maximum) in Ω, then $\lambda_1=0$. Arguing as in [21, Lemma A.1], it can be shown that the functionals k and K are differentiable with

$$\langle K'(u),v\rangle=\frac{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)-2}\frac{\nabla u}{K(u)}\cdot\nabla v\,dx}{\displaystyle\int_\Omega p(x)\left|\frac{\nabla u}{K(u)}\right|^{p(x)}dx}\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega),$$

$$\langle k'(u),v\rangle=\frac{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)-2}\frac{u}{k(u)}v\,dx}{\displaystyle\int_\Omega p(x)\left|\frac{u}{k(u)}\right|^{p(x)}dx}\quad\text{for all } u,v\in W_0^{1,p(x)}(\Omega).$$

Therefore, all the critical values of the quotient (1.4) are eigenvalues of Eq. (1.3) and vice versa. The m-th (variational) eigenvalue $\lambda^{(m)}_{p(x)}$ of (1.3) can be obtained as

$$\lambda^{(m)}_{p(x)}:=\inf_{K\in\mathcal{W}^{(m)}_{p(x)}}\sup_{u\in K}\|\nabla u\|_{p(x)},$$

where $\mathcal{W}^{(m)}_{p(x)}$ is the set of symmetric, compact subsets of $\{u\in W_0^{1,p(x)}(\Omega):\|u\|_{p(x)}=1\}$ such that $i(K)\geq m$, and i denotes the Krasnosel'skiĭ genus (or, actually, any other index satisfying the properties listed in Remark 1.4). In [21] existence and properties of the first eigenfunction were studied, while in [7] a numerical method to compute the first eigenpair of (1.3) was obtained and the symmetry breaking phenomena with respect to the constant case were observed. The growth rate of this sequence of eigenvalues was investigated in [31], getting a natural replacement for the growth estimate for the case p constant (cf. [22,23]),

$$\lambda_p^{(m)} \sim m^{p/N}, \qquad \lambda_p^{(m)} := \inf_{K\in\mathcal{W}_p^{(m)}} \sup_{u\in K} \|\nabla u\|_p^p,$$

where $\mathcal{W}_p^{(m)}$ is the set of symmetric, compact subsets of $\left\{u \in W_0^{1,p}(\Omega) : \|u\|_p = 1\right\}$ having index i greater than or equal to m.

In this paper we focus on the right continuity of the maps

$$\mathcal{E}_m : (C(\Omega), \|\cdot\|_\infty) \to \mathbb{R}, \qquad \mathcal{E}_m(p(\cdot)) := \lambda_{p(x)}^{(m)}, \quad m \geq 1.$$

We say that $\mathcal{E}_m$ is *right-continuous* if

$$\mathcal{E}_m(p_h(\cdot)) \to \mathcal{E}_m(p(\cdot)), \quad \text{as } h \to \infty,$$

whenever p, $(p_h) \subset \mathscr{C}$, $p_h \to p$ uniformly in Ω and $p(x) \leq p_h(x)$ for all $h \in \mathbb{N}$ and $x \in \Omega$.

We have the following main result.

Theorem 1.1. *$\mathcal{E}_m$ is right-continuous for all $m \geq 1$.*

The authors would like to thank Enea Parini for pointing out that the case $m = 1$, i.e. the stability of the first eigenvalue, was already treated in [8]. Even if not explicitly stated, also Theorem 1.2 of [8] holds when $(p_h)_+ < N$ and $p(x) \leq p_h(x)$ in Ω for all h. Indeed, without the last assumption, the theorem can fail, in irregular domains, even in the case of constant exponents, as proved in [28, Section 7], see Remark 1.3.

Remark 1.2. For a constant $p \in (1, N)$, problem (1.3) reduces to the well-known eigenvalue problem for the p-Laplacian operator (see e.g. [25,27]), namely

$$-\mathrm{div}(|\nabla u|^{p-2}\nabla u) = \lambda |u|^{p-2}u, \quad u \in W_0^{1,p}(\Omega).$$

In this particular case the continuity of variational eigenvalues has been investigated in [9,24,26,28–30] and, more recently, in [14] in the presence of a weight function in a possibly unbounded domain Ω. With exception of [24,26,28], all these contributions tackle the problem by studying the Γ-convergence of the norm functionals.

Remark 1.3. As pointed out by Lindqvist [28, see Section 7], already in the constant case, the convergence from *below* of the (p_h) to p does *not* guarantee the convergence of the eigenvalues, unless the domain Ω is sufficiently smooth.

Remark 1.4. The same result holds replacing the Krasnosel'skiĭ genus with a general index i with the following properties:

(i) $\mathrm{i}(K)$ is an integer greater than or equal to 1 and is defined whenever K is a nonempty, compact and symmetric subset of a topological vector space such that $0 \notin K$;

(ii) if X is a topological vector space and $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty, then there exists an open subset U of $X \setminus \{0\}$ such that $K \subseteq U$ and $\mathrm{i}(\widehat{K}) \leq \mathrm{i}(K)$ for any compact, symmetric and nonempty $\widehat{K} \subseteq U$;

(iii) if X, Y are two topological vector spaces, $K \subseteq X \setminus \{0\}$ is compact, symmetric and nonempty and $\pi : K \to Y \setminus \{0\}$ is continuous and odd, we have $\mathrm{i}(\pi(K)) \geq \mathrm{i}(K)$.

Examples are the Krasnosel'skiĭ genus and the $\mathbb{Z}_2$-cohomological index [17,18].

The paper is organized as follows. In Section 2 some notions about variable exponent spaces and Γ-convergence are recalled. Also preliminary results are proved in Section 2, in view of the last section. Finally, Section 3 contains the proof of Theorem 1.1 and a more general result which can be useful for future developments.

2. Preliminary results

The variable exponent Lebesgue space $L^{p(x)}(\Omega)$ consists of all measurable functions $u:\Omega\to\mathbb{R}$ having $\varrho_{p(x)}(u)<\infty$, where

$$\varrho_{p(x)}(u) := \int_\Omega |u(x)|^{p(x)}dx$$

is the $p(x)$-modular. $L^{p(x)}(\Omega)$ is endowed with the Luxemburg norm $\|\cdot\|_{p(x)}$ defined by

$$\|u\|_{p(x)} := \inf\Big\{\gamma>0 : \varrho_{p(x)}(u/\gamma)\le 1\Big\}.$$

The norm $\|\cdot\|_{p(x)}$ is in close relation with the $p(x)$-modular $\varrho_{p(x)}(\cdot)$, as shown for instance by unit ball property [19, Theorem 1.3] which we report here for completeness.

Proposition 2.1. *Let $p\in L^\infty(\Omega)$ with $1<p_-\le p_+<\infty$. Then, for all $u\in L^{p(x)}(\Omega)$ the following equivalence holds*

$$\|u\|_{p(x)}<1\,(=1;\,>1) \iff \varrho_{p(x)}(u)<1\,(=1;\,>1).$$

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ consists of all $L^{p(x)}(\Omega)$-functions having distributional gradient $\nabla u\in L^{p(x)}(\Omega)$, and is endowed with the norm

$$\|u\|_{1,p(x)} = \|u\|_{p(x)} + \|\nabla u\|_{p(x)}.$$

Under the smoothness assumption (1.2), we denote by $W_0^{1,p(x)}(\Omega)$ the closure of $C_c^\infty(\Omega)$ with respect to the norm $\|\cdot\|_{1,p(x)}$ and we endow $W_0^{1,p(x)}(\Omega)$ with the equivalent norm $\|\nabla\cdot\|_{p(x)}$. For further details on the variable exponent Lebesgue and Sobolev spaces, we refer the reader to [16]. We now recall from [13] the notion of Γ-convergence that will be useful in the sequel.

Definition 2.2. Let X be a metrizable topological space and let (f_h) be a sequence of functions from X to $\overline{\mathbb{R}}$. The *Γ-lower limit* and the *Γ-upper limit* of the sequence (f_h) are the functions from X to $\overline{\mathbb{R}}$ defined by

$$\Big(\Gamma-\liminf_{h\to\infty} f_h\Big)(u) = \sup_{U\in\mathcal{N}(u)}\Big[\liminf_{h\to\infty}\big(\inf\{f_h(v):v\in U\}\big)\Big],$$
$$\Big(\Gamma-\limsup_{h\to\infty} f_h\Big)(u) = \sup_{U\in\mathcal{N}(u)}\Big[\limsup_{h\to\infty}\big(\inf\{f_h(v):v\in U\}\big)\Big],$$

where $\mathcal{N}(u)$ denotes the family of all open neighborhoods of u in X. If there exists a function $f:X\to\overline{\mathbb{R}}$ such that

$$\Gamma-\liminf_{h\to\infty} f_h = \Gamma-\limsup_{h\to\infty} f_h = f,$$

then we write $\Gamma-\lim_{h\to\infty} f_h = f$ and we say that (f_h) *Γ-converges* to its *Γ-limit* f.

For any $p\in\mathscr{C}$, we define $\mathscr{E}_{p(x)}:L^1(\Omega)\to[0,\infty]$ as

$$\mathscr{E}_{p(x)}(u) := \begin{cases} \|\nabla u\|_{p(x)} & \text{if } u\in W_0^{1,p(x)}(\Omega),\\ +\infty & \text{otherwise}\end{cases} \tag{2.1}$$

and $g_{p(x)} : L^1(\Omega) \to [0, \infty)$ as

$$g_{p(x)}(u) := \begin{cases} \|u\|_{p(x)} & \text{if } u \in L^{p(x)}(\Omega), \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.3. *The following properties hold:*

(a) *$g_{p(x)}$ is even and positively homogeneous of degree 1;*
(b) *for every $b \in \mathbb{R}$ the restriction of $g_{p(x)}$ to $\{u \in L^1(\Omega) : \mathscr{E}_{p(x)}(u) \leq b\}$ is continuous.*

Proof. (a) follows easily from the definition of $g_{p(x)}$. (b) Let $(u_n) \subset \{u \in L^1(\Omega) : \mathscr{E}_{p(x)}(u) \leq b\}$ converge to u in $L^1(\Omega)$ and consider a subsequence (u_{h_n}). By (2.1), we know that (u_{h_n}) is bounded in $W_0^{1,p(x)}(\Omega)$ which is reflexive, hence there exists a subsequence $(u_{h_{n_j}})$ that converges weakly to $\bar{u}$ in $W_0^{1,p(x)}(\Omega)$. Since $W_0^{1,p(x)}(\Omega)$ is compactly embedded in $L^{p(x)}(\Omega)$, cf. [15, Proposition 2.2 and Lemma 5.5], $(u_{h_{n_j}})$ converges strongly to $\bar{u}$ in $L^{p(x)}(\Omega)$. By the arbitrariness of the subsequence (u_{h_n}), we get that the whole sequence $u_n \to \bar{u}$ in $L^{p(x)}(\Omega)$ and also in $L^1(\Omega)$. Therefore, $u = \bar{u}$ and the proof is concluded. $\square$

Lemma 2.4. *Let p, $(p_h) \subset \mathscr{C}$ be such that $p_h \to p$ pointwise. Then, for all $w \in C_c^1(\Omega)$*

$$\lim_{h\to\infty} \|\nabla w\|_{p_h(x)} = \|\nabla w\|_{p(x)}.$$

Proof. By means of [16, Corollary 3.5.4], we know that

$$\|\nabla w\|_{p(x)} \leq \liminf_{h\to\infty} \|\nabla w\|_{p_h(x)}.$$

It remains to prove that

$$\|\nabla w\|_{p(x)} \geq \limsup_{h\to\infty} \|\nabla w\|_{p_h(x)}.$$

If $\nabla w = 0$ in Ω, the conclusion is obvious, so we can assume that $\|\nabla w\|_{p(x)} > 0$. By hypothesis, $p_h \to p$ pointwise and $p_h(x) < N$ for all $h \in \mathbb{N}$ and $x \in \Omega$, hence for all $\alpha \in (0, 1)$

$$\left|\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right|^{p_h(x)} \to \left|\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right|^{p(x)} \quad \text{for all } x \in \Omega,$$
$$\left|\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right|^{p_h(x)} \leq 1 + \left(\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right)^N \in L^1(\Omega) \quad \text{for all } h \in \mathbb{N}.$$

Therefore, by the dominated convergence theorem, we obtain

$$\lim_{h\to\infty} \varrho_{p_h(x)}\left(\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right) = \varrho_{p(x)}\left(\frac{\alpha\nabla w}{\|\nabla w\|_{p(x)}}\right) \leq \alpha\varrho_{p(x)}\left(\frac{\nabla w}{\|\nabla w\|_{p(x)}}\right) = \alpha < 1.$$

Thus, for h sufficiently large $\varrho_{p_h(x)}\left(\alpha\nabla w/\|\nabla w\|_{p(x)}\right) < 1$, which in turn gives

$$\left\|\alpha\nabla w/\|\nabla w\|_{p(x)}\right\|_{p_h(x)} < 1$$

by Proposition 2.1. Whence

$$\limsup_{h\to\infty} \|\nabla w\|_{p_h(x)} \leq \frac{\|\nabla w\|_{p(x)}}{\alpha} \quad \text{for all } \alpha \in (0, 1)$$

and by the arbitrariness of α the assertion is proved. $\square$

Theorem 2.5. *Let p, $(p_h) \subset \mathscr{C}$ be such that $p_h \to p$ pointwise. Then*

$$\mathscr{E}_{p(x)}(u) \geq \left(\Gamma - \limsup_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u) \quad \text{for all } u \in L^1(\Omega). \tag{2.2}$$

Proof. Suppose that $\mathscr{E}_{p(x)}(u) < \infty$ (otherwise (2.2) is obvious) and take $b \in \mathbb{R}$ such that $b > \mathscr{E}_{p(x)}(u)$. Let $\delta > 0$ and $w \in C^1_c(\Omega)$ with $\|w - u\|_1 < \delta$ and $\|\nabla w\|_{p(x)} < b$, then $\|\nabla w\|_{p_h(x)} \to \|\nabla w\|_{p(x)}$ by Lemma 2.4. Therefore,

$$b > \lim_{h\to\infty} \mathscr{E}_{p_h(x)}(w),$$

and in turn

$$b > \limsup_{h\to\infty}(\inf\{\mathscr{E}_{p_h(x)}(v) : \|v - u\|_1 < \delta\}).$$

By the arbitrariness of $\delta > 0$ we get

$$b \geq \left(\Gamma - \limsup_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u)$$

and since $b > \mathscr{E}_{p(x)}(u)$ is arbitrary, we obtain (2.2). $\square$

Lemma 2.6. *Let $p, q : \Omega \to [1, \infty)$ be measurable functions with $p(x) \leq q(x)$ for a.a. $x \in \Omega$. Then $L^{q(x)}(\Omega) \hookrightarrow L^{p(x)}(\Omega)$ with embedding constant less than or equal to*

$$C(|\Omega|, p, q) := \left[\left(\frac{p}{q}\right)_+ + \left(1 - \frac{p}{q}\right)_+\right] \max\{|\Omega|^{(1/p-1/q)_+}, |\Omega|^{(1/p-1/q)_-}\}. \tag{2.3}$$

In particular, $C(|\Omega|, p, q_j) \to 1$ whenever $q_j \to p$ uniformly in Ω.

Proof. For all $u \in L^{q(x)}(\Omega)$, by Hölder's inequality [16, cf. (3.2.23)]

$$\|u\|_{p(x)} \leq \left[\left(\frac{p}{r}\right)_+ + \left(\frac{p}{q}\right)_+\right] \|1\|_{r(x)} \|u\|_{q(x)},$$

where $1/r := 1/p - 1/q$ a.e. in Ω. Moreover, by [16, Lemma 3.2.5], we get

$$\|1\|_{r(x)} \leq \max\{|\Omega|^{1/r_-}, |\Omega|^{1/r_+}\},$$

which concludes the proof. $\square$

Theorem 2.7. *Let p, $(p_h) \subset \mathscr{C}$ be such that $p(x) \leq p_h(x)$ for all $h \in \mathbb{N}$ and $x \in \Omega$, and $p_h \to p$ uniformly in Ω. Then*

$$\mathscr{E}_{p(x)}(u) \leq \left(\Gamma - \liminf_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u) \quad \text{for all } u \in L^1(\Omega). \tag{2.4}$$

Proof. If $\left(\Gamma - \liminf_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u) = +\infty$ there is nothing to prove. In the other case, take $b \in \mathbb{R}$ such that $b > \left(\Gamma - \liminf_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u)$. By virtue of [13, Proposition 8.1-(b)] there exists a sequence $(u_h) \subset L^1(\Omega)$ such that $u_h \to u$ in $L^1(\Omega)$ and

$$\left(\Gamma - \liminf_{h\to\infty} \mathscr{E}_{p_h(x)}\right)(u) = \liminf_{h\to\infty} \mathscr{E}_{p_h(x)}(u_h).$$

Hence, there is a subsequence (p_{h_n}) for which

$$\sup_{n\in\mathbb{N}} \mathscr{E}_{p_{h_n}(x)}(u_{h_n}) < b.$$

Let $(v_n) \subset C^1_c(\Omega)$ verify

$$\|v_n - u_{h_n}\|_1 < \frac{1}{n}, \quad \mathscr{E}_{p_{h_n}(x)}(v_n) < b \quad \text{for all } n \in \mathbb{N}.$$

Then $v_n \to u$ in $L^1(\Omega)$ and, by the embedding $W_0^{1,p_{h_n}(x)}(\Omega) \hookrightarrow W_0^{1,p(x)}(\Omega)$,

$$b > \|\nabla v_n\|_{p_{h_n}(x)} \geq \frac{\|\nabla v_n\|_{p(x)}}{C(|\Omega|, p, p_{h_n})} \quad \text{for all } n \in \mathbb{N},$$

where $C(|\Omega|, p, p_{h_n})$ is given in (2.3) with $q = p_{h_n}$ and $C(|\Omega|, p, p_{h_n}) \leq 2(1 + |\Omega|) < \infty$ for all n. Therefore, (v_n) is bounded in the reflexive space $W_0^{1,p(x)}(\Omega)$ and so there exists a subsequence (v_{n_j}) such that $v_{n_j} \rightharpoonup u$ in $W_0^{1,p(x)}(\Omega)$. By Lemma 2.6 and the uniform convergence of $(p_{h_{n_j}})$ to p,

$$\lim_{j\to\infty} C(|\Omega|, p, p_{h_{n_j}}) = 1,$$

and, together with the weak lower semicontinuity of the norm, we get

$$b \geq \liminf_{j\to\infty} \frac{\|\nabla v_{n_j}\|_{p(x)}}{C(|\Omega|, p, p_{h_{n_j}})} \geq \|\nabla u\|_{p(x)} = \mathscr{E}_{p(x)}(u).$$

In conclusion, by the arbitrariness of b, we obtain (2.4). □

Lemma 2.8. *Let p, $(p_h) \subset \mathscr{C}$ be such that $p_h \to p$ pointwise, $u \in L^{p(x)}(\Omega)$, $u_h \in L^{p_h(x)}(\Omega)$ for all h, and $u_h \to u$ a.e. in Ω. Then*

$$\|u\|_{p(x)} \leq \liminf_{h\to\infty} \|u_h\|_{p_h(x)}.$$

Proof. Suppose that $\liminf_{h\to\infty} \|u_h\|_{p_h(x)} < \infty$ (otherwise there is nothing to prove) and take any $\alpha \in \mathbb{R}$ such that $\alpha > \liminf_{h\to\infty} \|u_h\|_{p_h(x)}$. Then there exists a subsequence (p_{h_n}) for which $\|u_{h_n}\|_{p_{h_n}(x)} < \alpha$ for all j. Hence, $\varrho_{p_{h_n}(x)}(u_{h_n}/\alpha) < 1$ and by Fatou's Lemma

$$\int_\Omega \left|\frac{u}{\alpha}\right|^{p(x)} dx \leq \liminf_{n\to\infty} \int_\Omega \left|\frac{u_{h_n}}{\alpha}\right|^{p_{h_n}(x)} dx \leq 1.$$

Thus, by Proposition 2.1, $\|u/\alpha\|_{p(x)} \leq 1$, that is $\|u\|_{p(x)} \leq \alpha$. The conclusion follows by the arbitrariness of α. □

Lemma 2.9. *Let p, $(p_h) \subset \mathscr{C}$ and $p_h \to p$ uniformly in Ω. Then, for every sequence (u_h) such that $u_h \in W_0^{1,p_h(x)}(\Omega)$ for all h, and $\sup_{h\in\mathbb{N}} \|\nabla u_h\|_{p_h(x)} < \infty$, there exists a subsequence (u_{h_n}) for which*

$$\lim_{n\to\infty} \varrho_{p_{h_n}(x)}(u_{h_n}) = \varrho_{p(x)}(u).$$

Proof. Let $\varepsilon > 0$ be such that $\varepsilon < p_- - 1$ and $\varepsilon^2 + 2N\varepsilon < p_-^2$, then there is $\bar{h} \in \mathbb{N}$ for which

$$p(x) - \varepsilon < p_h(x) < p(x) + \varepsilon \quad \text{for all } x \in \Omega \text{ and } h \geq \bar{h}.$$

By Lemma 2.6,

$$\begin{array}{l} u_h \in W_0^{1,p(x)-\varepsilon}(\Omega), \\ \|\nabla u_h\|_{p(x)-\varepsilon} \leq 2(1 + |\Omega|)\|\nabla u_h\|_{p_h(x)} \leq 2b(1 + |\Omega|) < \infty \end{array} \quad \text{for all } h \geq \bar{h},$$

where $b := \sup_{h\in\mathbb{N}} \|\nabla u_h\|_{p_h(x)}$. Since $W_0^{1,p(x)-\varepsilon}(\Omega)$ is reflexive, there exists a subsequence (u_{h_n}) such that $u_{h_n} \rightharpoonup u$ in $W_0^{1,p(x)-\varepsilon}(\Omega)$. Now, it is easy to see that

$$\varepsilon^2 + 2N\varepsilon < p_-^2 \iff p(x) + \varepsilon < \frac{N(p(x) - \varepsilon)}{N - (p(x) - \varepsilon)} = (p(x) - \varepsilon)^* \quad \text{for all } x \in \Omega,$$

therefore $W_0^{1,p(x)-\varepsilon}(\Omega)$ is compactly embedded in $L^{p(x)+\varepsilon}(\Omega)$ (see [15, Theorem 5.7]) and so $u_{h_n} \to u$ in $L^{p(x)+\varepsilon}(\Omega)$. Now, by [3, Lemma B.1], there exist a subsequence, still denoted by (u_{h_n}), and a function $v \in L^{p(x)+\varepsilon}(\Omega)$ for which $u_{h_n} \to u$ and $|u_{h_n}| \le |v|$ a.e. in Ω. Whence, a.e.,

$$\lim_{n\to\infty} |u_{h_n}|^{p_{h_n}(x)} = |u|^{p(x)},$$
$$|u_{h_n}|^{p_{h_n}(x)} \le 1 + |v|^{p(x)+\varepsilon} \in L^1(\Omega) \quad \text{for all } n \in \mathbb{N}.$$

In conclusion, by the dominated convergence theorem we obtain

$$\lim_{n\to\infty} \int_\Omega |u_{h_n}|^{p_{h_n}(x)} dx = \int_\Omega |u|^{p(x)} dx,$$

namely the assertion. □

Theorem 2.10. *Let p, $(p_h) \subset \mathscr{C}$, $p_h \to p$ uniformly in Ω. Then, for every subsequence (p_{h_n}) and for every sequence $(u_n) \subset L^1(\Omega)$ verifying*

$$\sup_{n\in\mathbb{N}} \mathscr{E}_{p_{h_n}(x)}(u_n) < \infty,$$

there exists a subsequence (u_{n_j}) such that, as $j \to \infty$,

$$u_{n_j} \to u \quad \textit{in } L^1(\Omega),$$
$$g_{p_{h_{n_j}}(x)}(u_{n_j}) \to g_{p(x)}(u).$$

Proof. For $\varepsilon \in (0, p_- - 1)$ and for h_n sufficiently large, $W_0^{1,p_{h_n}(x)}(\Omega) \hookrightarrow W_0^{1,p(x)-\varepsilon}(\Omega)$, with embedding constant less than or equal to $2(1+|\Omega|)$ (cf. [16, Corollary 3.3.4]), then

$$\|\nabla u_n\|_{p(x)-\varepsilon} \le 2(1+|\Omega|)\|\nabla u_n\|_{p_{h_n}(x)} \le 2b(1+|\Omega|),$$

where

$$b := \sup_{n\in\mathbb{N}} \mathscr{E}_{p_{h_n}(x)}(u_n).$$

Since $W_0^{1,p(x)-\varepsilon}(\Omega)$ is reflexive, (u_n) admits a subsequence (u_{n_j}) weakly convergent to u in $W_0^{1,p(x)-\varepsilon}(\Omega)$. Thus, $u_{n_j} \to u$ in $L^1(\Omega)$ and up to a subsequence $u_{n_j} \to u$ a.e. in Ω. For the second part of the statement, we have to prove that $\|u_{n_j}\|_{p_{h_{n_j}}(x)} \to \|u\|_{p(x)}$. By Lemma 2.8 we know that

$$\|u\|_{p(x)} \le \liminf_{j\to\infty} \|u_{n_j}\|_{p_{h_{n_j}}(x)}.$$

Now, for every real number

$$\alpha < \limsup_{j\to\infty} \|u_{n_j}\|_{p_{h_{n_j}}(x)},$$

there exists a subsequence, still denoted by $(p_{h_{n_j}})$, for which $\alpha < \|u_{n_j}\|_{p_{h_{n_j}}(x)}$ for all j, and so

$$1 < \int_\Omega \left|\frac{u_{n_j}}{\alpha}\right|^{p_{h_{n_j}}(x)} dx,$$

by Proposition 2.1. Therefore, Lemma 2.9 yields

$$1 \le \lim_{j\to\infty} \varrho_{p_{h_{n_j}}(x)}\left(\frac{u_{n_j}}{\alpha}\right) = \varrho_{p(x)}\left(\frac{u}{\alpha}\right)$$

up to a subsequence, that is $\|u\|_{p(x)} \ge \alpha$ again by unit ball property. The conclusion follows by the arbitrariness of α. □

We need to show that the minimax values with respect to the $W_0^{1,p(x)}(\Omega)$-topology are equal to those with respect to the weaker topology $L^1(\Omega)$. To this aim, let $\mathcal{W}^{(m)}_{p(x)}$ be the family of those subsets K of

$$\{u \in W_0^{1,p(x)}(\Omega) : g_{p(x)}(u) = 1\}$$

which are compact and symmetric (i.e. $K = -K$), for which $i(K) \geq m$ with respect to the norm topology of $W_0^{1,p(x)}(\Omega)$, where i denotes the Krasnosel'skiĭ genus. Furthermore, denote by $\mathcal{K}^{(m)}_{s,p(x)}$ the family of compact and symmetric subsets K of

$$\{u \in L^1(\Omega) : g_{p(x)}(u) = 1\}$$

such that $i(K) \geq m$, with respect to the topology of $L^1(\Omega)$.

Theorem 2.11. *Let $p \in \mathscr{C}$ and $f_{p(x)} : L^1(\Omega) \to [0, \infty]$ be convex, even and positively homogeneous of degree 1. Suppose that there exists $\nu > 0$ such that*

$$f_{p(x)}(u) \geq \nu \mathscr{E}_{p(x)}(u) \quad \textit{for all } u \in L^1(\Omega).$$

Then, for every integer $m \geq 1$, we have

$$\inf_{K \in \mathcal{K}^{(m)}_{s,p(x)}} \sup_K f_{p(x)} = \inf_{K \in \mathcal{W}^{(m)}_{p(x)}} \sup_K f_{p(x)}. \tag{2.5}$$

In particular, for all $m \geq 1$

$$\inf_{K \in \mathcal{K}^{(m)}_{s,p(x)}} \sup_K \mathscr{E}_{p(x)} = \inf_{K \in \mathcal{W}^{(m)}_{p(x)}} \sup_K \mathscr{E}_{p(x)}. \tag{2.6}$$

Proof. By Proposition 2.3-(b) we know that for all $b \in \mathbb{R}$ the restriction of $g_{p(x)}$ to the set $\{v \in L^1(\Omega) : \mathscr{E}_{p(x)}(v) \leq b/\nu\} \supseteq \{v \in L^1(\Omega) : f_{p(x)}(v) \leq b\}$ is $L^1(\Omega)$-continuous. *A fortiori* the restriction of $g_{p(x)}$ to the same set is continuous with respect to the stronger topology $W_0^{1,p(x)}(\Omega)$ and (2.5) follows by [14, Corollary 3.3]. Finally, by definition, the function $\mathscr{E}_{p(x)}$ is convex, even and positively homogeneous of degree 1. Therefore, the second part of the statement follows immediately by taking $\nu = 1$. □

3. Proof of Theorem 1.1

Due to Proposition 2.3 and the first part of Theorem 2.11, the functionals $\mathscr{E}_{p(x)}$, $g_{p(x)}$, $\mathscr{E}_{p_h(x)}$ and $g_{p_h(x)}$ for all $h \in \mathbb{N}$ satisfy all the structural assumptions required in Section 4 of [14]. Moreover, by Theorems 2.5 and 2.7, we know that

$$\mathscr{E}_{p(x)}(u) = \Big(\Gamma - \lim_{h\to\infty} \mathscr{E}_{p_h(x)}\Big)(u) \quad \text{for all } u \in L^1(\Omega).$$

Therefore, together with Theorem 2.10, all the hypotheses of [14, Corollary 4.4] are verified and so we can infer that

$$\inf_{K \in \mathcal{K}^{(m)}_{s,p(x)}} \sup_{u \in K} \mathscr{E}_{p(x)}(u) = \lim_{h\to\infty} \Big(\inf_{K \in \mathcal{K}^{(m)}_{s,p_h(x)}} \sup_{u \in K} \mathscr{E}_{p_h(x)}(u) \Big).$$

Finally, by (2.6) the last equality reads as

$$\lambda^{(m)}_{p(x)} = \inf_{K \in \mathcal{W}^{(m)}_{p(x)}} \sup_{u \in K} \mathscr{E}_{p(x)}(u) = \lim_{h\to\infty} \Big(\inf_{K \in \mathcal{W}^{(m)}_{p_h(x)}} \sup_{u \in K} \mathscr{E}_{p_h(x)}(u) \Big) = \lim_{h\to\infty} \lambda^{(m)}_{p_h(x)}$$

which proves the assertion. □

Actually, the results presented in Section 2 allow us to prove a more general theorem. In order to state it, we need to recall the notion of asymptotic equicoercitivity.

Definition 3.1. A sequence (F_h) of functions from a metrizable topological space X to $\overline{\mathbb{R}}$ is said to be *asymptotically equicoercive* if, for every strictly increasing sequence (h_n) in $\mathbb{N}$ and every sequence (u_n) in X satisfying

$$\sup_{n\in\mathbb{N}} F_{h_n}(u_n) < +\infty,$$

there exists a subsequence (u_{n_j}) converging in X.

We also introduce $\mathcal{K}$, the family of nonempty compact subsets of $L^1(\Omega)$, and $d_{\mathcal{H}}$, the *Hausdorff distance* induced on $\mathcal{K}$ by the usual norm of $L^1(\Omega)$, that is

$$d_{\mathcal{H}}(K_1, K_2) = \max\Big\{\max_{u\in K_1} d(u, K_2), \max_{v\in K_2} d(v, K_1)\Big\}.$$

The $\mathcal{H}$*-topology* is the topology on $\mathcal{K}$ induced by $d_{\mathcal{H}}$.

Theorem 3.2. *Let p, $(p_h) \subset \mathscr{C}$ be such that $p_h \to p$ uniformly in Ω. Let $f, f_h : L^1(\Omega) \to [0,\infty]$ be such that*

(h_1) *f is even;*
(h_2) *f_h is convex, even, and positively homogeneous of degree 1 for all $h \in \mathbb{N}$;*
(h_3) *there exists $\nu > 0$ such that $f_h(u) \ge \nu \mathscr{E}_{p_h(x)}(u)$ for all $h \in \mathbb{N}$ and $u \in L^1(\Omega)$.*

For every integer $m \ge 1$, let $\mathcal{F}_h^{(m)}$, $\mathcal{F}^{(m)} : \mathcal{K} \to [0,\infty]$ be defined as

$$\mathcal{F}_h^{(m)}(K) := \begin{cases} \sup_K f_h & \text{if } K \in \mathcal{K}^{(m)}_{s,p_h(x)}, \\ +\infty & \text{otherwise,} \end{cases} \qquad \mathcal{F}^{(m)}(K) := \begin{cases} \sup_K f & \text{if } K \in \mathcal{K}^{(m)}_{s,p(x)}, \\ +\infty & \text{otherwise.} \end{cases}$$

If furthermore

$$f(u) = \Big(\Gamma - \liminf_{h\to\infty} f_h\Big)(u) \quad \text{for all } u \in L^1(\Omega), \tag{3.1}$$

then for all $m \ge 1$, the sequence $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive and we have

$$\mathcal{F}^{(m)}(K) = \Big(\Gamma - \liminf_{h\to\infty} \mathcal{F}_h^{(m)}\Big)(K) \quad \text{for all } K \in \mathcal{K}, \tag{3.2}$$

$$\inf_{K\in\mathcal{K}} \mathcal{F}^{(m)}(K) = \lim_{h\to\infty}\Big(\inf_{K\in\mathcal{K}} \mathcal{F}_h^{(m)}(K)\Big), \tag{3.3}$$

$$\inf_{K\in\mathcal{K}^{(m)}_{s,p(x)}} \sup_K f = \lim_{h\to\infty}\Bigg(\inf_{K\in\mathcal{K}^{(m)}_{s,p_h(x)}} \sup_K f_h\Bigg), \tag{3.4}$$

$$\inf_{K\in\mathcal{W}^{(m)}_{p(x)}} \sup_K f = \lim_{h\to\infty}\Bigg(\inf_{K\in\mathcal{W}^{(m)}_{p_h(x)}} \sup_K f_h\Bigg). \tag{3.5}$$

Proof. By (h_3), Theorem 2.10, (3.1), and Corollary 4.4 of [14], $(\mathcal{F}_h^{(m)})$ is asymptotically equicoercive and (3.2)–(3.4) hold. Now, by (h_2), (h_3) and Theorem 2.11

$$\inf_{K\in\mathcal{K}^{(m)}_{s,p_h(x)}} \sup_K f_h = \inf_{K\in\mathcal{W}^{(m)}_{p_h(x)}} \sup_K f_h \quad \text{for all } h \in \mathbb{N}.$$

Furthermore, by virtue of (3.1), [13, Theorem 11.1] and [13, Proposition 11.6], also f is convex and positively homogeneous of degree 1, thus Theorem 2.11 applies also to f, namely

$$\inf_{K\in\mathcal{K}^{(m)}_{s,p(x)}}\sup_K f=\inf_{K\in\mathcal{W}^{(m)}_{p(x)}}\sup_K f.$$

Therefore, (3.5) follows immediately by (3.4). □

Remark 3.3. Variable exponent spaces and the underlying energy functionals with $p(x)$-growth, such as $u\mapsto\int_\Omega|\nabla u|dx$, can be considered as a particular case of Musielak–Orlicz spaces and non-autonomous energy functionals. Recently, significant progresses were achieved in the framework of regularity theory for minimizers of a class of *double phase* integrands of the Calculus of Variations, see [4–6,11,12] and the references therein. The model case is

$$u\mapsto\int_\Omega(|\nabla u|^p+a(x)|\nabla u|^q)dx,\quad q>p>1,\ a(\cdot)\geq 0,$$

and it can be studied in the class of Musielak–Orlicz spaces, with Orlicz-type norm

$$\|u\|_{L^{\mathcal{H}}}:=\inf\Big\{\gamma>0:\int_\Omega\mathcal{H}\Big(x,\frac{|u(x)|}{\gamma}\Big)dx\leq 1\Big\},\qquad \mathcal{H}(x,s):=s^p+a(x)s^q,\quad s\geq 0,\ x\in\Omega.$$

For a given topological index, such as the Krasnosel'skiĭ genus or the $\mathbb{Z}_2$-cohomological index, we plan to investigate in a forthcoming paper the basic properties of the first eigenvalue, the asymptotic growth, and the stability of the nonlinear eigenvalues $\lambda^{(m)}_{a,p,q}$ of the double phase variational eigenvalue problem arising in this setting.

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