CONCAVITY AND PERTURBED CONCAVITY FOR p-LAPLACE EQUATIONS

MARCO GALLO AND MARCO SQUASSINA

Abstract. In this paper we study convexity properties for quasilinear Lane-Emden-Fowler equations of the type

$$\begin{cases} -\Delta_p u = a(x) u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

when $\Omega \subset \mathbb{R}^N$ is a convex domain. In particular, in the subhomogeneous case $q \in [0, p-1]$, the solution u inherits concavity properties from a whenever assumed, while it is proved to be concave up to an error if a is near to a constant. More general cases are also taken into account, including a wider class of nonlinearities. These results generalize some contained in [88] and [116].

Additionally, some results for the singular case $q \in [-1, 0)$ and the superhomogeneous case $q > p-1$, $q \approx p-1$ are obtained. Some properties for the p-fractional Laplacian $(-\Delta)^s_p$, $s \in (0,1)$, $s \approx 1$, are shown as well.

We highlight that some results are new even in the semilinear case $p = 2$; in some of these cases, we deduce also uniqueness (and nondegeneracy) of the critical point of u.

Contents

2020 *Mathematics Subject Classification.* 26B25, 35B09, 26B25, 35B99, 35D30, 35E10, 35J60, 35J62, 35R11.
Key words and phrases. Concavity of solutions, Perturbation results, Quasilinear equation, p-Laplacian, Nonautonomous equations, Fractional equation, Uniqueness of critical point, Nondegenerate critical point.
Fundings. The first author is supported by INdAM-GNAMPA Project “Mancanza di regolarità e spazi non lisci: studio di autofunzioni e autovalori”, codice CUP E53C23001670001. The second author is member of INdAM-GNAMPA.

1. Introduction

1.1. Some background

Qualitative properties of solutions of PDEs are a classical topic, and often the features of the domain and of the nonlinearity are inherited by the solutions. Consider the equation

$$\begin{cases} -\Delta u = a(x)g(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$: when for example Ω is the ball and $a(x)$ is radially symmetric and decreasing, a seminal result by [55, Theorem 1'] (see also [42, 110] for the p-Laplacian) states that the solution u itself is radially symmetric. Several generalizations of [55] have been taken into account considering, for example, sets which are symmetric only in one direction. In this paper, we are interested in the case of convex domains Ω, with no a priori assumption of symmetry. We will mainly focus on the power case $g(u) = u^q$ (namely, Lane-Emden-Fowler equations), but more general cases will be taken into account.

Generally, for a Dirichlet problem one may expect that the solutions of (1.1) are concave: when $a(x)$ is constant and Ω is the ball, this is the case for the torsion problem $-\Delta u = 1$ (i.e. $q = 0$) where the solution is explicit; the result anyway holds also in more general domains, like deformations of ellipses [69, 119] (see also [39] for other nontrivial examples)[1]. Actually, the result keeps holding also for singular equations $q \leq -1$ (in arbitrary domains, see Theorem 1.1 below). On the other hand, when $q > 0$, the solution are seen to be never concave, no matter what the domain is [3, Remark 4.1]; this is essentially due to the fact that $g(0) = 0$.

For general convex sets the situation is worse even for $q = 0$: if Ω has some flat part (in particular, in triangles), u is never concave [87, Theorem 18 in Section 7] (see also [86, Section 11], [119]). What one can obtain is that some *transformation* of u is concave: in the case of the Laplacian, Ω a general convex domain and $a(x)$ constant, the story so far can be summarized in Theorem 1.1 below. To state it, we recall that, for a function $u > 0$, u is *α-concave*, $\alpha \in \mathbb{R} \cup \{\pm\infty\}$, if:

- u is constant, if $\alpha = +\infty$;
- $\frac{1}{\alpha}u^\alpha$ is concave, if $\alpha \in (-\infty, 0) \cup (0, +\infty)$;
- $\log(u)$ is concave, if $\alpha = 0$;
- u is quasiconcave (i.e. $u^{-1}([k, +\infty))$ are convex for any $k \in \mathbb{R}$, see also (2.1)), if $\alpha = -\infty$.

In particular, if $\alpha = -1$, u is said *harmonic concave*. When $\alpha \in \mathbb{R}$ we define similarly strict and strong α-concavity (see Definition 2.4 for strong concavity). Moreover, we recall that u α-concave implies u β-concave for every $\beta \leq \alpha$. Notice that the definition for $\alpha = 0$ is coherent with $\frac{1}{\alpha}(u^\alpha - 1) \to \log(u)$ as $\alpha \to 0$.

Here and after, by writing $\partial\Omega \in C^{k,\alpha}$ for some $k \in \mathbb{N}$, we will implicitly assume $\alpha \in (0, 1]$; moreover, $2^* := \frac{2N}{N-2}$ for $N \geq 3$ and $2^* := +\infty$ for $N = 2$, stands for the Sobolev critical exponent. See Remark 7.3 for the definition of ground state.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex. Let u be a positive solution of*

$$\begin{cases} -\Delta u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $q \in \mathbb{R}$ and $\lambda > 0$. We have the following assertions.

- *If $q \in (-\infty, 0)$ and $\partial\Omega \in C^{2,\alpha}$, then u (unique) is strictly $\frac{1-q}{2}$-concave, and strongly $\frac{1-q}{2}$-concave in every $\Omega' \Subset \Omega$.*

[1] If $-\Delta u = 1$ in Ω, and u is shown to be concave in $\Omega_k := \{u \geq k\}$, then $v_k := u - k$ solves $-\Delta v_k = 1$ in Ω_k with $v_k = 0$ on $\partial\Omega_k$, and it is concave. Actually, a *propagation from the boundary* argument [39, 86, 119], typical of equations with $g(0) > 0$, ensures that concavity on $\partial\Omega_k$ is enough to achieve concavity in the whole Ω_k.

- *If $q \in [0,1]$, then u (unique) is strictly $\frac{1-q}{2}$-concave, and strongly $\frac{1-q}{2}$-concave in every $\Omega' \Subset \Omega$. If in addition Ω is strongly convex with $\partial\Omega \in C^{2,\alpha}$, then u is strongly $\frac{1-q}{2}$-concave.*
- *If $q \in (1, 2^* - 1)$, then*
  - *if $N = 2$, then the (unique) ground state solution is strictly $\frac{1-q}{2}$-concave, and strongly $\frac{1-q}{2}$-concave in every $\Omega' \Subset \Omega$;*
  - *if Ω is strongly convex with $\partial\Omega \in C^{2,\alpha}$, then there exists a solution which is strongly $\frac{1-q}{2}$-concave.*

The nonstrict concavity part of Theorem 1.1 is a summary of several results: $q = 0$ [108], $q = 1$ [24], $q \in (0,1)$ [88], $q > 1$ [104], and $q < 0$ [15]. The results is optimal in the following sense: for $q \in [-1, 0)$ [15, pages 329-330] (see also Remark 1.2 below), if Ω is the ball, then u is not α concave for any $\alpha > \frac{1-q}{2}$. For $q = 0$ [88, Remark 4.2.3 and Theorem 6.2], [15, page 328] there exists Ω (subset of a cone) such that u is not α-concave for any $\alpha > \frac{1}{2}$. For $q = 1$ [87, Theorem 15 in Section 7], [15, pages 328-329], for each $\alpha > 0$ there exists a suitably narrow Ω_α such that u is not α-concave: in particular it suggests that, for every Ω sufficiently good (e.g. with a finite number of edges) u could be α-concave for some $\alpha = \alpha(\Omega) > 0$ sufficiently small; this seems an open question. Let us moreover recall that, even if $g \geq 1$ and it is smooth (namely, a local perturbation of the torsion problem $g = 1$) and Ω is smooth and symmetric, the solutions might not even be quasiconcave [3, 68]; additionally, even if Ω is starshaped and close to a convex set, then the level sets of the solution of the torsion problem need not to be even connected [59]. Finally, we also mention that assuming Ω convex in a single direction is not sufficient to get (quasi)concavity in that direction [124].

Anyway, if Ω is chosen good enough, one can recover better concavity properties: when Ω is a ball and $q = 0$, the torsional function u is such that $\sqrt{\|u\|_\infty - u}$ is concave (actually, this property – stronger than concavity – characterizes ellipsoids [69]), while if $q = 1$ the eigenfunction is α-concave for some $\alpha \in (\frac{1}{N}, 1)$ (for instance, $\alpha > \frac{\sqrt{3}+2}{4} \approx 0.93$ when $N = 2$, see [104] for some explicit estimate of α).

Remark 1.2. It is easy to see that, whenever $a \equiv 1$, g is not too singular and a Hopf boundary lemma holds, solutions $u \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ of (1.1) are never α-concave with $\alpha > 1$: indeed, a straightforward computation shows that

$$-\Delta u^\alpha = -\alpha(\alpha - 1) u^{\alpha-2} |\nabla u|^2 + \alpha u^{\alpha-1} g(u);$$

by staying close to $\partial\Omega$ we have $|\nabla u| \geq C > 0$ while we may assume $u(x) = \varepsilon$ with ε small. Thus

$$-\Delta u^\alpha \leq -\alpha(\alpha - 1)\varepsilon^{\alpha-2} C^2 + \alpha \varepsilon^{\alpha-1} g(\varepsilon);$$

If $\varepsilon g(\varepsilon) = o(1)$ as $\varepsilon \to 0$, we have thus $-\Delta u^\alpha < 0$ near the boundary (that is, u^α is not concave). This remark in particular applies to $g(t) = t^q$ with $q \geq -1$, coherent with the above statements (see also Propositions 6.12 and 6.13).

Strict concavity of Theorem 1.1 has been investigated by several authors [1, 118], [34, Theorem 4.4 and Corollary 4.6], [15, Corollary 3.6]: the proofs are mainly based on continuation arguments and the famous constant rank theorem [34, Theorem 1.1], which has been subsequently generalized by [91] (see also [16]). The argument runs as follow: considered for any $q \neq 1$ (resp. $q = 1$) $w := -\mathrm{sign}(1-q) u^{\frac{1-q}{2}}$ (resp. $w := -\log(u)$), we have that w solves

$$\Delta w = -\frac{1-q}{w}\left(\frac{1+q}{(1-q)^2}|\nabla w|^2 + \frac{1}{2}\right) \qquad (\text{resp. } \Delta w = |\nabla w|^2 + \lambda). \tag{1.2}$$

In each case we have that the right hand side is positive (on the image of w, see [15, Lemma 3.1] for $q < -1$) with convex inverse in w, thus by [91] the Hessian of w has constant rank: since there exist points with full rank – near the extremal point [12, Lemma at page 207], or near the boundary (see Proposition 4.6) whenever this is strongly convex – we obtain the global strict concavity (actually, the Hessian matrix has full rank everywhere in Ω); see also [23, page 80] for an alternative argument based on developable graphs. In [98, Lemmas 2.5, 3.6 and 4.8] evolutive arguments are employed in presence of smooth strictly convex domains, and a strong concavity is obtained for $q \geq 0$; we mention also [1, 107] where estimates on the curvature of the level sets are given. We highlight that a remarkable consequence of the strict convexity is the uniqueness and nondegeneracy of the critical point (namely, a maximum) of the solution.

In the nonautonomous case, [88, Theorem 4.1] provided the following result: if u is a positive solution of the Dirichlet problem

$$\begin{cases} -\Delta u = a(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and $a(x)$ is $\theta \geq 1$ concave, then u is $\frac{\theta}{1+2\theta}$-concave; we see that for $\theta \to +\infty$ we recover the torsion problem. We recall also the results by [24, Theorem 6.1] and [118, Appendix], where eigenfunctions of $-\Delta u = (\lambda - V(x))u$ are shown to be log-concave if the potential V is convex and nonnegative (actually the concavity is strong if $D^2 V > 0$); see also [60] for some discussions in the superlinear case.

The theorem in [88] is sharp in the following sense [88, Theorem 6.2]: there exists Ω (subset of an open cone) such that, for any $\theta \in [1, +\infty]$, there exists *a* θ-concave function a_θ such that the solution u is not α-concave for any $\alpha > \frac{\theta}{1+2\theta}$. Moreover, in [87, Theorem 16 in Section 7], it is shown that the condition $\theta \geq 1$ cannot be relaxed to $\theta > 0$; it remains open anyway to show if the threshold $\theta = 1$ is sharp.

When $a(x)$ has no convexity property, we cannot expect concavity for u. A quantitative version of the convexity principle has been developed in [29] (see also [4, Proposition 3.2]): in a particular case it states that, if u is a positive solution of

$$\begin{cases} -\Delta u = a(x) u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

$q \in [0, 1)$, then there exists a convex function $\bar{v}$ such that

$$\|u^{\frac{1-q}{2}} - \bar{v}\|_\infty \leq C_q \|\nabla a\|_\infty,$$

thus $u^{\frac{1-q}{2}}$ is close to a convex function if a is close to a constant. Here $C_q \to +\infty$ as $q \to 1$: this is essentially related to the fact that the transformed equation of the eigenfunction problem (1.2) has a nonlinearity which is not strictly monotonic in t (see Section 6.1 for more comments); in this paper we also propose a different, parabolic, approach to deal with the eigenfunction problem, see Section 7.4. The results in [4, 29] apply also to more general operators where classical regularity of solutions holds.

Let us mention that a similar result has been achieved in regards of radial symmetry: indeed in [40] the authors show (roughly speaking) that the solutions of (1.1) in a ball satisfy

$$|u(x) - u(y)| \leq C \operatorname{def}(a)^\alpha \quad \text{for each } x, y \in B_1, \ |x| = |y|$$

where $\alpha \in (0, 1]$ and $\operatorname{def}(a)$ is a quantity which measures how far a is from being radially symmetric and decreasing.

The techniques on which the previous results are based mainly involve regularity of solutions and maximum principles on the *concavity function* related to a function $v : \Omega \to \mathbb{R}$ (namely, a transformation of the solution):

$$\mathcal{C}_v(x, y, t) := tv(x) + (1-t)v(y) - v\big(tx + (1-t)y\big)$$

for $x, y \in \Omega$, $t \in [0, 1]$. It is clear that v is concave if and only if $\mathcal{C}_v \leq 0$. When the solutions are sufficiently regular, the abovementioned results have been generalized also to fully nonlinear frameworks [17].

Anyway, due to the regularity restrictions, these techniques cannot be directly applied to p-Laplace equations: a classical idea, thus, is to regularize the operator, apply the result and pass to the limit. This procedure requires at least two delicate steps: the first is the uniqueness of the solution, which is needed to discuss concavity properties of a *fixed* solution. The second ingredient is the form of the regularization: as a matter of fact, we need a regularization process which preserves the concavity structure of the original equation. This is what has been done by [116] for $p \in (1, +\infty)$ and $q = 0, 1$, then generalized to $q \in [0, 1]$ by [22] (and more general cases, see (1.3) below). Namely they obtain that solutions of $-\Delta_p u = u^q$ are $\frac{p-1-q}{p}$-concave; this power turns to be relevant also for regularity information in singular equations, see Theorem 6.8. See also [104, Section 3] and [81, Section 3] for explicit computations in the ball.

Regarding strict concavity, very few is known in the case of the p-Laplacian: indeed, it seems that a direct application of the constant rank theorem is not generally the case when $p \neq 2$; nevertheless, in [23] the authors show that the concavity of the solutions is strict when $\Omega \subset \mathbb{R}^2$ (actually strong far from the boundary and the critical point). The proof is delicate: to reach the goal, the authors show first that the solution has a single critical point, and exploit this information to apply a constant rank theorem out of the critical point. Summing up, what is known [22, 23, 116] in the power case is the following result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and convex. Let u be a weak positive solution of*

$$\begin{cases} -\Delta_p u = \lambda u^q & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

for some $q \in [0, p-1]$ and $\lambda > 0$. Then u is $\frac{p-1-q}{p}$-concave. If $N = 2$ and $\partial\Omega \in C^2$, then u is strictly $\frac{p-1-q}{p}$-concave, and strongly $\frac{p-1-q}{p}$-concave in any $\Omega' \Subset \Omega \setminus \{\bar{x}\}$, where $\bar{x}$ is the unique critical point of the solution.

A different approach, based on concave envelopes of viscosity solutions, can be found in [41] in the case of the eigenfunction (see also [5]).

Before presenting our results, let us recall briefly what happens when $g(u)$ is assumed general in (1.1) ($a(x)$ constant). A first result was given by [88, Theorem 3.3], who showed α-concavity of solutions under some assumptions on $g(t)$ (see (6.2)). A more natural transformation has been studied by [22] (see also [34, 80, 89]): namely, under some suitable assumptions on g (see Remark 5.1), the authors showed that, given a positive solution u of

$$\begin{cases} -\Delta_p u = g(u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

and set

$$\varphi(t) := \int_1^t (G(\tau))^{-\frac{1}{p}} d\tau,$$

G primitive of g, it results that $\varphi(u)$ is concave (actually strictly concave when $N = 2$, [23]). This transformation seems to be relevant also in other frameworks, for instance the boundary behaviour of solutions of singular equations (see Proposition 6.13). Moreover, this approach is quite effective in the study of concavity of quasilinear equations of the type $-\mathrm{div}(\alpha(u)\nabla u) + \frac{\alpha(u)}{2}|\nabla u|^2 = g(u)$: indeed, when $\alpha \not\equiv const$, here power concavity seems to be not the right choice (even if g is a power), while a transformation of the type $\varphi(t) := \int_1^t \left(\frac{\alpha(\tau)}{G(\tau)}\right)^{1/p} d\tau$, shaped on α (and g) turns to be successful. Finally, we highlight that we will use this abstract φ to obtain a result on power transformations for the power singular equation (see Section 6.2).

We refer to [3] for more recent references on the topic of concavity.

1.2. Main results

Aim of the paper is to generalize some of the previous results to the case $p \in (1, +\infty)$, $q \in [0, p-1]$ and a (possibly) nonconstant, with or without concavity assumptions on it. We will further propose a general scheme which can be applied to more general nonlinearities $g(u)$ (and even $f(x, u)$), which extends also the semilinear setting proposed in [22]. Additional results will be considered as well (including the cases $q \in [-1, 0)$ and $q > p-1$ near $p-1$, as well as fractional equations), briefly commented below but fully presented in Section 7. For the sake of clarity we focus in the introduction only to some of the results of the paper.

Consider the Lane-Emden-Fowler equation

$$\begin{cases} -\Delta_p u = a(x) u^q & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.4}$$

In all the paper by solution we will mean *weak solution*. We start by a result on the exact concavity of (a power of) u, when a is assumed concave as well.

Theorem 1.4 (Exact concavity)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$. Let $q \in [0, p-1]$, and assume $a \in C^{1,\alpha}_{loc}(\Omega)$, $\alpha \in (0, 1)$, and $a > 0$ on Ω, a θ-concave with $\theta \geq 1$. Then the solutions of (1.4) are $\frac{\theta(p-1-q)}{1+\theta p}$-concave. If a is constant (i.e. ∞-concave), then the solutions are $\frac{p-1-q}{p}$-concave.*

Notice that, $\frac{\theta(p-1-q)}{1+\theta p} \to \frac{p-1-q}{p}$ as $\theta \to +\infty$; moreover for $p = 2$ the result is coherent with [88, Theorem 4.1], while for $p \neq 2$ and $a \equiv const$ it is coherent with [22, 116].

Remark 1.5. We highlight that the $C^{1,\alpha}$ regularity of the boundary is exploited in Theorem 1.4 (and its corollaries) only to gain the uniqueness of the solution. According to [13, 83], if one focuses only to ground state solutions, or if $q = p-1$, this assumption can be relaxed.

An application of the previous result is given by the (possibly nonradial) Hardy-Hénon type equations; see also [98, Remark 2.3] for further examples. Notice that the classical 2-norm $|x|$ is not even quasiconcave (in any subset of $\mathbb{R}^N$): see [6] for symmetry breaking results in the ball, while [10] for symmetry results for singular powers of Hardy-Leray type.

Example 1.6 (Hardy-Hénon type equation)**.** Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$ and $q \in [0, p-1]$. Assume that u is a solution of one of the following problems, with the conditions

$$u > 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega.$$

- Assume $\Omega \subset B_R(0)$ for some $R > 0$, and let $\sigma \geq 1$, $\omega \in [0, 1]$. Consider u solution of
$$-\Delta_p u = (R^\sigma - |x|^\sigma)^\omega u^q \quad \text{in } \Omega.$$
- Assume Ω is a subset of the half-plane $\{x \in \mathbb{R}^N \mid \sum_{i=1}^k x_i > 0\}$, $k \in \{1, \dots, N\}$, and let $\omega \in [0, 1]$. Consider u solution of
$$-\Delta_p u = \left(\sum_{i=1}^k x_i \right)^\omega u^q \quad \text{in } \Omega.$$
In particular, if $\Omega \subset \{x \in \mathbb{R}^N \mid x_i > 0 \text{ for each } i\}$, and u is a solution of
$$-\Delta_p u = |x|_1^\omega u^q \quad \text{in } \Omega$$
where $|x|_1$ is the 1-norm in $\mathbb{R}^N$.

- Assume $\Omega \subset \{(x,y) \in \mathbb{R}^2 \mid x, y > 0\}$ and $\omega_1, \omega_2 \geq 0$ with $\omega := \omega_1 + \omega_2 \in [0,1]$. Consider u solution of
$$-\Delta_p u = x^{\omega_1} y^{\omega_2} u^q \quad \text{ in } \Omega.$$

Then u is $\frac{p-1-q}{\omega+p}$-concave.

We move now to perturbed concavity: by assuming a near to a concave function (say, for simplicity, constant), then a power of u is near to a concave function with a comparable error. We present two possible approaches and results: see Remark 1.11 for a comparison. We highlight that the estimates on which the following result is based have an interest on their own (see Theorem 3.3).

Theorem 1.7 (Perturbed concavity I)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$. Assume $a \in C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$. Let $a_\infty \in (0, +\infty)$ be a constant and u_∞ be a positive solution of*
$$\begin{cases} -\Delta_p u_\infty = a_\infty & \text{in } \Omega, \\ u_\infty = 0 & \text{on } \partial\Omega. \end{cases}$$
Then the solution $u \in C^{0,\beta}(\overline{\Omega})$, $\beta \in (0,1]$, of (1.4) with $q = 0$ satisfies
$$\|u^{\frac{p-1}{p}} - u_\infty^{\frac{p-1}{p}}\|_\infty \leq C \|a - a_\infty\|_\infty^\kappa \tag{1.5}$$
for some $C = C(p, \Omega, a_\infty, \|a\|_{C^{0,\alpha}(\Omega)}) > 0$ and some $\kappa = \kappa(p, N, \beta) \in (0,1)$.

When u_∞ is merely concave, (1.5) cannot give precise information on the exact concavity of u. If u_∞ is assumed *strictly* concave, then some information can be deduced on ε-uniform concavity (see Corollary 6.3). Nevertheless, when u_∞ is strongly concave, and the convergence is proved to be C^2, the concavity of u_∞ is inherited by the solution u, for $a(x)$ sufficiently close to a_∞. Namely, in the semilinear case we combine the above result with the information on the limiting problem given by Theorem 1.1.

Corollary 1.8. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded, strongly convex, with $\partial\Omega \in C^{2,\alpha}$, and let $p = 2$. Consider $(a_n)_n : \Omega \to \mathbb{R}$, and assume that, for some $a_\infty > 0$ constant*
$$a_n \to a_\infty \quad \text{in } C^{0,\alpha}(\Omega) \text{ as } n \to +\infty.$$
Then the positive solution u_n of
$$\begin{cases} -\Delta u_n = a_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.6}$$
is such that u_n is strongly $\frac{1}{2}$-concave for $n \geq n_0 \gg 0$. In particular, for these values of n, the level sets of u_n are strictly convex and u_n has a single (and nondegenerate) critical point in Ω.

We comment now a different concavity perturbation result. We recall, for some $a \in L^\infty(\Omega)$, the *oscillation* related to a
$$\operatorname{osc}(a) := \sup_\Omega a - \inf_\Omega a.$$
Recall moreover the inner parallel set to Ω, for any $\delta > 0$,
$$\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}.$$

Theorem 1.9 (Perturbed concavity II)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be bounded, strongly convex and with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$. Let $q \in [0, p-1)$, and assume $a \in C^{1,\alpha}_{loc}(\Omega) \cap C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, and $a > 0$ on Ω. Then the solution of (1.4) satisfies*
$$\mathcal{C}_{u^{\frac{p-1-q}{p}}} \leq C \operatorname{osc}(a) \quad \text{on } \overline{\Omega} \times \overline{\Omega} \times [0,1]$$

where $C = C(u, \delta, a, p, q) > 0$ is given by

$$C := C_{p,q}\Big(\frac{\|u\|_\infty}{\min_{\overline{\Omega}_\delta} u}\Big)^{\frac{p-1-q}{p}}\Big(2 + \frac{\mathrm{osc}(a)}{\min_{\overline{\Omega}_\delta} a}\Big)\frac{1}{\min_{\overline{\Omega}_\delta} a}$$

and $C_{p,q} := \left(\frac{p-1-q}{q+1}\right)^{1/p}$; here $\delta > 0$ (small) is suitably chosen.

Corollary 1.10. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be bounded, strongly convex and with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$. Let $q \in [0, p-1)$, and assume $a_\infty > 0$ constant and $(a_n)_n \subset C^{1,\alpha}_{loc}(\Omega) \cap C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, with $\|a_n - a_\infty\|_\infty \to 0$ as $n \to +\infty$. Then the corresponding solutions $(u_n)_n$ verify*

$$\mathcal{C}_{u_n^{\frac{p-1-q}{p}}} \leq O(\|a_n - a_\infty\|_\infty) \to 0 \quad \textit{as } n \to +\infty.$$

Remark 1.11. As a further consequence of Theorem 1.9, by Hyers-Ulam Theorem [72] we know that there exists a concave function $\bar{v}$ such that

$$\|u^{\frac{p-1-q}{p}} - \bar{v}\|_\infty \lesssim \mathrm{osc}(a). \tag{1.7}$$

When $q = 0$, we can compare this result with the one in Theorem 1.7: both the results tell us that $u^{\frac{p-1}{p}}$ is near to a concave function $\bar{v}$. Theorem 1.7 gives an exact information on who $\bar{v}$ is, and requires only $a \in C^{0,\alpha}(\Omega)$ (and no restriction on the sign) since no approximation argument (see Section 5) is required to obtain the result. Theorem 1.9, instead, obtains $\bar{v}$ via an abstract result, but on the other hand gives a more accurate rate of convergence for the error, that is $\sim \mathrm{osc}(a)$ (instead of $\sim \mathrm{osc}(a)^{\theta\frac{\min\{1,p-1\}}{p}}$); moreover, Theorem 1.9 is valid also for $q \neq 0$. The two results seem thus complementary.

Remark 1.12. We believe that the condition $a \in C^{1,\alpha}(\mathbb{R}^N)$ in Theorem 1.9 is merely technical (while $a \in L^\infty(\mathbb{R}^N)$ is crucial), due to the regularity issues of the approximation process. Being $a \in C^1(\Omega)$, anyway, we can rephrase the conclusion of Theorem 1.9 as

$$\mathcal{C}_{u^{\frac{p-1-q}{p}}} \leq C\|\nabla a\|_\infty \quad \text{ on } \overline{\Omega} \times \overline{\Omega} \times [0,1]$$

for some constant $C = C_{p,q}\mathrm{diam}(\Omega)\Big(\frac{\|u\|_\infty}{\min_{\overline{\Omega}_\delta} u}\Big)^{\frac{p-1-q}{p}}\Big(2 + \frac{\mathrm{diam}(\Omega)|\nabla a|_\infty}{\min_{\overline{\Omega}_\delta} a}\Big)\frac{1}{\min_{\overline{\Omega}_\delta} a} > 0$.

In the paper we show also several results which are mainly based on perturbation techniques: in particular, we treat the singular case $q \in [-1, 0)$ by an approximation process. Moreover, by exploiting some uniform convergence to the eigenfunction problem, we show that if $q > p-1$ is sufficiently close to $p-1$, or $s \in (0,1)$ is sufficiently close to 1, then the solutions of $-\Delta_p u_q = u_q^q$ and

$$(-\Delta)^s_p u_s = \lambda_s u_s^{p-1}$$

actually enjoy some weaker form of log-concavity far from the boundary. Some comments on the literature of these topics and precise statements are given in the corresponding Sections 6.2–7.3. We highlight that some of the convergences provided have an interest on their own, and could be exploited for several other applications.

Main difficulties: the proofs of the main Theorems 1.4 and 1.9 rely on a suitable approximation process; among other technical difficulties, differently from [22, 29, 88, 116], the co-presence of the approximating components (the "$\varepsilon > 0$ parts", see Section 5) and the spatial-depending components (essentially $a(x)$) requires some fine management of the nonlinearities in play, in order to achieve the desired concavity and perturbed concavity results. A singular multiplicative decomposition will be employed moreover to deal with the case $a(x)u^q$, $q \neq 0$. The particular choice of the additive decomposition of the nonlinearity will also allow to unify some results contained in [88] and [22] (treating, for example, sum of powers). Regarding the singular case, the idea is to approximate the nonlinearity with $g_n(t) := (t + \frac{1}{n})^q$ and pass to the limit the abstract result which holds for general g; this approach is indeed different from the one proposed in the semilinear case by [15].

Outline of the paper: in Section 2 we recall some properties on approximation of domains, on α-concave functions and concavity functions, together with a weaker notion of ε-uniform concavity. In Section 3 we provide some tools on the difference of two solutions in quasilinear frameworks: this topic is of independent interest, but it has some consequences related to concavity of functions. Then in Section 4 we present some general results regarding the maxima of concavity functions on the boundary or in the interior: some of the results are known, others are refinement of known results. In Section 5 we develop an approximation argument for nonregular equations, which is suitable for achieving concavity properties: the discussions is set in a general framework, and some abstract consequences are presented in Section 5.1; then we provide the proofs of the main Theorems in Section 6, together with some comments on perturbed concavity for eigenfunctions and the case of singular equations, where a second approximation argument is set in motion. We collect then in Section 7 several additional results in other frameworks (superhomogeneous, p large, fractional, parabolic), which would be interesting to develop further in the future. We conclude the paper with a collection in Appendix A of some (partially known) tools for the p-Laplacian.

Notations: we define $|x|^2 := |x|_2^2 := \sum_i x_i^2$, $|x|_1 := \sum_i |x_i|$, $\|u\|_p^p := \int_\Omega |u|^p$ for $p \in (1, +\infty)$, and $\|u\|_\infty := \mathrm{supess}_{x\in\Omega}|u(x)|$. Moreover $[u]_{C^{0,\beta}} := \sup_{x,y\in\Omega,\, x\neq y} \frac{|u(x)-u(y)|}{|x-y|^\beta}$ and $\|u\|_{C^{0,\beta}} := \|u\|_\infty + [u]_{C^{0,\beta}}$ for $\beta \in (0,1]$, while D^2u denotes the Hessian matrix. We define also $\mathrm{osc}(a) := \sup_\Omega a - \inf_\Omega a$, $d(x, \partial\Omega) := \inf_{y\in\partial\Omega} |x-y|$, $\mathrm{int}(\Omega)$ the interior of Ω and $\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}$; we say that $\Omega_n \to \Omega$ in Hausdorff distance if $\max\{\sup_{x\in\Omega} d(x, \Omega_n), \sup_{x\in\Omega_n} d(x, \Omega)\} \to 0$ as $n \to +\infty$. The normal vector to the boundary ν will be always assumed to pointing inward. Definitions of strong convex domains, (harmonic, joint) concavity functions, and of ε-uniform concavity are given in Section 2 (see also Section 4.2); definition of ground state is given in Remark 7.3, while uniform ellipticity of operators is defined in (4.9). Definition of solution and of α-concavity are given above (see also Section 2).

2. Preliminary properties and definitions of concavity

We start by recalling some known properties on convex sets (which are always Lipschitz [64, Corollary 1.2.2.3]). We recall that a set is *strictly convex* if $(x, y) \subset \Omega$ for each $x, y \in \overline{\Omega}$, while it is *strongly convex* if the principal curvatures are well defined and strictly positive (see also [8, Definition 3.1.2] for discussions on weak curvatures). Moreover, for a general C^2 convex set, the curvatures are always nonnegative [71, Corollary 2.1.28]; the inequality cannot generally be improved even for strictly convex sets (for example $\{y > x^4\}$). We recall that any Lipschitz domain satisfies the uniform (interior and exterior) cone condition [70, Proposition 2.4.4 and Theorem 2.4.7] (see also [64, Theorem 1.2.2]), every domain Ω with $\partial\Omega \in C^{1,1}$ satisfies the uniform (interior and exterior) sphere condition [99]. For any $\delta > 0$, we set

$$\Omega_\delta := \{x \in \Omega \mid d(x, \partial\Omega) > \delta\}$$

often called *inner parallel bodies* or (referring to $\partial\Omega_\delta$) *surfaces parallel to the boundary*. We recall that every convex domain satisfies the unique nearest point property [71, Theorem 2.1.30], while every domain Ω with $\partial\Omega \in C^2$ has a neighborhood $\Omega \setminus \Omega_\delta$ where this property holds [53].

We state some properties on Ω_δ and some approximations of convex sets; see also [71, Lemma 2.3.2], [11, Theorem 5.1], [51, Theorem 5.1] and [8, Theorem 3.2.1] for other relevant approximations. Recall that quasiconcavity of v is equivalent to require

$$v\big(\lambda x + (1-\lambda)y\big) \geq \max\big\{v(x), v(y)\big\} \tag{2.1}$$

for each $x, y \in \Omega$, $\lambda \in [0,1]$.

Proposition 2.1. *The following properties hold.*

(i) *Let $\Omega \subset \mathbb{R}^N$ be open, bounded and convex. Then there exists $(\Omega^k)_{k\in\mathbb{N}}$, $\Omega^k \subseteq \Omega$, strongly convex, with $\partial\Omega^k \in C^\infty$, and such that $\Omega^k \subset \Omega^{k+1}$, $\Omega = \bigcup_k \Omega^k$ and $\Omega^k \to \Omega$ in Hausdorff distance as $k \to +\infty$.*

(ii) *Let $\Omega \subset \mathbb{R}^N$ be open, bounded, and convex [resp. strictly convex]. Then the distance function $d(\cdot, \partial\Omega)$ is Lipschitz and concave [resp. also strictly quasiconcave].[2] As a consequence, for any $\delta > 0$, Ω_δ is convex [resp. strictly convex].*

(iii) *Let $\Omega \subset \mathbb{R}^N$ be open and bounded, and let $k \in \mathbb{N} \cup \{\infty\}$. If $\partial\Omega \in C^k$, $k \geq 2$ then $d(\cdot, \partial\Omega) \in C^k(\Omega \setminus \Omega_{\delta_0})$ for some δ_0;[3] this means that, for $\delta > 0$ sufficiently small, $\partial\Omega_\delta \in C^k$ (and Ω_δ can be seen as a smooth manifold). The same holds for $k = 1$ by assuming in addition that $\partial\Omega$ satisfies the unique nearest point property (e.g., Ω is convex) and for a.e. $\delta > 0$ small.[4] Similar statements holds also for $C^{k,\alpha}$.*

Proof. Being the properties well known, we give just a sketch of the proof.

For (i), the claim is given in [93, Corollary 6.3.10] by considering Ω^k as the preimages of a suitable smooth, strongly concave exhaustion function (see also [19, Theorem 2.3], and [100, Theorem 1.4] for regularized distance function arguments). We obtain thus that Ω^k are increasing and cover Ω, and basic properties imply the Hausdorff convergence (see e.g. [70, Section 2.2.3.2]).

For (ii), we have the claim by [71, Theorem 2.1.24], [46, Theorem 5.4]. We give some details on the strict quasiconcavity: consider $x_1, x_2 \in \overline{\Omega}$, $x_1 \neq x_2$, and set $d_i := d(x_i, \partial\Omega) \geq 0$, $\bar{x} := \frac{x_1+x_2}{2} \in \Omega$, and $\bar{d} := \frac{d_1+d_2}{2} \geq 0$. Fix now a direction $\xi \in S^{N-1}$, and set $y_i^\xi := x_i + d_i\xi \in \overline{\Omega}$, $\bar{y}^\xi := \frac{y_1+y_2}{2} = \bar{x} + \bar{d}\xi$. By convexity, we have $\bar{y}^\xi \in \overline{\Omega}$, thus varying ξ we obtain $B(\bar{x}, \bar{d}) \subset \overline{\Omega}$, which implies $d(\bar{x}, \partial\Omega) \geq \bar{d}$; this gives mid-concavity (and thus concavity). We now distinguish:

- $d_1 \neq d_2$: in this case $d(\bar{x}, \partial\Omega) \geq \bar{d} > \min\{d_1, d_2\}$.
- $d_1 = d_2$: in this case $y_1^\xi \neq y_2^\xi$ for each ξ (otherwise $x_1 - x_2 = (d_1 - d_2)\xi = 0$, impossible) thus by strict convexity $\bar{y}^\xi \in \Omega$, that is $B(\bar{x}, \bar{d}) \subset \Omega$, and hence $d(\bar{x}, \partial\Omega) > \bar{d} = \min\{d_1, d_2\}$.

In both cases we have

$$d\left(\tfrac{x_1+x_2}{2}, \partial\Omega\right) > \min\left\{d(x_1, \partial\Omega), d(x_2, \partial\Omega)\right\}$$

that is, strict mid-quasiconcavity, which implies strict quasiconcavity, and in particular strict convexity of the level sets [9, Corollary 3.36].

For (iii), when $k \geq 2$, we rely on [56, Lemma 14.16], while for $k = 1$ we rely on [53]. See also [92] for $C^{k,\alpha}$ conclusions ($k \geq 2$) and [46, Theorem 5.7] for $C^{1,1}_{loc}$ conclusions.

By the end of the proof of [56, Lemma 14.16] we also see that the distance is *proper*, that is for small $\delta < \delta_0$ we have that $d(x, \partial\Omega) < \delta$ implies $\nabla d(\cdot, \partial\Omega) \neq 0$; when $k = 1$ we rely instead on Sard's lemma. By exploiting the implicit function theorem (as in the proof of [11, Theorem 5.1], see end of page 13 therein), we achieve that, for such δ, $\partial\Omega_\delta \in C^k$; see also the preimage theorem [109, Corollary in Section 1.2.2, page 37] in combination with [64, Theorem 1.2.1.5] or [93, Section 1.2]. For Lipschitz boundaries see [51, Theorem 4.1]. □

We recall some miscellanea results by [88, Properties 7 and 8, Lemma A.2] and [89, Lemma 3.2] (see also [103, Theorems 1 and 3, Corollary 2]).

---

[2] We observe that the distance function is not strictly concave (even if Ω is strictly convex): for instance consider x_1, x_2 sufficiently close to $\partial\Omega$, and such that both the points lie on the same perpendicular line to the boundary (namely, if $d(x_i, \partial\Omega) = |x_i - x_i^*|$ then $x_1^* = x_2^* =: x^*$); then clearly $\bar{x} = \frac{x_1+x_2}{2}$ is such that $d(\bar{x}, \partial\Omega) = |\bar{x} - x^*| = \frac{d(x_1,\partial\Omega)+d(x_2,\partial\Omega)}{2}$.

[3] We notice that a viceversa does not hold: a set satisfying $d(\cdot, \partial\Omega) \in C^k(\Omega \setminus \Omega_{\delta_0})$ is also known as *proximally* C^k.

[4] The size δ depends on the radius of the uniform interior sphere, or the *positive reach* of M, that is the size of the neighborhood where the unique nearest point property holds. The set of singular points of $d(\cdot, \partial\Omega)$ is also called the *ridge* of Ω.

We say that a function $h : \Lambda \subset \mathbb{R}^N \to \mathbb{R}$ (not necessarily positive) is *harmonic concave* if, for each $z_1, z_2 \in \Lambda$, $\lambda \in [0,1]$,

$$h\big(\lambda z_1 + (1-\lambda)z_2\big) \geq \begin{cases} \frac{h(z_1)h(z_2)}{\lambda h(z_2)+(1-\lambda)h(z_1)} & \text{if } \lambda h(z_2) + (1-\lambda)h(z_1) > 0, \\ 0 & \text{if } h(z_1) = h(z_2) = 0 \ . \end{cases}$$

If h is positive, this definition coincides with (-1)-concavity, see [88].

Proposition 2.2. *We have the following properties.*

- *Let $h > 0$ be α-concave, and $k > 0$ be β-concave, with $\alpha, \beta \in [0, +\infty]$. Then hk is γ-concave, where $\frac{1}{\gamma} = \frac{1}{\alpha} + \frac{1}{\beta}$.*
- *Let $\Omega \subset \mathbb{R}^N$ be convex and $I \subset \mathbb{R}$ be an interval, and let $h = h(t,x)$ be positive and such that $(x,t) \in \Omega \times I \mapsto t^2 h(x,t)$ is (jointly) concave. Then h is (jointly) harmonic concave.*
- *If h, k are α-concave functions for $\alpha \geq 1$, then $h + k$ is α-concave. If h is harmonic concave, then $h - k$ is harmonic concave for every constant $k \geq 0$.*
- *Let $h = h(x)$ be α-concave, then $(t,x) \mapsto h(x)$ is α (jointly) concave; similarly if $h = h(t)$.*

We consider now the *concavity function*, for a $v : \Omega \to \mathbb{R}$,

$$\mathcal{C}_v(x,y,\lambda) := \lambda v(x) + (1-\lambda)v(y) - v\big(\lambda x + (1-\lambda)y\big),$$

the *joint concavity function*, for a $h = h(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$, as

$$\mathcal{JC}_h\big((x,t),(y,s),\lambda\big) := \lambda h(x,t) + (1-\lambda)h(y,s) - h\big(\lambda x + (1-\lambda)y, \lambda t + (1-\lambda)s\big),$$

and, when $h > 0$, the *(jointly) harmonic concavity function* as

$$\mathcal{HC}_h\big((x,t),(y,s),\lambda\big) := \frac{h(x,t)h(y,s)}{\lambda h(y,s) + (1-\lambda)h(x,t)} - h\big(\lambda x + (1-\lambda)y, \lambda t + (1-\lambda)s\big).$$

Notice that (see [29])

$$\mathcal{HC} \leq \mathcal{JC}.$$

We recall some relations on $\mathcal{HC}$ which allows to deal with basic operations [29, Lemma A.1].

Proposition 2.3 ([29])**.** *Let f, g such that $f, g > 0$. Then, for any $x, y \in \Omega$, $t, s \in \mathbb{R}$, $\lambda \in [0,1]$,*

$$\mathcal{HC}_{f+g}\big((x,t),(y,s),\lambda\big) \geq \mathcal{HC}_f\big((x,t),(y,s),\lambda\big) + \mathcal{HC}_g\big((x,t),(y,s),\lambda\big)$$

and, if moreover $f - g > 0$,

$$\mathcal{HC}_{f-g}\big((x,t),(y,s),\lambda\big) \leq \mathcal{HC}_f\big((x,t),(y,s),\lambda\big) - \mathcal{HC}_g\big((x,t),(y,s),\lambda\big).$$

We finally recall some definitions of generalized concavity and some related properties (see also Proposition 7.4 for quasiconcavity).

Definition 2.4. We say that $v : \Omega \to \mathbb{R}$ is ε-*uniformly concave* with (continuous) modulus $\rho : (0, +\infty) \to (0, +\infty)$ if

$$v\left(\frac{x+y}{2}\right) \geq \frac{v(x) + v(y)}{2} + \rho(|x-y|) \qquad \text{for each } x, y \in \Omega,\ |x-y| \geq \varepsilon,$$

namely $\mathcal{C}_v(x,y,\frac{1}{2}) \leq -\rho(|x-y|)$ for $|x-y| \geq \varepsilon$. We say that v is ε-*strongly concave* with parameters $m > 0$ if $\rho(t) = \frac{1}{8}mt^2$.[5] If $\rho \equiv$ *const* we simply say that v is ε-uniformly concave.

[5] The "λ-version" actually reads as: $v(\lambda x + (1-\lambda)y) \geq \lambda v(x) + (1-\lambda)v(y) + \frac{1}{2}\lambda(1-\lambda)m|x-y|_2^2$.

We highlight that an ε-uniform concave function is *not* necessarily concave [63, Example 2.5]; anyway this class of functions enjoy several properties, see [63]. Clearly uniform concavity (i.e. 0-uniform concavity) implies both ε-uniform concavity (for each $\varepsilon > 0$) and concavity (by assuming v continuous). Similarly, strong concavity – i.e. $v - \frac{m}{2}|x|_2^2$ is concave – implies both ε-strong concavity and strict concavity. We further recall that when $v \in C^2(\Omega)$, strong concavity is equivalent to say that $D^2 v \succcurlyeq mI$, that is, $D^2 v - mI$ is positive semi-definite (i.e., the eigenvalues of $D^2 v$ lie in $[m, +\infty)$).

We are interested in properties inherited by a sequence of functions from their limit: we observe that, if $v_n \to v$ in $C^k(\Omega)$ and u is strictly convex, then it is not ensured that v_n is strictly convex for n large. Indeed, consider $v_n(x) := \begin{cases} x^{2k} & \text{for } x \notin [-\frac{1}{n}, \frac{1}{n}], \\ -x^{2k} + \frac{2}{n^{2k}} & \text{for } x \in [-\frac{1}{n}, \frac{1}{n}], \end{cases}$ and $v(x) := x^{2k}$; clearly (by substituting v_n with a smooth mollification) $v_n \to v$ in $C^{2k-1}([-1,1])$ and v is strictly convex, but v_n are not even convex. With the same example, we see that for $k = 1$ $v_n \to v$ in $C^1([-1,1])$ (but not in $C^2([-1,1])$) and $v(x) = x^2$ is strongly concave; thus C^1 convergence is not sufficient to inherit convexity from strong convexity.

On the other hand, by looking at the Hessian matrix, it is clear the following result.

Proposition 2.5. *Assume $v_n : \Omega \to \mathbb{R}$ converge in $C^2(\Omega)$ to v, which is assumed to be strongly concave on Ω. Then v_n is strongly concave for n sufficiently large.*

A weaker property, anyway, is inherited by the sequence also in case of L^∞-convergence, when v is assumed strictly concave.

Proposition 2.6. *We have the following properties.*

- *Assume Ω bounded and $v : \overline{\Omega} \to \mathbb{R}$ strictly concave, then v is uniformly concave (i.e., ε-uniformly concave for each $\varepsilon > 0$).*
- *Assume $v_n : \Omega \to \mathbb{R}$ converge in $L^\infty(\Omega)$ to v, which is assumed uniformly [resp. strongly] concave in $\overline{\Omega}$. Then, for each $\varepsilon > 0$, there exists $n_\varepsilon \gg 0$ such that, for each $n \geq n_\varepsilon$, u is ε-uniformly [resp. ε-strongly] concave.*

Proof. Set $\Lambda_\varepsilon := \big\{x, y \in \overline{\Omega},\ |x - y| \geq \varepsilon\big\}$, the first claim comes straightforwardly by considering $\rho := -\inf_{\Lambda_\varepsilon} \mathcal{C}_v(x, y, \frac{1}{2}) > 0$. Consider the second point. We have

$$\inf_{\Lambda_\varepsilon} \left(\mathcal{C}_v\left(x, y, \tfrac{1}{2}\right) + \tfrac{1}{2}\rho(|x - y|)\right) \leq -\tfrac{1}{2} \inf_{\Lambda_\varepsilon} \rho < 0.$$

Since $\mathcal{C}_{v_n}\left(x, y, \frac{1}{2}\right) \to \mathcal{C}_v\left(x, y, \frac{1}{2}\right)$ uniformly on Λ_ε, we have

$$\mathcal{C}_{v_n}\left(x, y, \tfrac{1}{2}\right) + \tfrac{1}{2}\rho(|x - y|) < -\tfrac{1}{4} \inf_{\Lambda_\varepsilon} \rho < 0$$

on Λ_ε, for n large. This gives the claim. □

We conclude this Section with some comments on a different notion of concavity.

Remark 2.7. A generalization of the concept of concavity was recently discussed in [18]: given L an operator on functions in the interval $(0,1)$, we say that u is *L-concave* if

$$u(tx + (1 - t)y) \geq \omega_{L,u}^{x,y}(t)$$

for each $x, y \in \Omega$ and $t \in [0,1]$, where

$$\begin{cases} L[\omega_{L,u}^{x,y}](t) = 0 & \text{in } (0,1), \\ \omega_{L,u}^{x,y}(0) = u(y), \quad \omega_{L,u}^{x,y}(1) = u(x) \end{cases}$$

in the viscosity sense (see also [71, Section VI]). It is clear that, if $L(\omega) = \omega''$ – the monodimensional counterpart of the Laplacian – then we recover the classical concavity. It is moreover straightforward to check that also with $L(\omega) = (|\omega'|^{p-2}\omega')'$ – the p-Laplacian in one dimension – we still recover the classical concavity.

We highlight that the α-concavity, in the sense that u^α is concave, is essentially equivalent to the L^α-concavity, where $L^\alpha(\omega) := \omega''\omega - (1-\alpha)(\omega')^2$ (or equivalently, the operator $\tilde{L}^\alpha := (|\omega|^{\alpha-1}\omega')'$, a quasilinear nonvariational operator). Here, as it is well known, L^α builds a *bridge* between concavity ($\alpha = 1$) and quasiconcavity ($\alpha \to -\infty$).

In [18], instead, the authors propose a different bridge, given by $L_\beta(\omega) := \beta\omega'' + (1-\beta)(\omega')^2$, for $\beta \in [0,1]$: here they recover concavity for $\beta = 1$ and quasiconcavity for $\beta = 0$. This class enjoys several properties: moreover, saying that a function is L_β-concave is essentially equivalent to saying that $\varphi_\beta(u) := -e^{-\frac{1-\beta}{\beta}u}$ is concave, and this transformation has a great relevance in financial mathematics [9, Chapter 8], [103]. For $\alpha \in [-\infty, 0]$ it is straightforward to check that

$$u \text{ is } L^\alpha\text{-concave} \iff \log(u) \text{ is } L_{\frac{1}{1-\alpha}}\text{-concave}.$$

Key differences between φ_β and the classical transformations, in the framework of Dirichlet problems, is given by the facts that: $\varphi'_\beta(t) \not\to +\infty$ as $t \to 0$ (see also Remark 4.8), and $t \mapsto \frac{\psi''_\beta(t)}{\psi'_\beta(t)} = -\frac{1}{t}$ is not nonincreasing ($\psi_\beta = \varphi_\beta^{-1}$), which are both of key importance. Notice moreover that power concavity is essentially the only class of transformations which are closed under scalar multiplication [103, Theorem 2].

We provide also an explicit example where φ_β does not work, since edges create problems: the solution of the torsion problem $-\Delta u = 1$ in the triangle of vertices $(\pm\sqrt{3}, 0)$, $(0,1)$ is given by $u(x,y) = \frac{y}{4}(1-y)^2 - 3x^2$; it is known that $\sqrt{u}$ is concave but u^α is not concave for any $\alpha > \frac{1}{2}$ [15, page 328]. We wonder if, for some large $r > 0$, $1 - e^{-ru}$ is concave: but it is straightforward to see that $y \mapsto 1 - e^{-ru(y,0)}$ is concave only if $r > \frac{8(3y-2)}{(1-y)^2(3y-2)^2}$, which is clearly impossible as $y \to 1$.

We suspect anyway that, in regular domains and for some class of problems, φ_β could work as a good transformation in Dirichlet problems.

3. Estimates on the difference of two solutions

In this Section we want to compare the solutions of two problems, where the space component may act differently; this comparison will in particular lead to the proof of Theorem 1.7 and its generalizations. Since the problem is quasilinear, we cannot directly work on a problem solved by the difference of the two solutions. We start by some considerations on general functions. We recall the following result by [96, Theorems 1.1 and 1.2].

Lemma 3.1 ([96]). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open and with the uniform interior cone condition. Consider a function $u \in C^{0,\beta}(\overline{\Omega}) \cap L^q(\Omega)$, $\beta \in (0,1]$ and $q \in [1,+\infty)$. Then*

$$\|u\|_\infty \leq C\|u\|_{C^{0,\beta}}^{1-\kappa_q}\|u\|_q^{\kappa_q} \tag{3.1}$$

where $\kappa_q = \kappa_{N,\beta,q} := \frac{\beta}{\beta+\frac{N}{q}} \in (0,1)$ and $C = C_{N,\beta,q} > 0$ is given as follows:

- *if $\Omega = \mathbb{R}^N$, then $C_{N,\beta,q} = \max\{\frac{N}{N+\beta}, \omega_N^{-\frac{1}{q}}\} > 0$ (here ω_N is the volume of the unit ball);*
- *if $\Omega \neq \mathbb{R}^N$, let r_0 be the radius of the (uniform) interior cone; then $C_{N,\beta,q} = \max\{\frac{N}{N+\beta}, \omega_K^{-\frac{1}{q}}, r_0^\beta\}$ (here ω_K is the volume of the cone scaled to radius 1).*[6]

As a consequence, if $u \in C^{k,\beta}(\overline{\Omega}) \cap W^{k,q}(\Omega)$, $k \in \mathbb{N}$, $\beta \in (0,1]$ and $q \in [1,+\infty)$, then

$$\|u\|_{C^k} \leq C\|u\|_{C^{k,\beta}}^{1-\kappa_q}\|u\|_{W^{k,q}}^{\kappa_q}.$$

As a corollary of the Lemma we gain a preliminary estimate on the difference of two general functions. We write, for $p \in (1,+\infty)$, $\beta \in (0,1]$,

$$\kappa_{p^*} = \frac{\beta}{\beta+\frac{N}{p^*}} = \frac{p\beta}{p\beta+N-p} \in (0,1), \quad \kappa_{2^*} = \frac{\beta}{\beta+\frac{N}{2^*}} = \frac{2\beta}{2\beta+N-2} \in (0,1).$$

[6] Notice that $\kappa_{N,\beta,q} \to 1$ and $C_{N,\beta,q} \to \max\{\frac{N}{N+\beta}, r_0^\beta\} > 0$ as $q \to +\infty$.

Corollary 3.2. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be open and with the uniform interior cone condition, and let $p \in (1,+\infty)$. Let $u, v \in C^{0,\beta}(\overline{\Omega}) \cap W^{1,p}(\Omega)$, $\beta \in (0,1]$, then*

$$\|u-v\|_\infty \leq C\left(\|u\|_{C^{0,\beta}} + \|v\|_{C^{0,\beta}}\right)^{1-\kappa_{p^*}} \|\nabla u - \nabla v\|_p^{\kappa_{p^*}} \tag{3.2}$$

with $C = C(N,\beta,p) > 0$. Similarly, if $u, v \in C^{0,\beta}(\overline{\Omega})$ with $u - v \in D^{1,2}(\Omega)$, then

$$\|u-v\|_\infty \leq C\left(\|u\|_{C^{0,\beta}} + \|v\|_{C^{0,\beta}}\right)^{1-\kappa_{2^*}} \|\nabla u - \nabla v\|_2^{\kappa_{2^*}} \tag{3.3}$$

with $C = C(N,\beta) > 0$.

We want to prove Theorem 1.7. We discuss here a more general case.

Theorem 3.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open and with the uniform interior cone condition, and let $p \in (1,+\infty)$. Consider $a_1, a_2 \in L^q(\Omega)$, $q \in (1,+\infty]$, and the problems*

$$\begin{cases} -\Delta_p u_1 = a_1(x) & \text{in } \Omega, \\ u_1 = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p u_2 = a_2(x) & \text{in } \Omega, \\ u_2 = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.4}$$

with positive solutions $u_1, u_2 \in C^{0,\beta}(\overline{\Omega})$ for some $\beta \in (0,1]$.

- *If $p \geq 2$ assume $q \geq \frac{p}{p-2}$. Then*
$$\|u_1 - u_2\|_\infty \leq C\|a_1 - a_2\|_q^{\frac{\kappa_{p^*}}{p-1}}$$
where $C = C(p,q,\Omega,\|u_1\|_{C^{0,\beta}(\Omega)}, \|u_2\|_{C^{0,\beta}(\Omega)}) > 0$.
- *If $p \leq 2$ assume $q \geq 2$ and $u_1, u_2 \in W^{1,\infty}(\Omega)$. Then*
$$\|u_1 - u_2\|_\infty \leq C\|a_1 - a_2\|_q^{\kappa_{2^*}}$$
where $C = C(p,q,\Omega,\|u_1\|_{C^{0,\beta}(\Omega)}, \|u_2\|_{C^{0,\beta}(\Omega)}, \|\nabla u_1\|_\infty, \|\nabla u_2\|_\infty) > 0$.

Proof. By definition of weak solution we have, for any $\varphi \in W_0^{1,p}(\Omega)$,

$$\int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot \nabla\varphi = \int_\Omega (a_1(x) - a_2(x))\varphi;$$

we choose $\varphi = u_1 - u_2 \in W_0^{1,p}(\Omega)$, so that

$$\int_\Omega \left(|\nabla u_1|^{p-2}\nabla u_1 - |\nabla u_2|^{p-2}\nabla u_2\right) \cdot (\nabla u_1 - \nabla u_2) \leq \|a_1 - a_2\|_q \|u_1 - u_2\|_r$$

for $\frac{1}{q} + \frac{1}{r} = 1$, $r \leq \max\{p, 2\}$. Let us now distinguish two cases.

If $p \geq 2$, by [105, Section 12(I)], Hölder and Poincaré inequalities we have

$$2^{2-p}\|\nabla u_1 - \nabla u_2\|_p^p \leq \|a_1 - a_2\|_q \|u_1 - u_2\|_r \leq |\Omega|^{\frac{p-r}{pr}} C_{\Omega,p} \|a_1 - a_2\|_q \|\nabla u_1 - \nabla u_2\|_p;$$

here $C_{\Omega,p}$ is the best Poincaré constant on $W_0^{1,p}(\Omega)$. Thus

$$\|\nabla u_1 - \nabla u_2\|_p \leq \left(2^{2-p}|\Omega|^{\frac{p-r}{pr}} C_{\Omega,p}\right)^{\frac{1}{p-1}} \|a_1 - a_2\|_q^{\frac{1}{p-1}}.$$

Set $C_1 := \|u_1\|_{C^{0,\beta}(\Omega)} + \|u_2\|_{C^{0,\beta}(\Omega)}$ by (3.2) we obtain

$$\|u_1 - u_2\|_\infty \lesssim C_1^{1-\kappa_{p^*}} \|a_1 - a_2\|_q^{\frac{\kappa_{p^*}}{p-1}}.$$

If $1 < p \leq 2$ then, by [105, Section 12(VII)] we have

$$(p-1)\int_\Omega \frac{|\nabla u_1 - \nabla u_2|^2}{\left(1 + |\nabla u_1|^2 + |\nabla u_2|^2\right)^{\frac{2-p}{2}}} \leq \|a_1 - a_2\|_q \|u_1 - u_2\|_r \leq |\Omega|^{\frac{2-r}{2r}} C_{\Omega,2} \|a_1 - a_2\|_q \|\nabla u_1 - \nabla u_2\|_2.$$

Set $C_2 := 1 + \|\nabla u_1\|_\infty + \|\nabla u_2\|_\infty$ we have

$$C_2^{-\frac{2-p}{2}}(p-1)\|\nabla u_1 - \nabla u_2\|_2^2 \leq |\Omega|^{\frac{2-r}{2r}} C_{\Omega,2}\|a_1 - a_2\|_q \|\nabla u_1 - \nabla u_2\|_2$$

thus

$$\|\nabla u_1 - \nabla u_2\|_2 \leq \left(\tfrac{1}{p-1} C_2^{\frac{2-p}{2}} |\Omega|^{\frac{2-r}{2r}} C_{\Omega,2}\right) \|a_1 - a_2\|_q.$$

Therefore, exploiting (3.3),

$$\|u_1 - u_2\|_\infty \lesssim C_1^{1-\kappa_{2^*}} C_2^{\frac{2-p}{2}} \|a_1 - a_2\|_q^{\kappa_{2^*}}.$$

□

Corollary 3.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open and with the uniform interior cone condition, and let $p \in (1, +\infty)$. Let $a_1, a_2 \in L^\infty(\Omega)$ and $u_1, u_2 \in L^\infty(\Omega)$ solutions of (3.4). Then*

- *Let $p \geq 2$, and assume Ω satisfies the uniform (interior and exterior) cone condition. Then*

$$\|u_1 - u_2\|_\infty \leq C\|a_1 - a_2\|_\infty^{\frac{\kappa_{p^*}}{p-1}}$$

  where $C = C(p, \Omega, \|a_1\|_\infty, \|a_2\|_\infty) > 0$.
- *Let $p \leq 2$ and assume $a_i \in C^{0,\alpha}(\Omega)$, $\partial\Omega \in C^{1,\alpha}$. Then*

$$\|u_1 - u_2\|_\infty \leq C\|a_1 - a_2\|_\infty^{\kappa_{2^*}}$$

  where $C = C(p, \Omega, \|a_1\|_{C^{0,\alpha}}, \|a_2\|_{C^{0,\alpha}}) > 0$.

Proof. By [30, Theorem 2.3] (see also [121, Corollary 4.2]) we have the $C^{0,\beta}(\overline{\Omega})$ regularity, while by [101, Theorem 1] (see also [66, Corollary 1.1] and [95, Section 4.3, Theorem 5.2]) we have the $C^{1,\beta}(\overline{\Omega})$ (and thus $W^{1,\infty}$) regularity. Hence u satisfies the assumptions of Theorem 3.3 and we achieve the claim. □

Remark 3.5. We observe that, as a first consequence of Corollary 3.4, we get the uniqueness result of Lemma A.1 in the specific case $f(x,t) \equiv a(x)$, and Corollary A.3 in the specific case $f_n(x,t) \equiv a_n(x) \geq a_0$.

We deal now with concavity properties that can be deduced by this perturbation result.

Corollary 3.6. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, and let $p \in (1, +\infty)$. Let $a_1, a_2 \in L^\infty(\Omega)$ and u_1, u_2 as in Corollary 3.4. Assume $a_2 > 0$ to be θ-concave, $\theta \geq 1$. Then u_2 is $\frac{\theta(p-1)}{1+\theta p}$-concave and the following holds.*

- *Let $p \geq 2$. Then*

$$\|u_1^{\frac{\theta(p-1)}{1+\theta p}} - u_2^{\frac{\theta(p-1)}{1+\theta p}}\|_\infty \leq C\|a_1 - a_2\|_\infty^{\kappa_{2^*}\frac{\theta(p-1)}{1+\theta p}}$$

  where $C = C(p, \Omega, \|a_1\|_\infty, \|a_2\|_\infty) > 0$.
- *Let $p \leq 2$ and assume $a_i \in C^{0,\alpha}(\Omega)$, $\partial\Omega \in C^{1,\alpha}$. Then*

$$\|u_1^{\frac{\theta(p-1)}{1+\theta p}} - u_2^{\frac{\theta(p-1)}{1+\theta p}}\|_\infty \leq C\|a_1 - a_2\|_\infty^{\kappa_{2^*}\frac{\theta(p-1)}{1+\theta p}}$$

  where $C = C(p, \Omega, \|a_1\|_{C^{0,\alpha}}, \|a_2\|_{C^{0,\alpha}}) > 0$.

The same results apply for $\theta = \infty$ by substituting $\frac{\theta(p-1)}{1+\theta p}$ with $\frac{p-1}{p}$.

Proof. The claim comes from Corollary 3.4, Theorem 1.4 and the fact that, for $\gamma \in (0,1)$,

$$\|u_1^\gamma - u_2^\gamma\|_\infty \le \|u_1 - u_2\|_\infty^\gamma. \qquad \square$$

Proof of Theorem 1.7. Arguing e.g. as in Section 5, the solutions are bounded. The result is thus a consequence of Corollary 3.6, by considering a_2 constant. $\square$

We postpone the proof of Corollary 1.8 and its generalizations to Section 6.

Remark 3.7. We see that Corollary 3.4 can be applied to deduce some information on log-concavity. Indeed, as in the proof of Lemma A.1, we first have $\frac{u_1}{u_2}, \frac{u_2}{u_1} \in L^\infty(\Omega)$. Thus, by the mean value theorem, there exists $\lambda^* \in (0,1)$ such that

$$\begin{aligned}|\log(u_1) - \log(u_2)| &= \tfrac{1}{2}\left|\log\left(\tfrac{u_1}{u_2}\right) - \log\left(\tfrac{u_2}{u_1}\right)\right| = \frac{1}{2(\lambda^* \frac{u_1}{u_2} + (1-\lambda^*)\frac{u_2}{u_1})}\left|\tfrac{u_1}{u_2} - \tfrac{u_2}{u_1}\right| \\ &\le C\left|\tfrac{u_1}{u_2} + \tfrac{u_2}{u_1}\right| |u_1 - u_2| \le C'|u_1 - u_2|.\end{aligned}$$

Hence by Corollary 3.4, we have

$$\|\log(u_1) - \log(u_2)\|_\infty \le C\|a_1 - a_2\|_\infty^{\frac{\kappa_{p^*}}{p-1}}$$

if $p \ge 2$, while

$$\|\log(u_1) - \log(u_2)\|_\infty \le C\|a_1 - a_2\|_\infty^{\kappa_{2^*}}$$

if $p \le 2$. Here $C = C(p, \Omega, \|a_1\|_{C^{0,\alpha}}, \|a_2\|_{C^{0,\alpha}}, \|\frac{u_1}{u_2}\|_\infty, \|\frac{u_2}{u_1}\|_\infty) > 0$.

We notice that, clearly, $\log(u_1), \log(u_2) \notin L^\infty(\Omega)$, but the difference does so. On the other hand, we have, for each $\delta > 0$, $\log(u_1), \log(u_2) \in L^\infty(\Omega_\delta)$, thus, if $a_n \to a_0$ uniformly, then

$$\log(u_n) \to \log(u_0) \quad \text{in } L^\infty(\Omega_\delta);$$

if for example one is able to deduce that $\log(u_0)$ is strictly or strongly concave in Ω_δ, then some concavity information on $\mathcal{C}_{\log(u_n)}$ for n large can be deduced as well, see Propositions 2.5 and 2.6.

4. General lemmas about concavity

4.1. Concavity on the boundary

In this Section we discuss the possibility of the concavity function $\mathcal{C}_v$ to have a maximum on the boundary of its domain. Here we do not use the equation, but only the information on the boundary; see anyway [15, Lemma 3.3] where also the equation is exploited (in a singular framework).

We start by some result which deals precisely with the boundary of (a general) Ω, but gives no good quantitative information to be inherited by functions approximating the solution.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 1$, be open, bounded and convex, and let $v : \overline{\Omega} \to \mathbb{R}$. Let $(\bar{x}, \bar{y}, \bar{\lambda}) \in \overline{\Omega} \times \partial\Omega \in [0,1]$ be a global maximum of $\mathcal{C}_{v|\overline{\Omega}\times\overline{\Omega}\times[0,1]}$. Assume*

- *if $[\bar{x}, \bar{y}] \subset \partial\Omega$ then*
$$v = const \quad on\ [\bar{x}, \bar{y}]; \tag{4.1}$$
- *if $[\bar{x}, \bar{y}] \not\subset \partial\Omega$ then*
$$\mathcal{C}_v(\bar{x}, \bar{y}, \lambda) < 0 \quad for\ \lambda \approx 0^+. \tag{4.2}$$

If $(\bar{x}, \bar{y}, \bar{t})$ is a global maximum of $\mathcal{C}_{v|\overline{\Omega}\times\overline{\Omega}\times[0,1]}$, then $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{t}) = 0$ or $(\bar{x}, \bar{y}) \in \mathrm{int}\left(\overline{\Omega} \times \overline{\Omega}\right)$.

In particular, if Ω is strictly convex, then $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{t}) = 0$ may happen only if $\bar{x} = \bar{y}$ or $\bar{\lambda} \in \{0,1\}$.

Proof. We can assume $\bar\lambda \in (0,1)$, otherwise $\mathcal{C}_v(\bar x,\bar y,\bar\lambda) = 0$. Set $\bar z := \bar\lambda\bar x + (1-\bar\lambda)\bar y$. If $[\bar x,\bar y] \subset \partial\Omega$, then $\bar z \in \partial\Omega$ and hence $u(\bar x) = u(\bar y) = u(\bar z)$ and $\mathcal{C}_v(\bar x,\bar y,\bar\lambda) = 0$. Thus we can assume $[\bar x,\bar y] \not\subset \partial\Omega$.

Assume by contradiction that $(\bar x,\bar y) \in \partial\left(\overline{\Omega}\times\overline{\Omega}\right)$, thus one among $\bar x$ and $\bar y$ belongs to $\partial\Omega$; w.l.o.g., say $\bar y \in \partial\Omega$. Roughly speaking, by the assumptions v is concave along the segment $[\bar x,\bar y]$, near $\bar y$; by moving $\bar y$ a little closer to $\bar x$, but keeping their convex combination the same, we expect that this procedure increases the value of $\mathcal{C}_v$, which would be an absurd. Let us see this in details.

By (4.2), for some small $\lambda^* \in (0,\bar\lambda)$, $(\lambda^* \approx 0)$ we have

$$\mathcal{C}_v(\bar x,\bar y,\lambda^*) < 0.$$

Set $\bar y^* := \lambda^*\bar x + (1-\lambda^*)\bar y$, $(\bar y^* \approx \bar y)$ it means

$$v(\bar y^*) > \lambda^* v(\bar x) + (1-\lambda^*)v(\bar y). \tag{4.3}$$

Choose $\mu \in (0,1)$ in such a way $\mu\bar x + (1-\mu)\bar y^* = \bar\lambda\bar x + (1-\bar\lambda)\bar y$, that is

$$\mu := \frac{\bar\lambda - \lambda^*}{1-\lambda^*} \in (0,1);$$

notice that $\bar y^*$, rather than $\bar y$, is closer to $\bar x$, while μ maintains the middle combination constant. Moreover (4.3) clearly implies

$$\mu v(\bar x) + (1-\mu)v(\bar y^*) > \bar\lambda v(\bar x) + (1-\bar\lambda)v(\bar y)$$

from which $\mathcal{C}_v(\bar x,\bar y^*,\mu) > \mathcal{C}_v(\bar x,\bar y,\bar\lambda)$, that is the claim. $\square$

Remark 4.2. In order to apply the lemma, we notice that (4.2) is equivalent to check

$$\frac{v\big(\bar y + \lambda(\bar x - \bar y)\big) - v(\bar y)}{\lambda} > v(\bar x) - v(\bar y) \quad \text{for } \lambda \approx 0^+.$$

Thus it is sufficient to verify

$$\limsup_{\lambda\to 0^+} \frac{v\big(\bar y + \lambda(\bar x - \bar y)\big) - v(\bar y)}{\lambda} > v(\bar x) - v(\bar y) \tag{4.4}$$

or equivalently, if $v \in C^1(\overline{\Omega})$

$$\partial_{\bar x - \bar y} v(\bar y) \equiv \nabla v(\bar y)\cdot(\bar x - \bar y) > v(\bar x) - v(\bar y)$$

or similarly

$$v(\bar x) < \nabla v(\bar y)\cdot(\bar x - \bar y) + v(\bar y).$$

Notice that, saying

$$v(x) < \nabla v(\bar y)\cdot(x - \bar y) + v(\bar y) \quad \text{for } x \in \overline{\Omega}\setminus\{\bar y\}$$

means that v lies strictly beneath the plane tangent to v in the boundary point $\bar y \in \partial\Omega$.

Remark 4.3. By assuming $v \in C^1(\Omega)$ and the stronger assumption (which clearly implies (4.4))

$$\limsup_{\lambda\to 0^+} \frac{v\big(\bar y + \lambda(\bar x - \bar y)\big) - v(\bar y)}{\lambda} = +\infty \tag{4.5}$$

the proof of Lemma 4.1 can be simplified: indeed, in the case $[\bar x,\bar y] \not\subset \partial\Omega$, being $\bar y + \varepsilon(\bar y - \bar x) \in \Omega$ for ε small, by exploiting

$$\mathcal{C}_v(\bar x,\bar y,\bar\lambda) \ge \mathcal{C}_v\big(\bar x,\bar y + \varepsilon(\bar y - \bar x),\bar\lambda\big) \quad \text{for } \varepsilon \text{ small},$$

we obtain

$$(1-\bar\lambda)\frac{v(\bar y) - v\big(\bar y + \varepsilon(\bar y - \bar x)\big)}{\varepsilon} \ge \frac{v\big(\bar\lambda\bar x + (1-\bar\lambda)\bar y\big) - v\big(\bar\lambda\bar x + (1-\bar\lambda)\bar y + \varepsilon(1-\bar\lambda)(\bar y - \bar x)\big)}{\varepsilon}.$$

Thus, sending $\varepsilon \to 0^+$, we obtain

$$-\infty \ge (1-\bar\lambda)\nabla v\big(\bar\lambda\bar x + (1-\bar\lambda)\bar y\big)\cdot(\bar y - \bar x);$$

being $\bar{\lambda}\bar{x} + (1-\bar{\lambda})\bar{y} \in \Omega$, we have a contradiction.

We can finally deal with transformations of a functions u; notice that we are not requiring φ to be monotone nor concave.

Corollary 4.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded and convex. Let $k \in \mathbb{R}$ and u be a function with*

$$u = k \ \text{ and } \ \partial_\nu u > 0 \quad \text{ on } \partial\Omega;$$

here ν is the interior normal vector. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be such that $\varphi \in C^1(\mathbb{R} \setminus \{k\})$ and

$$\limsup_{t \to k} \varphi'(t) = +\infty.$$

Set $v := \varphi(u)$. If $(\bar{x}, \bar{y}, \bar{\lambda})$ is a global maximum of $\mathcal{C}_{v|\overline{\Omega}\times\overline{\Omega}\times[0,1]}$, then $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{\lambda}) = 0$ or $(\bar{x}, \bar{y}) \in \operatorname{int}\big(\overline{\Omega} \times \overline{\Omega}\big)$. In particular, if Ω is strictly convex, then $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{t}) = 0$ may happen only if $\bar{x} = \bar{y}$ or $\bar{\lambda} \in \{0, 1\}$.

Proof. Assume $y \in \partial\Omega$, $x \in \overline{\Omega}$, we want to ensure condition (4.4) (actually, (4.5)). Fixed such points, by $\partial_{\nu(y)} u(y) > 0$, we have that also $\partial_{x-y} u(y) > 0$ on $\partial\Omega$ (here we use that $[x, y] \not\subset \partial\Omega$, thus $x - y \not\perp \nu$ is pointing inward). Thus

$$\ell_{x,y} := \partial_{x-y} u(y) = \lim_{\lambda \to 0^+} \frac{u(y + \lambda(x-y)) - u(y)}{\lambda} > 0;$$

in particular, $u(y + \lambda(x - y)) > u(y)$ for λ small. As a consequence, by the mean value theorem for each λ small there exists $t(\lambda) \in \big(u(y + \lambda(x - y)), u(y)\big)$ (and thus $t(\lambda) \to u(y) = k$ as $\lambda \to 0$) such that

$$\begin{aligned}
\limsup_{\lambda \to 0^+} \frac{v\big(y + \lambda(x-y)\big) - v(y)}{\lambda} &= \limsup_{\lambda \to 0^+} \frac{\varphi\big(u\big(y + \lambda(x-y)\big)\big) - \varphi(u(y))}{\lambda} \\
&= \limsup_{\lambda \to 0^+} \left(\varphi'(t(\lambda)) \frac{u\big(y + \lambda(x-y)\big) - u(y)}{\lambda}\right) = +\infty > v(x) - v(y).
\end{aligned}$$

This concludes the proof. $\square$

We move now to a result which gives better information on the concavity function in a tubular neighborhood of the boundary, whenever this is assumed strongly convex; namely, $v = \varphi(u)$ is shown to be strictly convex near the boundary. We refer to [113, Propositions 2.2, 2.3 and 2.4] (see also [90, Lemmas 2.1 and 2.4], [34, Lemma 4.3], [116, Proposition 3.2]).

Proposition 4.5 ([113])**.** *Let $\Lambda \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded, $\partial\Omega \in C^{2,\alpha}$, and strictly convex. Let moreover $v \in C^1(\overline{\Lambda})$ be such that*

$$v(x) > \nabla v(y) \cdot (x - y) + v(y) \tag{4.6}$$

$y \in \partial\Lambda$ and $x \in \overline{\Lambda}$, $x \neq y$. Then all the global maxima of $\mathcal{C}_{v|\overline{\Lambda}\times\overline{\Lambda}\times[0,1]}$ lie in $\operatorname{int}\big(\overline{\Lambda} \times \overline{\Lambda}\big)$.

Proposition 4.6 ([113])**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded, $\partial\Omega \in C^{2,\alpha}$, and strongly convex. Let moreover $u \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \Omega_\eta)$ for some $\eta > 0$, such that*

$$u > 0 \ \text{ in } \Omega, \quad u = 0 \ \text{ on } \partial\Omega, \quad \partial_\nu u > 0 \ \text{ on } \partial\Omega.$$

Let $\varphi \in C^2((0, +\infty), \mathbb{R})$ be such that

$$\varphi'' < 0 < \varphi' \ \text{ near } 0, \quad \lim_{t \to 0^+} \varphi'(t) = +\infty, \quad \lim_{t \to 0^+} \frac{\varphi(t)}{\varphi'(t)} = \lim_{t \to 0^+} \frac{\varphi'(t)}{\varphi'(t)} = 0.$$

Set $v = \varphi(u)$. Then there exists $\delta \in (0, \eta)$ such that

$$D^2 v(x) \quad \text{is definite negative for } x \in \Omega \setminus \Omega_\delta \tag{4.7}$$

and (4.6) holds for $y \in \Omega \setminus \Omega_\delta$ and $x \in \Omega$, $x \neq y$.

In the previous result, $D^2 v$ denotes the Hessian matrix of v. The strategy of application of the previous two results is the following: while Ω is the domain of reference, Λ will be a smaller domain, where the inequality (4.6) holds actually up to the boundary $\partial\Lambda \subset \Omega \setminus \Omega_\delta$. Moreover, this information is inherited by C^2 approximating sequences.

Proposition 4.7 ([113]). *Let $\Lambda \subset \mathbb{R}^N$, $N \geq 1$, be convex, bounded and satisfying the interior sphere condition. Let moreover v_ε be such that, as $\varepsilon \to 0$, $v_\varepsilon \to v$ in $C^1(\overline{\Lambda}) \cap C^2(\overline{\Lambda} \setminus \Lambda_\eta)$ for some $\eta > 0$. Then, for ε sufficiently small, all the global maxima of $\mathcal{C}_{v_\varepsilon |\overline{\Lambda}\times\overline{\Lambda}\times[0,1]}$ lie in* $\mathrm{int}\left(\overline{\Lambda} \times \overline{\Lambda}\right)$.

We notice that an information of the type (4.7), for some class of problems (requiring, in particular, $p = 2$, $f(x,t) \equiv g(t)$ and $g(0) > 0$, such as the semilinear torsion problem) automatically implies the concavity of the solution on the whole domain [39, 86, 119].

Remark 4.8. We make some final comments on the case when φ' does not blow up in 0, but can be taken arbitrary large according to u and Ω; see Remark 2.7 for a related framework. We consider $\varphi = \varphi_r \in C^1([0,+\infty))$ with

$$\begin{aligned} &\varphi_r' > 0, \qquad \varphi_r'(0) \to +\infty \text{ as } r \to +\infty, \\ &\frac{\varphi_r'(0)}{\varphi_r(t) - \varphi_r(0)} \to +\infty \quad \text{as } r \to +\infty, \text{ for any } t > 0, \\ &\frac{\varphi_r'(t)}{\varphi_r''(t)} \to 0 \quad \text{as } r \to +\infty \text{ and } t \to 0. \end{aligned}$$

Some model cases are given by

$$\varphi_r(t) = 1 - e^{-rt}, \quad \varphi_r(t) = \left(t + \tfrac{1}{r}\right)^\gamma,\ \gamma \in (0,1), \quad \varphi_r(t) = \log\left(t + \tfrac{1}{r}\right). \tag{4.8}$$

First, looking at the proof of (4.7) in [90, Lemma 2.4, fact 2] we see that, by assuming

$$\left| \frac{\varphi_r'(u(x))}{\varphi_r''(u(x))} \right|$$

sufficiently small (for x near the boundary and $r \gg 0$), we have the claim for some r which depends only on u and Ω. Move now to the proof of Corollary 4.4: let $(\bar{x}, \bar{y}, \bar{\lambda}) \in \overline{\Omega} \times \partial\Omega \times [0,1]$ be the maximum point of $\mathcal{C}_v$. If $\bar{x} \in \overline{\Omega} \setminus \Omega_\delta$, since by (4.7) v is concave in this strip, $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{\lambda})$ would be here nonpositive. Thus we can assume $\bar{x} \in \Omega_\delta$, that is $|\bar{x} - \bar{y}| > \delta$. As a consequence

$$\begin{aligned} \limsup_{\lambda \to 0^+} \frac{v\big(\bar{y} + \lambda(\bar{x} - \bar{y})\big) - v(\bar{y})}{\lambda} &= \limsup_{\lambda \to 0^+} \frac{\varphi_r\big(u\big(\bar{y} + \lambda(\bar{x} - \bar{y})\big)\big) - \varphi_r(u(\bar{y}))}{\lambda} \\ &= \limsup_{\lambda \to 0^+} \left(\varphi'(t(\lambda)) \frac{u\big(\bar{y} + \lambda(\bar{x} - \bar{y})\big) - u(\bar{y})}{\lambda} \right) = \varphi_r'(0) \nabla u(\bar{y}) \cdot (\bar{x} - \bar{y}). \end{aligned}$$

Observed that

$$\omega_\delta := \min_{y \in \partial\Omega,\, x \in \overline{\Omega}_\delta} \nabla u(y) \cdot (x - y) > 0$$

and that

$$\varphi_r(u(\bar{x})) - \varphi_r(u(\bar{y})) = \varphi_r(u(\bar{x})) - \varphi_r(0) \leq \varphi_r(\|u\|_\infty) - \varphi_r(0)$$

by choosing r large such that

$$\frac{\varphi_r'(0)}{\varphi_r(\|u\|_\infty) - \varphi_r(0)} > \frac{1}{\omega_\delta}$$

we obtain

$$\limsup_{\lambda \to 0^+} \frac{v\big(\bar{y} + \lambda(\bar{x} - \bar{y})\big) - v(\bar{y})}{\lambda} > v(\bar{x}) - v(\bar{y}).$$

Thus, for $r \gg 0$, depending on u and Ω (smooth, strongly convex), we can state that: if $(\bar{x}, \bar{y}, \bar{\lambda})$ is a global maximum of $\mathcal{C}_{v|\overline{\Omega}\times\overline{\Omega}\times[0,1]}$, then $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{\lambda}) = 0$ or $(\bar{x}, \bar{y}) \in \mathrm{int}\left(\overline{\Omega} \times \overline{\Omega}\right)$.

Anyway, the standard properties, discussed in Section 4.2, which allow to gain concavity principle in the interior of Ω here do not hold (e.g. for (4.8)), thus we do not further develop here this topic.

4.2. **Concavity and perturbed concavity in the interior**

We deal now with the information on the concavity function in the interior of Ω; here the equation solved by u plays its role. We start by recalling some results on exact concavity, see [88, Theorem 3.1] and [113, Proposition 2.1].

Theorem 4.9. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, and $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of*

$$-\sum_{i,j} a_{ij}(\nabla v)\partial_i\partial_j v = b(x, v, \nabla v) \quad \text{in } \Omega.$$

Assume a_{ij} is uniformly elliptic, that is

$$\sum_{i,j} a_{ij}(\xi)\eta_i\eta_j \geq C|\eta|^2 \quad \text{for each } \xi \in (\nabla v)(\Omega) \text{ and } \eta \in \mathbb{R}^N \tag{4.9}$$

and

$$t \in v(\Omega) \mapsto b(x,t,\xi) \text{ nonincreasing} \qquad \text{for } \xi \in (\nabla v)(\Omega),$$
$$(x,t) \in \Omega \times v(\Omega) \mapsto b(x,t,\xi) \text{ jointly harmonic concave} \qquad \text{for } \xi \in (\nabla v)(\Omega).$$

Assume moreover one of the following:

- *$t \in v(\Omega) \mapsto b(x,t,\xi)$ strictly decreasing for $\xi \in (\nabla v)(\Omega)$,*
- *$(x,t) \in \Omega \times v(\Omega) \mapsto b(x,t,\xi)$ strictly jointly harmonic concave for $\xi \in (\nabla v)(\Omega)$,*
- *a_{ij} and b smooth enough, namely $a_{ij} \in C^1(\mathbb{R}^N)$, $\nabla_x b, \nabla_\xi b \in L^\infty_{loc}(\Omega \times \mathbb{R} \times \mathbb{R}^N)$.*

Let $(\bar{x}, \bar{y}, \bar{\lambda}) \in \Omega \times \Omega \times [0,1]$ be a maximum of $\mathcal{C}_{v|\overline{\Omega}\times\overline{\Omega}\times[0,1]}$. Then

$$\nabla v(\bar{x}) = \nabla v(\bar{y}) = \nabla v\big(\bar{\lambda}\bar{x} + (1-\bar{\lambda})\bar{y}\big)$$

and $\mathcal{C}_v(\bar{x}, \bar{y}, \bar{\lambda}) \leq 0$.

Proof. The same result for b strictly decreasing is given in [88, Theorem 3.1] (see also [79, Theorem 3.13]), from whose proof it is clear that b strictly harmonic concave works as well. To deal with the nonstrict case, the problem under $a_{ij} \in C^{1,\alpha}_{loc}(\mathbb{R}^N)$, $b(\cdot, t, \cdot) \in C^{1,\alpha}_{loc}(\Omega \times \mathbb{R}^N)$ for some $\alpha \in (0,1)$ has been considered in [90, Lemma 1.5 and Theorem 1.3] and [35, Lemma 3.2], where the proof is based on a perturbation argument (see also [29, proof of Proposition 2.8]). A different and more direct proof, which allows to assume less regularity, has been given in [113, Proposition 2.1] (see also [62, Theorem 2.1]). $\square$

We recall now some theorems about perturbed concavity. To state the theorem we will use the following notation: for $u : \Omega \to \mathbb{R}$ and $b : \Omega \times \mathbb{R} \to \mathbb{R}$ we write

$$\begin{aligned}
\mathcal{JC}_{b(\cdot,u(\cdot))}(x,y,\lambda) :=& \mathcal{JC}_b\big((x,u(x)),(y,u(y)),\lambda\big) \\
=& \lambda b(x,u(x)) + (1-\lambda)b(y,u(y)) - b\big(\lambda x + (1-\lambda)y, \lambda u(x) + (1-\lambda)u(y)\big),
\end{aligned}$$

$$\begin{aligned}
\mathcal{HC}_{b(\cdot,u(\cdot))}(x,y,\lambda) :=& \mathcal{HC}_b\big((x,u(x)),(y,u(y)),\lambda\big) \\
=& \frac{b(x,u(x))b(y,u(y))}{\lambda b(y,u(y)) + (1-\lambda)b(x,u(x))} - b\big(\lambda x + (1-\lambda)y, \lambda u(x) + (1-\lambda)u(y)\big).
\end{aligned}$$

The following theorem is given in [29, Lemmas 2.3 and 2.9]; see also [4, Theorems 2.2 and 2.3].[7] See Section 6.1 for some comments on the case $\mu = 0$.

[7] We highlight that in [4, 29] a different notation for $\mathcal{JC}$ and $\mathcal{HC}$ has been used: the concavity function has opposite sign (and also a reflection in λ in [4]), while the equation has the form $a_{ij}(\nabla v) = b(x, v, \nabla v)$. To obtain what we state now, it is sufficient to consider $v \rightsquigarrow -v$, $a_{ij}(\xi) \rightsquigarrow a_{ij}(-\xi)$, $b(x,t,\xi) \rightsquigarrow b(x,-t,-\xi)$; in particular, the monotonicity in t of the source changes.

Theorem 4.10 ([29]). *Let $\Omega \subset \mathbb{R}^N$ be convex and open, and let $v \in C^2(\Omega) \cap C(\overline{\Omega})$ be a solution of*

$$-\sum_{i,j} a_{ij}(\nabla v)\partial_{ij} v = b(x, v, \nabla v)$$

with a_{ij} symmetric and uniformly elliptic. Assume that $\mathcal{C}_v$ assume a positive interior maximum $(\bar{x}, \bar{y}, \bar{\lambda}) \in \Omega \times \Omega \times [0,1]$, in particular, set $\bar{z} := \bar{\lambda}\bar{x} + (1-\bar{\lambda})\bar{y}$, we have $v(\bar{z}) < \bar{\lambda} v(\bar{x}) + (1-\bar{\lambda})v(\bar{y})$. Then

$$\nabla v(\bar{x}) = \nabla v(\bar{y}) = \nabla v(\bar{z}) =: \bar{\xi}.$$

Assume moreover that b strictly decreases over the interested segment, that is

$$\partial_t b(\bar{z}, t, \bar{\xi}) \leq -\mu < 0 \quad \text{for } t \in [v(\bar{z}), \bar{\lambda} v(\bar{x}) + (1-\bar{\lambda})v(\bar{y})].$$

Then

$$\mathcal{C}_v(\bar{x}, \bar{y}, \bar{\lambda}) \leq \frac{1}{\mu} \mathcal{H}\mathcal{C}_{b(\cdot, v(\cdot), \bar{\xi})}(\bar{x}, \bar{y}, \bar{\lambda}) \leq \frac{1}{\mu} \mathcal{J}\mathcal{C}_{b(\cdot, v(\cdot), \bar{\xi})}(\bar{x}, \bar{y}, \bar{\lambda}).$$

Remark 4.11 (Comparison with mid-concavity). It is known that if v is continuous on an open set, then

$$\mathcal{C}_v(x, y, \lambda) \leq 0 \ \text{ for each } \lambda \in [0,1] \iff \mathcal{C}_v(x, y, \tfrac{1}{2}) \leq 0;$$

that is, v is concave if and only if it is *mid-concave*

$$\mathcal{C}^m_v(x, y) := \frac{v(x) + v(y)}{2} - v\left(\frac{x+y}{2}\right) \leq 0.$$

It is thus possible to develop the above theory with λ fixed to $\frac{1}{2}$, as done e.g. in [62, 76, 79], but we use here the full $\mathcal{C}_v$, since the statements on the perturbed concavity are stronger. We anyway highlight some differences now.

The arguments of Remark 4.3 still apply, thus Corollary 4.4 holds true. In [62, Theorem 2.1] Theorem 4.9 is shown for $\mathcal{C}^m$, but the argument can be extended to $\mathcal{C}$ [113, proof of Proposition 2.1]. See also [76, Theorems 3.4 and 3.5] for a result on viscosity solutions.

Moreover, in [62, Lemma 3.2] (see also [79, Lemma 3.12]), when v negatively explodes on the boundary (that is the case of $v = \log(u)$), they provide a tool which ensures that $\mathcal{C}^m_v$ cannot get positive while approaching the boundary, in the sense: for any $x_n, y_n \in \Omega$

$$d((x_n, y_n), \partial(\Omega \times \Omega)) \to 0 \implies \limsup_{n \to +\infty} \mathcal{C}^m_v(x_n, y_n) \leq 0. \tag{4.10}$$

To get this result the fact that Ω is strictly convex and the restriction to $\lambda = \frac{1}{2}$ seem essential.

This information is not necessary when working with exact concavity, since the argument are set on a fixed Ω_δ, where $v = \varphi(u)$ is bounded. On the other hand, this a priori information on the whole Ω is indeed used for perturbed concavity: thus, to get a perturbed concavity result on $\log(u)$ we need to work with $\mathcal{C}^m$; see Theorem 6.7. By [114, Corollary 1], from a bound on $\mathcal{C}^m_{\log(u)}$ we can obtain a bound also on $\mathcal{C}_{\log(u)}$ up to doubling the error; this still allows to apply Hyers-Ulam Theorem [72]. See Remark 5.5 for details. A different approach is given in Remark 4.8.

5. The approximation argument

We construct now our approximation argument, inspired by [22, 113, 116]. Here we need to deal also with the dependence on x. We keep the argument for a general $f = f(x,t)$

$$\begin{cases} -\Delta_p u = f(x, u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

with some additional assumptions:

(f1) Sub p-linearity: $|f(x,t)| \leq C(t^{p-1} + 1)$.

(f2) Hopf boundary lemma holds: for the sake of simplicity we will assume $f(x,t) > 0$ for $t > 0$, but more general cases could be treated.[8]

(f3) Regularity of solutions, ensured by: $f \in C^{1,\alpha}_{loc}(\Omega, (0,+\infty)) \cap C^{0,\alpha}_{loc}(\overline{\Omega}, [0,+\infty))$.

Moreover in this Section we require:

- Ω strongly convex and with $\partial\Omega \in C^{2,\alpha}$,

and

(fs) $|f(x,t)| \le C(1 + t^{q-1})$ for some $q \in (0, p-1)$, to ensure coercivity and uniqueness.

We refer to Remark 5.2 for the eigenfunction case $q = p-1$, while to Section 5.1 for the case of Ω general convex.

We define $F(x,t) := \int_0^t f(x,\tau)d\tau$. The approximation argument is developed in several steps.

Step 1. First we observe that, under our assumptions – in particular (fs) – actually a positive solution of the problem exists by a global minimization process: see e.g. [95, Theorem 2.1, Section 5.2]. If $p > N$ clearly $u \in C^{0,1-\frac{N}{p}}(\Omega) \subset L^\infty(\Omega)$. Assume thus $p \le N$ (see also Remark 5.2). Set

$$G(x,t,\xi) := \frac{1}{p}|\xi|^p - F(x,t),$$

u is a minimizer of $v \mapsto \int_\Omega G(x,v,\nabla v)$. Thanks to [95, Theorem 3.2, Section 5.3, pag 328], we have that the global minimizer u is in $L^\infty(\Omega)$.

Step 2. More generally, by [30, Theorem 2.1] (see also [66, Propositions 1.2 and 1.3]), all the solutions belong to $L^\infty(\Omega)$, and thus, being $\partial\Omega \in C^{1,\alpha}$, thanks to [101, Theorem 1] (see also [66, Corollary 1.1]), all the solutions are in $C^{1,\beta}(\overline{\Omega})$ for some $\beta = \beta(p,N) \in (0,1)$; thus uniqueness holds by Lemma A.1. In particular, all the solutions are minima.

Step 3. We already observed that $u \in C^{1,\beta}(\overline{\Omega})$. More precisely, we have

$$\|u\|_{C^{1,\beta}(\overline{\Omega})} \le C(p, N, \|u\|_\infty, \|f(\cdot,u)\|_\infty, \Omega).$$

By Hopf Lemma [122, Theorem 5] (see also [116, Lemma A.3], [66, Proposition 2.2]) we have

$$\partial_\nu u > 0 \quad \text{on } \partial\Omega;$$

thus there exists $\eta > 0$ such that

$$\inf_{\Omega_\eta} u > 0, \qquad \inf_{\Omega \setminus \Omega_\eta} |\nabla u| > 0.$$

We observe that, being Ω convex, we have $u \in W^{2,2}_{loc}(\Omega \setminus \Omega_\eta)$ by [78] (see also [95, Section 4.3, Theorem 5.2]). Thus, being $\partial\Omega \in C^{2,\alpha}$ and $f \in C^{0,\alpha}_{loc}(\overline{\Omega}, [0,+\infty))$ we obtain thanks to [95, Section 4.6, Theorem 6.3]

$$u \in C^2(\overline{\Omega} \setminus \Omega_\eta).$$

We consider moreover $\delta \in (0,\eta)$ sufficiently small to be fixed (see Step 7) such that

$$\Omega_\eta \subset \Omega_\delta \subset \Omega;$$

with smooth boundaries. Similarly, for $k = 1...5$, we may assume $\Omega_{\delta/k}$ convex and smooth (see Proposition 2.1); notice that we actually can substitute these sets with nicer suitable approximations, if needed.

Step 4. Consider the functional

$$J(v) := \frac{1}{p}\int_\Omega |\nabla v|^p - \int_\Omega F(x,v)$$

[8] For example, $f(x,t) \ge -\zeta(t)$ with $\zeta(0) = 0$, ζ continuous, nondecreasing and one of the following holds: $\zeta(t) = 0$ for some $t \in [0,t_0)$ or $\int_0^1 \frac{1}{(\zeta(t)t)^{1/p}} = +\infty$; in particular, one can assume $f(x,t) \ge -Ct^{p-1}$.

to which u is a critical point. We consider a regularization $I_\varepsilon \approx J$ defined as follow: let us introduce a function $K = K(t)$ – to be fixed, see Step 8 – such that

- $K \geq 0$, and $K > 0$ in $(0, +\infty)$,
- $K \in C^1((0, +\infty))$, and $K^{2/p} \in C^{1,\alpha}_{loc}((0, +\infty))$,
- $|K'(t)| \leq C(1 + |t|^{p-1})$ for $t > 0$.

Then define, for each $\varepsilon > 0$

$$I_\varepsilon(v) := \frac{1}{p} \int_\Omega \left(\varepsilon K(v)^{2/p} + |\nabla v|^2\right)^{p/2} - \int_\Omega F(x, v);$$

clearly $I_0 \equiv J$. Similarly to J, for each $\varepsilon > 0$ we gain the existence of $u_\varepsilon \in W^{1,p}(\Omega)$, global minimizer of I_ε.

Step 4.1. By exploiting the equicoercivity and the fact that u_ε are minima, we obtain for $\varepsilon \in (0, 1)$

$$C\|u_\varepsilon\|^p_{W^{1,p}_0(\Omega)} \leq I_\varepsilon(u_\varepsilon) \leq I_\varepsilon(0) \leq I_1(0)$$

thus equibounded, which means that $u_\varepsilon \rightharpoonup \bar{u}$ in $W^{1,p}_0(\Omega)$. As a consequence, exploiting that J is lower semicontinuous, $J \leq I_\varepsilon$, u_ε are minima, $I_\varepsilon \to J$ (by dominated convergence theorem) and u is a minimum, we obtain

$$J(\bar{u}) \leq \liminf_{\varepsilon \to 0} J(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u_\varepsilon) \leq \liminf_{\varepsilon \to 0} I_\varepsilon(u) = J(u) \leq J(\bar{u}).$$

Being $J(u) = J(\bar{u})$ and the minimum unique, we have $\bar{u} = u$ and thus $u_\varepsilon \rightharpoonup u$ in $W^{1,p}_0(\Omega)$. Moreover we have, by the above computation, $J(u_\varepsilon) \to J(u)$ and (by the p-subhomogeneous growth) $\int_\Omega F(x, u_\varepsilon) \to \int_\Omega F(x, u)$, which together imply $\|\nabla u_\varepsilon\|_p \to \|\nabla u\|_p$. Thus we have

$$u_\varepsilon \to u \quad \text{in } W^{1,p}_0(\Omega).$$

If $p > N$ we have $W^{1,p}(\Omega) \hookrightarrow C^{0,1-\frac{N}{p}}(\Omega)$, thus $\|u_\varepsilon\|_{C^{0,1-\frac{N}{p}}(\Omega)}$ is equibounded. In the following two steps, hence, we can restrict to $p \leq N$.

Step 4.2. Set

$$G_\varepsilon(x, t, \xi) := \frac{1}{p}\left(\varepsilon K(t)^{2/p} + |\xi|^2\right)^{p/2} - F(x, t),$$

u_ε is a minimizer of $v \mapsto \int_\Omega G_\varepsilon(x, v, \nabla v)$. Due to the assumptions on K, by [95, Theorem 3.2, Section 5.3, pag 328] we have that the minimizers u_ε are in $L^\infty(\Omega)$ for $\varepsilon \in (0, 1)$. More precisely

$$\|u_\varepsilon\|_\infty \leq C(\|u_\varepsilon\|_{p^*}, p, \text{meas}(\Omega)),$$

thus, by Step 4.1, they are equibounded. As a consequence

$$F(x, u_\varepsilon) \leq C_1, \quad K(u_\varepsilon) \leq C_2 \quad \text{in } \Omega$$

for suitable $C_1, C_2 > 0$.

Step 4.3. By Step 4.2 we have

$$\frac{1}{p}|\xi|^p - C \leq G_\varepsilon(x, t, \xi) \leq C|\xi|^p + C$$

for $|t| \leq \sup_\varepsilon \|u_\varepsilon\|_\infty$, which by [54, Theorem 3.1] directly implies that the minimizers u_ε are Hölder continuous. To obtain a more explicit equibound, we see u_ε as solutions of the equation

$$-\text{div}_\xi\big((\nabla_\xi G_\varepsilon)(x, u_\varepsilon, \nabla u_\varepsilon)\big) + (\partial_t G_\varepsilon)(x, u_\varepsilon, \nabla u_\varepsilon) = 0 \quad \text{in } \Omega$$

that is

$$-\text{div}\big(A^\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\big) = f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \quad \text{in } \Omega$$

where

$$A^\varepsilon(t, \xi) := (\varepsilon K(t)^{2/p} + |\xi|^2)^{\frac{p-2}{2}}\xi,$$

$$f_\varepsilon(x,t,\xi) := f(x,t) - \frac{\varepsilon}{p}(\varepsilon + |\xi|^2 K(t)^{-2/p})^{\frac{p-2}{2}} K'(t).$$

We can thus apply [95, Theorem 4.1, Section 5.4] (see also [121, Corollary 4.2], [30, Theorem 2.3]) and obtain the existence of a $\beta_0 = \beta_0(p, K, \sup_\varepsilon \|u_\varepsilon\|_\infty) \in (0,1)$ and $C = C(p, K, \sup_\varepsilon \|u_\varepsilon\|_\infty, \delta)$, such that

$$\|u_\varepsilon\|_{C^{0,\beta_0}(\overline{\Omega_{\delta/5}})} \leq C;$$

we can assume $\beta_0 < \beta$.[9]

Step 4.4. By exploiting the convergence in Step 4.1 and the uniform estimate in Step 4.3 (or at the end of Step 4.1), we obtain

$$u_\varepsilon \to u \quad \text{in } C^{0,\beta}(\overline{\Omega_{\delta/5}}).$$

By this convergence and Step 3, there exists $C = C(\delta)$ such that

$$\tfrac{1}{C} \leq u_\varepsilon \leq C \qquad \text{in } \Omega_{\delta/4}$$

for ε small. As a consequence

$$\tfrac{1}{C_1} \leq F(x, u_\varepsilon) \leq C_1, \quad \tfrac{1}{C_2} \leq K(u_\varepsilon) \leq C_2 \qquad \text{in } \Omega_{\delta/4}$$

for suitable $C_1, C_2 > 0$.

Step 5. Set $a_{ij}^\varepsilon(t,\xi) := \partial_{\xi_j} A_i^\varepsilon(t,\xi)$, and assumed $\partial_t K(u_\varepsilon) \leq C_2$, by Step 4.3 and Step 4.4 we can apply [49, 120] (see also [101], [95, Section 4.6]) to get the existence of $\beta_2 \in (0,1)$ such that $u_\varepsilon \in C^{1,\beta_2}(\Omega)$ and

$$\|u_\varepsilon\|_{C^{1,\beta_2}(\overline{\Omega_{\delta/3}})} \leq C;$$

here $\beta_2 = \beta_2(\|u\|_\infty, \|f(\cdot,u)\|_\infty, C_2, p)$ and $C = C(\delta, \|u\|_\infty, \|f(\cdot,u)\|_\infty, C_2, p)$. We can assume $\beta_2 \leq \beta$ and $\beta_2 < \alpha$. By Ascoli-Arzelà theorem,

$$u_\varepsilon \to u \quad \text{in } C^{1,\beta_2}(\overline{\Omega_{\delta/3}}).$$

By this convergence and Step 3,

$$\inf_{\Omega \setminus \Omega_\eta} |\nabla u_\varepsilon| > 0, \qquad \sup_{\Omega_{\delta/3}} |\nabla u_\varepsilon| \leq C'$$

for $\varepsilon > 0$ small.

Step 6. By [95, Section 4.6, Theorem 6.4] we conclude that there exists $\beta_3 \in (0,1)$, $\beta_3 = \beta_3(C_1, C_2, \alpha, \delta)$, such that

$$u_\varepsilon \in C^{2,\beta_3}(\Omega_{\delta/3});$$

we can assume $\beta_3 \leq \beta_2$. Moreover

$$\|u_\varepsilon\|_{C^{2,\beta_3}(\overline{\Omega_{\delta/2}} \setminus \Omega_\delta)} \leq C$$

for some $C = C(C_1, C_2, \alpha, \delta)$. From which, by Ascoli-Arzelà theorem, we obtain

$$u_\varepsilon \to u \quad \text{in } C^1(\overline{\Omega_{\delta/2}}) \cap C^2(\overline{\Omega_{\delta/2}} \setminus \Omega_\delta).$$

Step 7. Let us consider an invertible transformation $\varphi = \varphi(t) \in C^{2,\alpha}_{loc}((0,+\infty))$, such that

$$\lim_{t\to 0^+} \varphi'(t) = +\infty, \quad \varphi'' < 0 < \varphi' \text{ near } 0, \quad \lim_{t\to 0^+} \frac{\varphi(t)}{\varphi'(t)} = \lim_{t\to 0^+} \frac{\varphi'(t)}{\varphi''(t)} = 0.$$

Notice that $\varphi^{-1} : \mathbb{R} \to (0,+\infty)$, but we are not requiring φ to be positive, nor well defined in 0.

Step 7.1. Set $v := \varphi(u)$, from the estimate in Step 4 we get

$$\tfrac{1}{C'} \leq v \leq C' \qquad \text{in } \Omega_{\delta/4},$$

[9] Notice that actually $\|u_\varepsilon\|_{C^{0,\beta_0}(\overline{\Omega})} \leq C' = C'(p, K, \sup_\varepsilon \|u_\varepsilon\|_\infty, \Omega)$ depending on the geometry of Ω.

where $C' = C'(\delta)$. Moreover, since $u \in C(\overline{\Omega})$, together with $u > 0$ in Ω, $u = 0$ on $\partial\Omega$ and $\partial_\nu u > 0$ on $\partial\Omega$, by Corollary 4.4 we obtain that $\mathcal{C}_v$ cannot attain a maximum (over $\overline{\Omega} \times \overline{\Omega} \times [0,1]$) on $\partial(\Omega \times \Omega) \times [0,1]$.[10]

Moreover, since $u \in C^1(\overline{\Omega}) \cap C^2(\overline{\Omega} \setminus \Omega_\eta)$ by Step 3, and applying Proposition 4.6, we obtain that, for δ sufficiently small,

$$\begin{cases} D^2 v < 0 & \text{in } \Omega \setminus \Omega_\delta, \\ v(x) - v(x_0) < \nabla v(x_0) \cdot (x - x_0) & \text{for each } x_0 \in \Omega \setminus \Omega_\delta,\ x \in \Omega \setminus \{x_0\}. \end{cases} \tag{5.2}$$

Step 7.2. Set $v_\varepsilon := \varphi(u_\varepsilon)$. From the estimate in Step 4 and 7.1 we get

$$\tfrac{1}{C'} \le v_\varepsilon \le C' \quad \text{in } \Omega_{\delta/4}, \qquad |\nabla v_\varepsilon| \le C'' \quad \text{in } \Omega_{\delta/3}$$

for some $C' = C'(\delta) > 0$ and $C'' = C''(\delta) > 0$; this, combined with the convergence in Step 6, gives

$$v_\varepsilon \to v \quad \text{in } C^1(\overline{\Omega_{\delta/2}}) \cap C^2(\overline{\Omega_{\delta/2}} \setminus \Omega_\delta).$$

Thanks to Proposition 4.7, $\mathcal{C}_{v_\varepsilon}$ cannot attain its positive maximum (over $\overline{\Omega_{\delta/2}} \times \overline{\Omega_{\delta/2}} \times [0,1]$) on $\partial(\Omega_{\delta/2} \times \Omega_{\delta/2}) \times [0,1]$, for ε small.

Step 8. Let us consider now the equation satisfied by v_ε. Call $\psi := \varphi^{-1}$, we have $u_\varepsilon = \psi(v_\varepsilon)$. We make the following choice

$$K(\psi(t)) \equiv (\psi'(t))^p,$$

i.e. $K(t) := \frac{1}{(\varphi'(t))^p}$. With this choice, set

$$H_\varepsilon(\xi) := \left(\varepsilon + |\xi|^2\right)^{p/2}$$

we have

$$-\mathrm{div}\big((\nabla H_\varepsilon)(\nabla v_\varepsilon)\big) = B_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon)$$

where

$$B_\varepsilon(x,t,\xi) := p\frac{f(x,\psi(t))}{(\psi'(t))^{p-1}} + pH_\varepsilon(\xi)^{\frac{p-2}{p}}\left((p-1)|\xi|^2 - \varepsilon\right)\frac{\psi''(t)}{\psi'(t)}.$$

To discuss the properties of B_ε, we notice that $(p-1)|\xi|^2 - \varepsilon$ has a variable sign when $\varepsilon \neq 0$. We thus simplify the argument by choosing[11]

$$f(x,t) \equiv h(x,t)g(t) + k(x,t) \tag{5.3}$$

where g and ψ are related by

$$\frac{g(\psi(t))}{(\psi'(t))^{p-1}} = \frac{\psi''(t)}{\psi'(t)} \tag{5.4}$$

that is (up to additive constants and positive multiplicative constants) $\varphi(t) = \int_1^t (G(s))^{-\frac{1}{p}} ds$, where we are assuming

$$G(s) := \int_0^s g(\tau)d\tau > 0 \qquad \text{for } s > 0.$$

Moreover we assume[12]

$$h(x,t) > 0 \quad \text{for } t > 0,\ x \in \Omega;$$

notice that, differently from f, we do not require h to be finite in $t = 0$. Thus B_ε takes the form

$$B_\varepsilon(x,t,\xi) = p\frac{\psi''(t)}{\psi'(t)}\Big(h(x,\psi(t)) + H_\varepsilon(\xi)^{\frac{p-2}{p}}\left((p-1)|\xi|^2 - \varepsilon\right)\Big) + p\frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}.$$

[10] Notice that, up to now, v may be unbounded and $\mathcal{C}_v$ not well defined on the boundary.

[11] Other possible choices could be set up according to the specific case. For instance, $f(x,t) = \zeta(g(t))$ for some suitable function ζ.

[12] Notice that, in the regular case $p = 2$, that is $\varepsilon = 0$, we can allow $h = 0$.

Discussions on the assumptions.

We discuss now some properties of the right-hand side B_ε. We are interested only in

$$(x,t,\xi)\in\Omega_{\delta/2}\times\in v_\varepsilon(\Omega_{\delta/2})\times\nabla v_\varepsilon(\Omega_{\delta/2})$$

where $t\in v_\varepsilon(\Omega_{\delta/2})\in[\frac{1}{C'},C']$ while $|\nabla v_\varepsilon(\Omega_{\delta/2})|\in[-C'',C'']$ (for ε small) thanks to Step 7.2.

For these values of (x,t,ξ) we have

$$h(x,\psi(t))+H_\varepsilon(\xi)^{\frac{p-2}{p}}\left((p-1)|\xi|^2-\varepsilon\right)>0$$

for ε small. Actually we will require

$$\Theta+H_\varepsilon(\xi)^{\frac{p-2}{p}}\left((p-1)|\xi|^2-\varepsilon\right)>0$$

for some

$$0<\Theta<\inf_{\Omega_{\delta/2}\times\in[\frac{1}{C'},C']}h(x,\psi(t)) \tag{5.5}$$

to be fixed (independent on ε) and $\varepsilon>0$ sufficiently small. In particular, we can rewrite B_ε as

$$B_\varepsilon(x,t,\xi)=p\frac{\psi''(t)}{\psi'(t)}\Big(\Theta+H_\varepsilon(\xi)^{\frac{p-2}{p}}\left((p-1)|\xi|^2-\varepsilon\right)\Big)+p\big(h(x,\psi(t))-\Theta\big)\frac{\psi''(t)}{\psi'(t)}+p\frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}.$$

• *Positivity.* If we assume

$$\frac{\psi''(t)}{\psi'(t)}\geq 0,\quad \frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}\geq 0,$$

then

$$B_\varepsilon(x,t,\xi)\geq 0 \tag{5.6}$$

strict, if one of the two above is strict.

• *Monotonicity.* If we assume

$$t\mapsto\frac{\psi''(t)}{\psi'(t)},\quad t\mapsto\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t))-\Theta\big)\quad\text{and}\quad t\mapsto\frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}\qquad\text{nonincreasing,}$$

then

$$t\mapsto B_\varepsilon(x,t,\xi)\quad\text{nonincreasing;} \tag{5.7}$$

if one of the three above is strictly decreasing, then

$$t\mapsto B_\varepsilon(x,t,\xi)\quad\text{strictly decreasing.} \tag{5.8}$$

If moreover

$$\partial_t\left(\frac{\psi''(t)}{\psi'(t)}\right)\leq-\tilde{\mu}<0,$$

for some $\tilde{\mu}>0$, then there exists $\mu=\mu(\delta)>0$ such that

$$\partial_t B_\varepsilon(x,t,\xi)\leq-\mu<0. \tag{5.9}$$

• *Concavity.* To discuss the harmonic concavity of the sum, we rely on Proposition 2.2. Indeed, if we assume

$$t\mapsto t^2\frac{\psi''(t)}{\psi'(t)},\quad (x,t)\mapsto t^2\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t))-\Theta\big)\quad\text{and}(x,t)\mapsto t^2\frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}\quad\text{jointly concave,}$$

then

$$(x,t)\mapsto t^2B_\varepsilon(x,t,\xi)\quad\text{jointly concave.}$$

If we assume B_ε positive (see (5.6)), by Proposition 2.2 we obtain

$$(x,t)\mapsto B_\varepsilon(x,t,\xi)\quad\text{jointly harmonic concave.} \tag{5.10}$$

Now that the machinery is set on, we can move to the discussions of the main theorems.

Remark 5.1. We observe that, if

$$f(x,t) \equiv g(t),$$

monotonicity and harmonic concavity of B_ε are ensured by requiring only

$$t \mapsto \frac{\psi''(t)}{\psi'(t)} \quad \text{nonincreasing and harmonic concave.}$$

A sufficient condition is given in [22] (see also [34, Theorem 4.4], [80, Corollary 2]) which reads as follow:

(i) $G^{1/p}$ concave,
(ii) $\frac{g}{G}$ harmonic concave.

See [22, Section 4.5] for details. This ensures that $\varphi(t) = \int_1^t (G(s))^{-\frac{1}{p}} ds$ is concave. We will see that the possibility of $k(x,t)$ of not being zero (see (5.3)) will allow more general cases than [22], even in the autonomous semilinear case (see Corollary 6.6).

Remark 5.2. We comment here the case of the p-linear growth of f, when it has the particular form

$$f(x,t) = a(x)|t|^{p-2}t.$$

We show here how to adapt the previous steps to this case.

Step 1 and 4, existence: both existence results for J and I_ε can be made by considering (recall that $a \in L^\infty(\Omega)$)

$$\inf\left\{ J(u) \mid u \in W_0^{1,p}(\Omega),\ \int_\Omega a(x)u^p = 1 \right\}, \quad \inf\left\{ I_\varepsilon(u) \mid u \in W_0^{1,p}(\Omega),\ \int_\Omega a(x)u^p = 1 \right\};$$

we notice that the functionals are coercive on the subspace where the weighted p-norm is prescribed. Thus we can find Lagrange multipliers λ and λ_ε and solutions u and u_ε (see e.g [14, Theorem 6.3.2]).

Step 2 and 4.1, uniqueness and convergence: we observe that, by compact embeddings, $u_\varepsilon \rightharpoonup \bar{u}$ in $W_0^{1,p}(\Omega)$ implies $u_\varepsilon \to \bar{u}$ in $L^p(\Omega)$, thus (being a bounded) $\int_\Omega a(x)u_\varepsilon^p \to \int_\Omega a(x)\bar{u}^p$; in particular, $\int_\Omega a(x)\bar{u}^p = 1$. To get $u = \bar{u}$ we can thus repeat the same arguments, observing that we have uniqueness up to a scaling (thanks to Lemma A.1) and the noninvariant constraint $\int_\Omega a(x)u^p = 1$.

Step 4.2, equiboundedness for the pertubed problem: u_ε satisfies the equation

$$-\mathrm{div}\big(A^\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\big) = f_\varepsilon(x, u_\varepsilon, \nabla u_\varepsilon) \quad \text{in } \Omega$$

where $A_\varepsilon(t,\xi) = (\varepsilon K(t)^{2/p} + |\xi|^2)^{\frac{p-2}{2}}\xi$ but now

$$f_\varepsilon(x,t,\xi) := \lambda_\varepsilon a(x)t^{p-1} - \frac{\varepsilon}{p}(\varepsilon + |\xi|^2 K(t)^{-2/p})^{\frac{p-2}{2}} K'(t).$$

First, we observe the equiboundedness of the Lagrange multipliers λ_ε: indeed, for any positive $\varphi \in W_0^{1,p}(\Omega)$, λ_ε is given by

$$\lambda_\varepsilon = \frac{1}{\int_\Omega a(x)u_\varepsilon^{p-1}\varphi}\left(\int_\Omega A^\varepsilon(u_\varepsilon, \nabla u_\varepsilon)\cdot\nabla\varphi + \frac{\varepsilon}{p}\int_\Omega \left(\varepsilon + |\nabla u_\varepsilon|^2 K(u_\varepsilon)^{-2/p}\right)^{\frac{p-2}{2}} K'(u_\varepsilon)\varphi \right).$$

Assuming, when $p > 2$, $K(t)^{-\frac{p-2}{p}}K'(t) \leq 1 + t^\sigma$ for some $\sigma < \frac{2N}{N-p}$, since $u_\varepsilon \to u$ in $W_0^{1,p}(\Omega)$ and almost everywhere, by exploiting that u_ε and ∇u_ε are dominated by L^p functions, the Young inequality for products and the dominated convergence theorem, we obtain that the right-hand side is bounded.

Next, by assuming, when $p > 2$, $K(t)^{-\frac{p-2}{p}}K'(t) \leq 1 + t^\sigma$ for some $\sigma < \frac{N+p}{N-p}$, we can apply [95, Theorem 7.1, Section 4.7] and get $u_\varepsilon \in L^\infty(\Omega)$ with

$$\|u_\varepsilon\|_\infty \leq C$$

where $C = C(N, p, |\Omega|, \|u_\varepsilon\|_{p^*})$, and thus equibounded.

The above extra conditions on K will be satisfied by our choice of K in Section 6.1.

5.1. Proof of general results

We show now that the conclusions of Theorems 1.4 and 1.9 holds in more general cases; we will furnish the statements of general results for the sake of completeness, even if they appear a bit cumbersome.

5.1.1. *Exact concavity.*

First, we observe that, exploiting the boundedness of $|\nabla u_\varepsilon|$ by Step 5, the equation verified by v_ε

$$-\sum_{i,j} \tilde{a}^{\varepsilon}_{ij}(\nabla v_\varepsilon)\partial_{ij} v_\varepsilon = B_\varepsilon(x, v_\varepsilon, \nabla v_\varepsilon),$$

where $\tilde{a}^{\varepsilon}_{ij}(\xi) := p(\varepsilon + |\xi|^2)^{\frac{p-4}{2}}\big((p-2)\xi_i\xi_j + (\varepsilon + |\xi|^2)\delta_{ij}\big)$, is uniformly elliptic.

Then, by assuming (5.8) (or (5.7), notice that the regularity assumed on f implies the one desired) and (5.10), since by Step 6 the solutions verify $v_\varepsilon \in C^2(\Omega_{\delta/2})$, we can apply Theorem 4.9 to gain that $\mathcal{C}_{v_\varepsilon}$ cannot assume maximum in $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. Combined with Step 7.2, we see that $\mathcal{C}_{v_\varepsilon} \in C(\overline{\Omega_{\delta/2}} \times \overline{\Omega_{\delta/2}} \times [0,1])$ cannot have a positive maximum, which means that $\mathcal{C}_{v_\varepsilon} \le 0$ on $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. By the (pointwise) convergence obtained in Step 4, we obtain that $\mathcal{C}_v \le 0$ on $\Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. This is on the other hand true for any $\delta > 0$ (small): this means that $\mathcal{C}_v \le 0$ on $\Omega \times \Omega \times [0,1]$, that is $\varphi(u)$ is concave.

To pass from a smooth, strongly convex domain to a general convex Ω, we make an approximation process (based on the uniqueness and minimality of the solution) as in [22, Section 4.1] (see also [116, Section 5]); we give some details in the case of the eigenfunction. First, being Ω convex, from [121, Corollary 4.2] (see also [38, Corollary 3.7]) all the solutions belong to $C(\overline{\Omega})$ and thus, being $\partial\Omega \in C^{1,\alpha}$, thanks to [101, Theorem 1] all the solutions are in $C^{1,\beta}(\overline{\Omega})$; thus uniqueness (up to scaling) holds by Lemma A.1. Let now $u \in W^{1,p}_0(\Omega) \cap C(\overline{\Omega})$ be the solution on Ω (obtained by minimization with constraint $\int_\Omega a(x)u^p = 1$) and let $\Omega^k \subset \Omega$ be a sequence of smooth strongly convex sets which approximate Ω in the Hausdorff distance (see Proposition 2.1), and define $J_k(v) := \frac{1}{2}\int_{\Omega^k} |\nabla v|^p_p - \int_{\Omega^k} a(x)v^p$. We consider $u_k \in W^{1,p}_0(\Omega^k)$ critical points of J_k obtained by constrained minimization $\int_{\Omega^k} a(x)u_k^p = 1$, and extend u_k to Ω by $u = 0$ on $\Omega \setminus \Omega^k$.

We clearly have $J_k(u_k) = J(u)$, moreover by

$$\frac{1}{p}\|u_k\|_{W^{1,p}_0(\Omega)^p} \le J(u_k) = J_k(u_k) \le J_k(u_1) \le J_1(u_1) = J(u_1)$$

we have that $u_k \rightharpoonup \bar{u}$ in $W^{1,p}_0(\Omega)$ and strongly in $L^p(\Omega)$; in particular $\int_\Omega a(x)\bar{u}^p = 1$.

Let now $\varepsilon_k := 2\sup_{\Omega\setminus\Omega^k} u$, and define $v_k := (\int_{\Omega^k} a(x)(u-\varepsilon_k)_+)^{-1}(u-\varepsilon_k)_+$. Notice that $\varepsilon_k \to 0$, since $\max_{\overline{\Omega}\setminus\Omega^k} u = u(x_k)$ with $d(x_k, \partial\Omega) \le \sup_{x\in\Omega} d(x, \Omega_k) \to 0$. Since $v_k \to u$ in $W^{1,p}_0(\Omega)$ we obtain

$$J(\bar{u}) \le \liminf_k J(u_k) = \liminf_k J_k(u_k) \le \liminf_k J_k(v_k) = J(u) \le J(\bar{u})$$

from which, by uniqueness of the minimizer, $\bar{u} = u$. As a consequence, if the concavity claim holds for each u_k, then it is so (by pointwise convergence) also for u.

We have thus proved the following result.

Theorem 5.3 (Exact concavity). *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be open, bounded and convex, with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1,+\infty)$. Let u be a solution of (5.1), and let f and φ satisfy (f1)–(f3) (5.3), (5.4), and one among (fs) and $f(x,t) = a(x)|t|^{p-2}t$. Assume moreover that (5.7) and (5.10) hold. Then $\varphi(u)$ is concave.*

5.1.2. *Perturbed concavity*

Assume (5.9). We assume here

$$\varphi \in C(\overline{\Omega}); \tag{5.11}$$

see Remark 5.5 for more general φ unbounded, and Section 6.1 for the logarithmic case.

By Step 7.1 we have that the maximum (x_0, y_0, λ_0) of $\mathcal{C}_v$ cannot be attained on the boundary, thus it belongs to $\Omega \times \Omega \times [0,1]$; in particular, there exists $\delta > 0$ sufficiently small such that $(x_0, y_0, \lambda_0) \in \Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. Clearly, (x_0, y_0, λ_0) is also the maximum point over $\overline{\Omega_{\delta/2}} \times \overline{\Omega_{\delta/2}} \times [0,1]$.

Let now $(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon) \in \overline{\Omega_{\delta/2}} \times \overline{\Omega_{\delta/2}} \times [0,1]$ be the point of maximum of $\mathcal{C}_{v_\varepsilon}$; by Step 7.2, for $\varepsilon > 0$ small, it cannot be on the boundary, thus $(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon) \in \Omega_{\delta/2} \times \Omega_{\delta/2} \times [0,1]$. Moreover, by the convergence in Step 7.2, we have $(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon) \to (x_0, y_0, \lambda_0)$; notice that $(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon)$, differently from (x_0, y_0, λ_0), depends also on δ. Then, by Theorem 4.10

$$\nabla v_\varepsilon(x_\varepsilon) = \nabla v_\varepsilon(z_\varepsilon) = \nabla v_\varepsilon(y_\varepsilon) =: \xi_\varepsilon,$$

where $z_\varepsilon := \lambda_\varepsilon x_\varepsilon + (1-\lambda_\varepsilon) y_\varepsilon$. By (5.9) in particular we have

$$\partial_t B_\varepsilon(z_\varepsilon, t, \xi_\varepsilon) \le -\mu < 0 \quad \text{for } t \in [v_\varepsilon(z_\varepsilon), \lambda_\varepsilon v_\varepsilon(x_\varepsilon) + (1-\lambda_\varepsilon) v_\varepsilon(y_\varepsilon)].$$

Moreover

$$\mathcal{C}_{v_\varepsilon}(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon) \le \frac{1}{\mu} \mathcal{HC}_{B_\varepsilon(\cdot, v_\varepsilon(\cdot), \xi_\varepsilon)}(x_\varepsilon, y_\varepsilon, \lambda_\varepsilon).$$

If (5.9) is uniform in ε (i.e. μ is uniform), we can pass to the limit and obtain

$$\mathcal{C}_{v}(x_0, y_0, \lambda_0) \le \frac{1}{\mu} \mathcal{HC}_{B_0(\cdot, v(\cdot), \xi_0)}(x_0, y_0, \lambda_0).$$

We need to estimate $\mathcal{HC}_{B_0(\cdot, v(\cdot), \xi_0)}(x_0, y_0, \lambda_0)$. We observe

$$B_0(x,t,\xi) = p\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t)) + (p-1)|\xi|^p\big) + p\frac{k(x,\psi(t))}{(\psi'(t))^{p-1}}.$$

For the sake of simplicity, in this Section we assume[13]

$$k(x,t) \equiv 0 \quad \text{and} \quad h(x,t) \equiv h(x). \tag{5.12}$$

We also assume

$$t \mapsto \frac{\psi''(t)}{\psi'(t)} \quad \text{harmonic concave} \tag{5.13}$$

and write

$$\frac{1}{\rho(t)} := \frac{\psi''(t)}{\psi'(t)}, \quad b_\xi(x) := p\big(h(x) + (p-1)|\xi|^p\big);$$

[13] Otherwise we can, respectively, estimate the $\mathcal{HC}_{B_0(\cdot, v(\cdot), \xi_0)}(x, y, \lambda)$ by exploiting Proposition 2.3, and make some error estimates for $h(x,t)$, or we factorize $h(x,t)$ as $c(x)h_1(x,t)$, where c has small oscillations and h_1 has some joint concavity property.

here ρ is convex and positive for $t > 0$, and b is locally bounded and positive. Namely (we omit the subscripts) we have

$$\begin{aligned}
&\mathcal{HC}_{B_0(\cdot,v(\cdot),\xi_0)}(x,y,\lambda)\\
&\equiv\frac{\frac{b(x)}{\rho(v(x))}\frac{b(y)}{\rho(v(y))}}{\lambda\frac{b(y)}{\rho(v(y))}+(1-\lambda)\frac{b(x)}{\rho(v(x))}}-\frac{b(\lambda x+(1-\lambda)y)}{\rho(\lambda v(x)+(1-\lambda)v(y))}\\
&=\frac{1}{\rho(\lambda v(x)+(1-\lambda)v(y))}\Big(b(x)b(y)\frac{\rho(\lambda v(x)+(1-\lambda)v(y))}{\lambda b(y)\rho(v(x))+(1-\lambda)b(x)\rho(v(y))}-b(\lambda x+(1-\lambda)y)\Big)\\
&\leq\frac{1}{\rho(\lambda v(x)+(1-\lambda)v(y))}\left(\frac{\mathfrak{M}^2_{\delta,\xi}}{\mathfrak{m}_{\delta,\xi}}\frac{\rho(\lambda v(x)+(1-\lambda)v(y))}{\lambda\rho(v(x))+(1-\lambda)\rho(v(y))}-\mathfrak{m}_{\xi,\delta}\right)\\
&\leq\frac{1}{\rho(\lambda v(x)+(1-\lambda)v(y))}\left(\frac{\mathfrak{M}^2_{\delta,\xi}-\mathfrak{m}^2_{\delta,\xi}}{\mathfrak{m}_{\delta,\xi}}\right)
\end{aligned}$$

where

$$\mathfrak{m}_{\delta,\xi}:=\min_{x\in\Omega_{\delta/2}}b_\xi(x)\quad \mathfrak{M}_{\delta,\xi}:=\max_{x\in\Omega_{\delta/2}}b_\xi(x).$$

Set moreover (see Step 7)

$$m_\delta:=\min_{x\in\overline{\Omega}_{\delta/2}}\rho(v(x))=\min_{x\in\Omega_{\delta/2}}\frac{\psi'(v(x))}{\psi''(v(x))}$$

we have $m_\delta > 0$, and thus

$$\mathcal{HC}_{B_0(\cdot,v(\cdot),\xi_0)}(x_0,y_0,\lambda_0)\leq\frac{1}{m_\delta}\left(2+\frac{\mathfrak{M}_{\delta,\xi}-\mathfrak{m}_{\delta,\xi}}{\mathfrak{m}_{\delta,\xi}}\right)(\mathfrak{M}_{\delta,\xi}-\mathfrak{m}_{\delta,\xi});$$

therefore

$$\mathcal{C}_v(x_0,y_0,\lambda_0)\leq\frac{1}{\mu}\frac{p}{m_\delta}\left(2+\frac{\mathfrak{M}_\delta-\mathfrak{m}_\delta}{\mathfrak{m}_\delta}\right)(\mathfrak{M}_\delta-\mathfrak{m}_\delta)$$

where

$$\mathfrak{m}_\delta:=\min_{x\in\overline{\Omega}_{\delta/2}}h(x),\quad \mathfrak{M}_\delta:=\max_{x\in\overline{\Omega}_{\delta/2}}h(x)$$

(notice indeed that $\mathfrak{M}_{\delta,\xi}-\mathfrak{m}_{\delta,\xi}=\mathfrak{M}_\delta-\mathfrak{m}_\delta$ and that $\mathfrak{m}_{\delta,\xi}\geq\mathfrak{m}_\delta$). This means

$$\mathcal{C}_v\leq\frac{1}{\mu}\frac{p}{m_\delta}\left(2+\frac{\mathfrak{M}_\delta-\mathfrak{m}_\delta}{\mathfrak{m}_\delta}\right)(\mathfrak{M}_\delta-\mathfrak{m}_\delta)\quad\text{ in }\overline{\Omega}\times\overline{\Omega}\times[0,1]. \tag{5.14}$$

Additionally, if $h\in L^\infty(\Omega)$ and $\inf_\Omega h>0$, then the maximum and minimum of h on Ω_δ can be estimated with the ones in Ω, that we call respectively $\mathfrak{M}$ and $\mathfrak{m}$

$$\mathcal{C}_v\leq\frac{1}{\mu}\frac{p}{m_\delta}\left(2+\frac{\mathfrak{M}-\mathfrak{m}}{\mathfrak{m}}\right)(\mathfrak{M}-\mathfrak{m})\quad\text{ in }\overline{\Omega}\times\overline{\Omega}\times[0,1];$$

notice that $\mathfrak{M}-\mathfrak{m}=\operatorname{osc}(h)$. Moreover, if $h\in W^{1,\infty}(\Omega)$ then

$$\mathcal{C}_v\leq\frac{1}{\mu}\frac{p}{m_\delta}\left(2+\frac{\operatorname{diam}(\Omega)|\nabla h|_\infty}{\mathfrak{m}}\right)\operatorname{diam}(\Omega)|\nabla h|_\infty\quad\text{ in }\overline{\Omega}\times\overline{\Omega}\times[0,1].$$

Theorem 5.4 (Perturbed concavity). *Let $\Omega\subset\mathbb{R}^N$, $N\geq 2$, be bounded, strongly convex and with $\partial\Omega\in C^{2,\alpha}$, and let $p\in(1,+\infty)$. Let u be a solution of* (5.1)*, and let f and φ satisfy* (f1)–(f3) (5.3)*,* (5.4)*, and* (fs)*. Assume moreover that* (5.9) *(μ uniform in ε),* (5.11)*,* (5.12)*,* (5.13) *hold. Then $v=\varphi(u)$ satisfies* (5.14)*.*

Remark 5.5. Consider the case φ negatively unbounded in the origin (e.g. the logarithm), i.e.

$$\lim_{t\to 0}\varphi(t)=-\infty.$$

Assume Ω strictly convex and with smooth boundary. By [62, Lemma 3.2] we have that the mid-concavity function $\mathcal{C}_v^m$ is nonpositive near the boundary (in the sense of (4.10)), thus it is bounded from above. If $\mathcal{C}_v^m\le 0$ we are done, otherwise it admits a (positive) maximum point $(x_0,y_0)\in\Omega\times\Omega$. Then we argue as in the general case (essentially with $\lambda_0=\frac{1}{2}$, by adapting Theorem 4.10 to $\mathcal{C}_v^m$) and obtain

$$\mathcal{C}_v^m\le \frac{1}{\mu}\frac{p}{m_\delta}\left(2+\frac{\mathfrak{M}_\delta-\mathfrak{m}_\delta}{\mathfrak{m}_\delta}\right)(\mathfrak{M}_\delta-\mathfrak{m}_\delta)\quad \text{ in } \overline{\Omega}\times\overline{\Omega}.$$

By [114, Corollary 1] we can estimate the concavity function in terms of the mid-concavity function up to a factor 2, that is

$$\mathcal{C}_v\le \frac{2}{\mu}\frac{p}{m_\delta}\left(2+\frac{\mathfrak{M}_\delta-\mathfrak{m}_\delta}{\mathfrak{m}_\delta}\right)(\mathfrak{M}_\delta-\mathfrak{m}_\delta)\quad \text{ in } \overline{\Omega}\times\overline{\Omega}\times[0,1].$$

To this relation we can apply Hyers-Ulam Theorem [72] (as in Remark 1.11).

We observe that this argument cannot be applied to the case of eigenfunctions (and logarithmic transformation): indeed, in this case $\mu=0$ and the above estimate cannot be set. See Section 6.1 for a different approach. On the other hand, different problems could be treated in this way, such as showing that the solutions of suitable perturbations of the eigenfunction equation, e.g.

$$f(x,u)=a(x)u^{p-1}+\sigma u^q,$$

with $q\in[0,p-1)$ and σ small, are almost log-concave; see e.g. [4, Proposition 3.4]. We leave the details to the interested reader. Notice that it is natural to consider, in the choice of the transformation, the biggest power, which in this case is $p-1$ (see Corollary 6.6 and [106]).

6. Applications

As we saw in Theorems 5.3 and 5.4, the machinery of the previous Sections allows to treat the case of a general g (in the spirit of [22, 113], see Remark 5.1 and Corollary 6.6 below); we focus here our attention on the case $g(t)=t^q$, that is

$$f(x,t)=a(x)t^q$$

with $q\in(0,p-1]$, and $a(x)\ge 0$. We write ($k\equiv 0$)

$$f(x,t)=\big(a(x)t^{-q_1}\big)t^{q_2}\equiv h(x,t)g(t)$$

with $q=q_2-q_1$, $q_1\in[0,p-1-q]$ and $q_2\in[q,p-1]$. When $q_2\neq p-1$, by (5.4) φ can be chosen as

$$\varphi(t)=\zeta t^\gamma \tag{6.1}$$

with $\gamma:=\frac{p-1-q_2}{p}\in(0,1)$ and $\zeta:=\zeta(p,q_2)=\left(\frac{p-1-q_2}{q_2+1}\right)^{1/p}$; for the exact concavity $\zeta>0$ plays no role, while for the perturbed concavity it is involved in the coefficients of the estimate. We leave the logarithmic case $q_2=p-1$ for later, see Remark 6.1. By (6.1) we have $\psi(t)=\frac{1}{\zeta^{1/\gamma}}t^{1/\gamma}$ and

$$\frac{\psi''(t)}{\psi'(t)}=\frac{1-\gamma}{\gamma}\frac{1}{t}$$

which is positive, decreasing, harmonic concave, and such that $t^2\frac{\psi''(t)}{\psi'(t)}=\frac{1-\gamma}{\gamma}t$ is concave. Moreover

$$\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t))-\Theta\big)=\frac{1-\gamma}{\gamma}\frac{1}{t}\left(\frac{1}{\zeta^{-q_1/\gamma}}a(x)t^{-q_1/\gamma}-\Theta\right)$$

where we recall (see (5.5)) that Θ is chosen in such a way $\frac{1}{\zeta^{-q_1/\gamma}}a(x)t^{-q_1/\gamma}-\Theta$ is far from zero. In particular the above quantity is always positive. With no loss of generality we discuss monotonicity and concavity of

$$\kappa(x,t) := \frac{1}{t}\big(a(x)t^{-q_1/\gamma}-\tilde{\Theta}\big)$$

where $\tilde{\Theta} := \zeta^{-q_1/\gamma}\Theta$.

Monotonicity: to met the conditions for (5.9) we compute

$$\partial_t \kappa(x,t) = \frac{1}{t^2}\Big(-\Big(\frac{q_1}{\gamma}+1\Big)a(x)t^{-q_1/\gamma-1}+\tilde{\Theta}\Big);$$

by assuming $q_1<\gamma$ and $\tilde{\Theta}$ sufficiently small (depending also on q_1, q_2, γ), we have the claim (both (5.7) and (5.9)). Namely, choosing $\tilde{\Theta} < \frac{1}{2}\big(\frac{q_1}{\gamma}+1\big)(\inf_{\Omega_{\delta/2}} a)\|v\|_\infty^{-q_1/\gamma-1}$ (and consequently ε small), we have

$$\partial_t \kappa(x,t) \leq -\frac{1}{2}\Big(\frac{q_1}{\gamma}+1\Big)\Big(\inf_{\Omega_{\delta/2}} a\Big)\|v\|_\infty^{-q_1/\gamma-1}$$

and thus

$$\partial_t B_\varepsilon(x,t,\xi) \leq -\frac{p}{2}\frac{1-\gamma}{\gamma}\frac{1}{\zeta}\Big(\frac{q_1}{\gamma}+1\Big)\Big(\inf_{\Omega_{\delta/2}} a\Big)\|u\|_\infty^{-q_1-\gamma} =: -\mu.$$

Concavity: to met the condition for (5.10) we discuss the joint concavity of

$$(x,t)\mapsto t^2\kappa(t) = a(x)t^{1-q_1/\gamma}-\tilde{\Theta}t;$$

clearly it is sufficient to discuss the joint concavity of

$$(x,t)\mapsto a(x)t^{1-q_1/\gamma}.$$

We restrict to $q_1\in[0,\gamma]$ and assume $x\mapsto a(x)$ θ-concave, for some $\theta\in[0,+\infty]$.

Consider first $q_1\in(0,\gamma)$: we observe that $t\mapsto t^{1-q_1/\gamma}$ is $\omega=\frac{\gamma}{\gamma-q_1}\in(0,+\infty)$ concave. By Proposition 2.2 we have $(x,t)\mapsto a(x)t^{1-q_1/\gamma}$ is jointly $(\frac{1}{\theta}+\frac{1}{\omega})^{-1}$-concave. By imposing $(\frac{1}{\theta}+\frac{1}{\omega})^{-1}=1$ we get the relation among p, θ and γ:

$$\gamma = \frac{\theta(p-1-q)}{1+\theta p},$$

$q_1=\frac{p-1-q}{1+\theta p}$, $q_2=\frac{p-1+\theta pq}{1+\theta p}$; in particular, by the restriction $q_1<\gamma$, we have $\theta>1$. If now $q_1=\gamma$, then $\omega=\infty$ and the product function is concave if $\theta=1$; in this case $\gamma=q_1=\frac{p-1-q}{1+p}$, $q_2=\frac{p-1+pq}{1+p}$. If instead $q_1=0$, then $\omega=1$ and the product function is concave if $\theta=\infty$ (i.e. a is constant); in this case $\gamma=\frac{p-1-q}{p}$, $q_2=q$. In all the above relations we have that $\gamma>0$ forces $p<q-1$.

Under these assumptions we have (5.10).

Remark 6.1. Consider now $q_2=p-1$ (possibly $p<q-1$), i.e. $q_1=p-1-q$. Then $\varphi(t)=\log(t)$ (up to constants), and $\psi(t)=e^t$, thus $\frac{\psi''(t)}{\psi'(t)}=1$ which is positive, nonincreasing, and concave. Moreover

$$\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t))-\Theta\big) = a(x)e^{-q_1 t}-\Theta.$$

The function is nonincreasing (thus we have (5.7)), but never α-concave, $\alpha>0$, if $q_1\neq 0$. Thus we assume $q_1=0$ (that is, $q_2=q=p-1$) and hence $\frac{\psi''(t)}{\psi'(t)}\big(h(x,\psi(t))-\Theta\big)=a(x)-\Theta$ which is (jointly) concave if $a(x)$ is so. Again, under these assumptions we have (5.10).

We are now ready to prove the main theorems.

Proof of Theorem 1.4. The conclusion comes from the previous observations and Theorem 5.3. □

Proof of Corollary 1.8. Since the problem is linear, we have

$$-\Delta(u_n - u) = a_n(x) - a_\infty \quad \text{in } \Omega$$

with $u_n - u = 0$ on $\partial\Omega$. It is well known that [52, Theorem 2.30]

$$\|u_n - u\|_{C^{2,\alpha}} \le C\|a_n - a\|_{C^{0,\alpha}};$$

thus, in particular, $u_n \to u$ in $C^2(\Omega)$. Hence by Theorem 1.1, Corollary 3.6 and Proposition 2.5, we have the claim. $\square$

Arguing as in Corollary 1.8, a weaker version can be stated also in general convex domains.

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 2$, be open, bounded, convex, and let $p = 2$. Consider $(a_n)_n : \Omega \to \mathbb{R}$, and assume that, for some $a_\infty > 0$ constant*

$$a_n \to a_\infty \quad \text{in } C^{0,\alpha}(\Omega) \text{ as } n \to +\infty.$$

Then, for each $\delta > 0$, the positive solution u_n of (1.6) is such that u_n is strongly $\frac{1}{2}$-concave in Ω_δ, for $n \ge n_0(\delta) \gg 0$. In particular, for these values of n, the level sets of u_n are strictly convex in Ω_δ and u_n has a single (and nondegenerate) critical point in Ω_δ.

A weaker version of Corollary 1.8 can be stated also in the quasilinear setting.

Corollary 6.3. *Let $\Omega \subset \mathbb{R}^2$, be open, bounded, and strictly convex, with $\partial\Omega \in C^{1,\alpha}$, and let $p \in (1, +\infty)$. Consider $(a_n)_n : \Omega \to \mathbb{R}$, equibounded in $C^{0,\alpha}(\Omega)$, $\alpha \in (0,1)$, and assume that, for some $a_\infty > 0$ constant*

$$a_n \to a_\infty \quad \text{in } L^\infty(\Omega) \text{ as } n \to +\infty.$$

Then, for any $\varepsilon > 0$, the positive solution u_n of

$$\begin{cases} -\Delta_p u_n = a_n(x) & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

is such that u_n is ε-uniformly $\frac{1}{2}$-concave for $n \ge n_0(\varepsilon) \gg 0$.

Proof. By Theorem 1.3 we have that $v_\infty = \sqrt{u_\infty}$ is strictly concave in Ω. By Hopf boundary lemma [122, Theorem 5], and Corollary 4.4 we observe that, being Ω strictly convex, $\mathcal{C}_{v_\infty}(x, y, \frac{1}{2})$ cannot be zero if $|x - y| \ge \varepsilon$. Thus v_∞ is strictly concave in $\overline{\Omega}$, and the claim follows by Corollary 3.6 and Proposition 2.6. $\square$

We propose now another application of Corollary 3.6, where a_2 is nonconstant. Similar statements hold for the Examples 1.6.

Corollary 6.4 (Hardy-Hénon type equation)**.** *Let Ω open, bounded and convex be a subset of $\{x \in \mathbb{R}^N \mid x_i > 0 \text{ for each } i\}$, $N \ge 2$, and let $p \in (1, +\infty)$. Consider $\omega \in [0,1]$ and the positive solutions u and v of*

$$\begin{cases} -\Delta_p u = |x|_2^\omega & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \qquad \begin{cases} -\Delta_p v = |x|_1^\omega & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases}$$

Then $v^{\frac{p-1}{\omega+p}}$ is concave (strictly, if $p = 2$) and

$$\|u^{\frac{p-1}{\omega+p}} - v^{\frac{p-1}{\omega+p}}\|_\infty \le C\left((N^\omega - 1)\|\max\{|x|_2^\omega, |x|_1^\omega\}\|_{L^\infty(\Omega)}\right)^{\kappa_{p^*}\frac{1}{\omega+p}}$$

if $p \ge 2$, $C = C(p, \Omega, \||x|_1^\omega\|_\infty, \||x|_2^\omega\|_\infty) > 0$, while

$$\|u^{\frac{p-1}{\omega+p}} - v^{\frac{p-1}{\omega+p}}\|_\infty \le C\left((N^\omega - 1)\|\max\{|x|_2^\omega, |x|_1^\omega\}\|_{L^\infty(\Omega)}\right)^{\kappa_{2^*}\frac{p-1}{\omega+p}}$$

if $p \le 2$ and $\partial\Omega \in C^{1,\alpha}$, $C = C(p, \Omega, \||x|_1^\omega\|_{C^{0,\omega}}, \||x|_2^\omega\|_{C^{0,\omega}}) > 0$. Here $|x|_1$ is the 1-norm in $\mathbb{R}^N$ and $|x| = |x|_2$ is the Euclidean norm. In particular the difference goes to zero as $\omega \to 0$.

Proof. By the equivalence of the norms in $\mathbb{R}^N$ we know that $|x|_1 \le N|x|_2 \le N^{3/2}|x|_1$, thus

$$\||x|_2^\omega - |x|_1^\omega\|_{L^\infty(\Omega)} \le (N^\omega - 1)\|\max\{|x|_2^\omega, |x|_1^\omega\}\|_{L^\infty(\Omega)}.$$

The claim comes from Example 1.6 and Corollary 3.6. ☐

Remark 6.5. We notice that, for each $\omega \in (0,1]$, highlighting the dependence on ω, we have that $v_\omega^{\frac{p-1}{\omega+p}}$ is concave, and moreover $\|v_\omega^{\frac{p-1}{\omega+p}}\|_\infty \le C$. Thus by [115, Theorem 10.9], up to a subsequence, we have $v_\omega^{\frac{p-1}{\omega+p}} \to \bar{v}$ as $\omega \to 0$ in $L^\infty_{loc}(\Omega)$, for some $\bar{v}$ concave: by Corollary 6.4 we thus obtain that, up to a subsequence, $u_\omega^{\frac{p-1}{\omega+p}}$ converge to a concave function as $\omega \to 0$.

Proof of Theorem 1.9. The conclusion comes from the observations at the beginning of the Section and Theorem 5.4. In particular, by choosing $q_1 = 0$ we obtain ($v = u^\gamma$)

$$\mathcal{C}_{u^\gamma}(x_0, y_0, \lambda_0) \le p\frac{1}{\mu_{a,u}}\frac{1}{m_{\delta,u}}\left(2 + \frac{\mathrm{osc}(a)}{\mathfrak{m}_\delta}\right)\mathrm{osc}(a)$$

where $\gamma = \frac{p-1-q}{p}$, $\zeta = \left(\frac{p-1-q}{q+1}\right)^{1/p}$ and

$$m_{\delta,u} := \frac{\gamma}{1-\gamma}\min_{\overline{\Omega_{\delta/2}}} u(x), \qquad \mathfrak{m}_\delta := \min_{x \in \overline{\Omega_{\delta/2}}} a(x), \quad \mu_{a,u,\delta} := \frac{p}{2}\frac{1-\gamma}{\gamma}\frac{1}{\zeta}\mathfrak{m}_\delta\|u\|_\infty^{-\gamma}.$$

That is

$$\mathcal{C}_{u^\gamma}(x_0, y_0, \lambda_0) \le \zeta\left(\frac{\|u\|_\infty}{\min_{\overline{\Omega_{\delta/2}}} u(x)}\right)^\gamma \left(2 + \frac{\mathrm{osc}(a)}{\mathfrak{m}_\delta}\right)\frac{\mathrm{osc}(a)}{\mathfrak{m}_\delta}.$$

☐

Proof of Corollary 1.10. First we observe that

$$\|a_n - a_\infty\|_\infty = O(\mathrm{osc}(a_n)).$$

Moreover, we have, by the equiboundedness of $\|a_n\|_\infty$ and regularity theory [30, Theorem 2.1], $\|u_n\|_\infty \le C$ and, by Corollary A.3, $\min_{\overline{\Omega_{\delta/2}}} u_n \ge C > 0$. The claim thus follows by Theorem 1.9. ☐

As a final result in this Section, we show an application in the case $k \not\equiv 0$ (see (5.3)), which allows to include some cases covered by [88], but not by [22] (see Remark 5.1). Namely, consider

$$f(x,t) \equiv g(t) + k(t),$$

and as an example, we focus on

$$f(x,t) = t^q + t^r$$

with $q, r \in [0, p-1)$. We thus have (up to constants) $\varphi(t) = t^{\frac{p-q-1}{p}}$. To gain

$$t \mapsto \frac{k(\psi(t))}{(\psi'(t))^{p-1}} \quad \text{nonincreasing}, \quad t \mapsto t^2\frac{k(\psi(t))}{(\psi'(t))^{p-1}} \quad \text{concave},$$

we need

$$t \mapsto t^{\frac{pr-(p-1)q-(p-1)}{p-q-1}} \quad \text{nonincreasing}, \quad t \mapsto t^{\frac{pr-(p+1)q+(p-1)}{p-q-1}} \quad \text{concave},$$

which is given by

$$r \le q, \quad (p+1)q - pr \le p-1.$$

Notice that for $p = 2$ the second relation gives $3q - 2r \le 1$, which actually is the condition to impose in order to have

$$t \mapsto t^\gamma f(t) \quad \text{strictly decreasing}, \quad t \mapsto t^{\frac{3\gamma-1}{\gamma}} f(t^{1/\gamma}) \quad \text{concave}, \tag{6.2}$$

$\gamma = \frac{1-q}{2}$, requested in [88, Theorem 3.3]. We have thus shown the following result.

Corollary 6.6. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded and convex, and let $p \in (1, +\infty)$. Let $q, r \in (0, p-1)$ and u be a positive solution of*

$$\begin{cases} -\Delta_p u = u^q + u^r & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

with $r \leq q$ and

$$(p+1)q - pr \leq p - 1.$$

Then $u^{\frac{p-1-q}{p}}$ is concave.

We highlight that it is the bigger exponent who leads the choice of the transformation function. Moreover, it remains open to determinate if the restrictions on q, r in Corollary 6.6 are sharp.

6.1. Weighted eigenfunctions

We consider

$$f(x,t) = a(x)|t|^{p-2}t.$$

The approximation argument has been commented in Remark 5.2, while exact concavity has been considered in Remark 6.1. We give now some insights on perturbed concavity.

When $p = 2$ and $\varphi(t) = \log(t)$ the equation solved by $v = \varphi(u)$ is given by

$$-\Delta v = |\nabla v|^2 + 1;$$

thus even in the semilinear case we see that the main issue is given by the fact that the nonlinearity of the transformed equation does not have a derivative in t far from zero. This problem has been tackled in different ways: in [4, Proposition 3.4] the authors study a perturbed equation, with an additional term which allows to gain the desired assumption on the nonlinearity (see Remark 5.5). In [29, Proposition 2.8] they consider instead an approximation process, by staying far from the boundary; this approach seems a bit heavy to be implemented in a nonregular quasilinear setting, since a double approximation process should be set up. Another possibility is to pass to the parabolic equation, where a trick in time could be employed, see Section 7.4.

Here we propose a different approach, by estimating the concavity function of the transformation

$$\varphi_\sigma(t) := \log(u - \sigma)$$

in the set where it is well defined, i.e. $\{u > \sigma\}$; notice that, since we cannot ensure the quasiconcavity of u, this set is generally not a priori convex. Near the boundary we have instead an information on $\log(u)$. For the sake of simplicity we present the argument for $p = 2$, but it can be easily adapted to any $p \in (1, +\infty)$.

Theorem 6.7. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded, $\partial\Omega \in C^{2,\alpha}$, and strictly convex. Then there exists $\sigma_0 > 0$ such that, for each $\sigma \in (0, \sigma_0]$, set*

$$\Lambda_\sigma := \{u > \sigma\}$$

we have[14]

$$\mathcal{C}^m_{\log(u)}(x,y) \leq \max_{\Lambda_\sigma \times \Lambda_\sigma} \mathcal{C}^m_{\log(u)} \quad \textit{for } (x,y) \in \big(\Omega \times \Omega\big) \setminus \big(\Lambda_\sigma \times \Lambda_\sigma\big)$$

and

$$\mathcal{C}_{\log(u-\sigma)}(x,y,\lambda)(x,y) \leq \frac{1}{\sigma} C_{a,u,\delta} \operatorname{osc}(a)$$

for any $(x,y) \in \Lambda_\sigma \times \Lambda_\sigma$ with $[x,y] \subset \Lambda_\sigma$ and $\lambda \in [0,1]$, where $C_{a,u,\delta} := e^{\|u\|_\infty}\left(2 + \frac{\operatorname{osc}(a)}{\mathfrak{m}_\delta}\right)\frac{1}{\mathfrak{m}_\delta}$.

[14] Notice that, arguing as in Remark 5.5 we have also

$$\mathcal{C}_{\log(u)}(x,y,\lambda) \leq 2 \max_{\Lambda_\sigma \times \Lambda_\sigma} \mathcal{C}_{\log(u)} \quad \text{for } (x,y,\lambda) \in \big(\Omega \times \Omega\big) \setminus \big(\Lambda_\sigma \times \Lambda_\sigma\big) \times [0,1].$$

Proof. By [62, Lemma 3.2] (see also (4.10)) $\mathcal{C}^m_{\log(u)}$ does not attain a maximum near the boundary $\partial(\Omega\times\Omega)$. Thus there exists Ω_{δ_0} such that

$$\mathcal{C}^m_{\log(u)}(x,y) \le \max_{\Omega_{\delta_0}\times\Omega_{\delta_0}} \mathcal{C}^m_{\log(u)} \quad \text{for } (x,y)\in \big(\Omega\times\Omega\big)\setminus\big(\Omega_{\delta_0}\times\Omega_{\delta_0}\big).$$

For σ sufficiently small we have $\Omega_{\delta_0}\subset\Lambda_\sigma$. Consider the transformation $\varphi_\sigma:(\sigma,+\infty)\to(-\infty,\|u\|_\infty]$, $\varphi_\sigma(t):=\log(t-\sigma)$, with inverse $\psi_\sigma:(-\infty,\|u\|_\infty]\to(\sigma+\infty)$ given by $\psi_\sigma(t)=e^t+\sigma$. Set $v_\sigma:=\varphi_\sigma(u)$. The function v_σ satisfies

$$-\Delta v_\sigma = \frac{\psi''_\sigma(v_\sigma)}{\psi'_\sigma(v_\sigma)}|\nabla v_\sigma|^2 + \frac{f(x,\psi_\sigma(v_\sigma))}{\psi'_\sigma(v_\sigma)}$$

where $\frac{\psi''_\sigma(t)}{\psi'_\sigma(t)}\equiv 1$, which is positive and nonincreasing, and

$$\frac{f(x,\psi_\sigma(t))}{\psi'_\sigma(t)} = a(x)\frac{e^t+\sigma}{e^t} = a(x)\left(1+\sigma e^{-t}\right)$$

which is positive and decreasing in t. In particular $\partial_t\Big(\frac{f(x,\psi_\sigma(t))}{\psi'_\sigma(t)}\Big) = -\sigma a(x)e^{-t}$ and thus

$$\partial_t\left(\frac{f(x,\psi_\sigma(t))}{\psi'_\sigma(t)}\right) \le -\sigma e^{-\|u\|_\infty}\mathfrak{m}_\delta =: -\mu_\sigma < 0$$

which implies, by Theorem 4.10,

$$\mathcal{C}_{\log(u-\sigma)}(x,y,\lambda)(x,y,\lambda) \le \frac{1}{\mu_\sigma}\left(2+\frac{\operatorname{osc}(a)}{\mathfrak{m}_\delta}\right)\operatorname{osc}(a)$$

for each $[x,y]\subset\{u>\sigma\}$. □

6.2. Singular equations

We consider now singular equations, that is $q<0$.[15] To deal with this case, we need to introduce a regularized problem (with respect to the source), apply previous results and then pass to the limit. We start by recalling the following result [37, Theorems 1.3 and 1.5] (see therein the definition of solution); see also [38, Theorem 2.2] and [57, Teorema 2.3] for the case $a(x)\equiv 1$ and [20] for the case $p=2$.

Theorem 6.8 ([37])**.** *Let $\Omega\subset\mathbb{R}^N$, $N\ge 2$, be an open, bounded and $\partial\Omega\in C^{2,\alpha}$, and let $p\in(1,+\infty)$. Assume $q\in(-\infty,0)$ and $a\in L^\infty(\Omega)$. Then there exists a unique solution u of (1.4). If $q\in[-1,0)$ then $u\in W^{1,p}_0(\Omega)$, while if $q\in(-\infty,-1)$ then $u^{\frac{p-1-q}{p}}\in W^{1,p}_0(\Omega)$.*

We focus on the case $a(x)\equiv 1$ and $q\in[-1,0)$. Consider, for any $\eta>0$, the regularized equation

$$\begin{cases} -\Delta_p u_\eta = (u_\eta+\eta)^q & \text{in } \Omega,\\ u_\eta>0 & \text{in } \Omega,\\ u_\eta = 0 & \text{on } \partial\Omega,\end{cases}$$

so that we have

$$f(x,t)\equiv g(t) = (t+\eta)^q.$$

In this case, it is hard to deduce concavity properties of powers of u_η, which makes the approach due to [88, 90] difficult to apply. We will base the proof thus on the abstract approach developed here and in [22]. We obtain the following result.

Theorem 6.9. *Let $\Omega\subset\mathbb{R}^N$, $N\ge 2$, be an open, bounded, convex and $\partial\Omega\in C^{2,\alpha}$, and let $p\in(1,+\infty)$. Let $u\in W^{1,p}_0(\Omega)$ be the solution of (1.4) with $q\in[-1,0)$ and $a(x)\equiv 1$. Then $u^{\frac{p-1-q}{p}}$ is concave.*

---

[15] We thank Lorenzo Brasco for having brought to our attention this problem.

Proof. By [37, Lemma 4.5 and proof of Theorem 4.6] we have that

$$u_\eta \to u \quad \text{in } W^{1,p}_0(\Omega);$$

in particular almost everywhere and u_η is dominated.

Assume first $q \in (-1, 0)$. We see that $g(t) = (t + \eta)^q$ satisfies the assumptions of Section 5.1 (see in particular Remark 5.1), since

$$G(t) = \frac{(t + \eta)^{q+1} - \eta^{q+1}}{q + 1}$$

and thus, being $q \in (-1, 0)$, $G^{1/p}$ is concave and $\frac{g}{G}$ is harmonic concave (since $\frac{G}{g}$ is convex). Therefore, for each $\eta > 0$ we have $\varphi_\eta(u_\eta)$ concave, where

$$\varphi_\eta(t) := \int_1^t \left((s + \eta)^{q+1} - \eta^{q+1}\right)^{-1/p} ds;$$

we notice that φ_η cannot be expressed in terms of elementary functions. By $u_\eta \to u$ (notice that $\left((s + \eta)^{q+1} - \eta^{q+1}\right)^{-1/p} \le s^{-\frac{q+1}{p}}$) we have (up to constants)

$$\varphi_\eta(u_\eta) \to u^{\frac{p-1-q}{p}}$$

almost everywhere. Thus $u^{\frac{p-1-q}{p}}$ is concave.

Consider now $q = -1$; we have

$$G(t) = \log(t + \eta) - \log(\eta)$$

and again $G^{1/p}$ is concave and $\frac{g}{G}$ is harmonic concave. Therefore, for each $\eta > 0$ we have $\varphi_\eta(u_\eta)$ concave, where

$$\varphi_\eta(t) := \int_1^t (\log(s + \eta) - \log(\eta))^{-1/p} ds;$$

again φ_η does not have an elementary expression. Being (observe that $\frac{\log(\frac{1}{\eta})}{\log(s+\eta)-\log(\eta)} \le C_t$ for $s \in (t, +\infty)$)

$$\left(\log(\tfrac{1}{\eta})\right)^{\frac{1}{p}} \varphi_\eta(t) \sim t - 1 \quad \text{as } \eta \to 0$$

we have (up to constants)

$$\left(\log(\tfrac{1}{\eta})\right)^{\frac{1}{p}} \varphi_\eta(u_\eta) \to u$$

and hence u is concave (notice $\frac{p-1-q}{p} = 1$ when $q = -1$). $\square$

Remark 6.10. Being $\varphi_\eta(u_\eta)$ concave on the bounded set Ω, converging pointwise to $u^{\frac{p-1-q}{p}}$, then a known result [115, Theorem 10.8] automatically implies that $\varphi_\eta(u_\eta) \to u^{\frac{p-1-q}{p}}$ in $L^\infty_{loc}(\Omega)$.

Remark 6.11. (i) The case $q < -1$, among the others, was treated in the semilinear framework $p = 2$, in a different way by [15], by dealing with the boundary through a direct use of the equation itself and a maximum principle on the function $x \mapsto |\nabla u|^2 + \frac{2}{q+1}u^{q+1}$ (see [15, Lemmas 3.1 and 3.3]). In our setting, the range $q < -1$ creates more difficulties for the applicability of the concavity results; in particular $t \mapsto t^{\frac{p-1-q}{p}}$ is no more concave with singular derivative.

We notice that also regularity issues arise when $q < -1$ (see Theorem 6.8 and Proposition 6.12 below). In particular, the solution does not generally belong to $W^{1,p}_0(\Omega)$, but $u^{\frac{p-1-q}{p}}$ does, where the power is exact the same of the expected concavity.

(ii) Consider now $a(x)$ nonconstant. To apply the arguments of Section 5.1, the very first thing to check is if $t \mapsto t^2\frac{\psi''(t)}{\psi'(t)}$ is concave, which is more difficult than showing $t \mapsto \frac{\psi'(t)}{\psi''(t)}$ convex (as done in

[22]); both leads to the harmonic concavity of $\frac{\psi''(t)}{\psi'(t)}$, but in our case we need the above property to deal with sums (see (5.10)). We have

$$t^2\frac{\psi''(t)}{\psi'(t)} = \left(\varphi^2 G^{1/p}\tfrac{g}{G}\right)(\psi(t))$$

which seems not easy to handle (recall that, in our case, φ and ψ are not explicit).

With the idea of a possible future development (in the directions $a \not\equiv const$ or $q < -1$), we recall and show the following results.

Proposition 6.12 (Boundary regularity)**.** *Let $\Omega \subset \mathbb{R}^N$ be open and bounded. Let u be a solution (1.4). Then the following results hold.*
- *Case $q \in (-\infty, 0)$ and $a(x) \equiv 1$: assume $\partial\Omega \in C^{0,1}$. Then $u \in C(\overline{\Omega})$.*
- *Case $q \in (-1, 0)$ and $\inf_\Omega a > 0$: assume $\partial\Omega \in C^{2,\alpha}$. Then $u \in C^{1,\alpha}(\overline{\Omega})$.*

Proof. The case $q \in (-\infty, 0)$ with $a(x) \equiv const$ is contained in [38, Corollary 2.4]. Focus now on $q \in (-1, 0)$; first, we observe that $u \in L^\infty(\Omega)$ [20, Lemma 5.5 and Remark 1.1]. Let $\underline{u} \in C^{1,\alpha}(\overline{\Omega})$ be a positive solution of the Dirichlet problem

$$-\Delta_p \underline{u} = \delta \quad \text{in } \Omega$$

for some $\delta \in (0, 1)$ chosen in such a way

$$\|\underline{u}\|_\infty \leq (\inf a)^{-\frac{1}{q}}.$$

Arguing as in [65, Section 2.3.3, Claim 1] there exists $k > 0$ such that

$$\underline{u}(x) \geq k d(x, \partial\Omega) \quad \text{for } x \in \Omega.$$

Being by construction

$$-\Delta_p \underline{u} = \delta \leq 1 \leq a(x)\underline{u}^q$$

and $u \in C(\overline{\Omega})$, thanks to the comparison principle [111, Theorem 1.5] we obtain $\underline{u} \leq u$, and thus

$$u^q \leq \underline{u}^q \leq k^q d^q.$$

We have thus the claim by [67, Lemma 3.1]. ☐

Finally we recall a Hopf boundary lemma. By the unique nearest point property, for a C^2 domain we can define near the boundary a generalized normal vector as

$$\nu(x) := \frac{x - \hat{x}}{|x - \hat{x}|}$$

where $\hat{x}$ is the unique point on $\partial\Omega$ nearest to x. By [122, Theorem 5] (see also [116, Lemma A.3]), [36, Theorem 1.2] and [57, Corollario 2.3] we have the following result.

Proposition 6.13 (Hopf boundary Lemma)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$. Let $u \in C(\overline{\Omega})$ be a solution of (1.4).*
- *Case $q \in [-1, 0)$ and $a \geq 0$: assume Ω satisfies the interior ball condition and $u \in C^1(\overline{\Omega})$. Then*

$$\partial_\nu u(x) > 0 \quad \textit{for each } x \in \partial\Omega.$$

Moreover, if $q = -1$ and $a(x) \equiv 1$,

$$\lim_{x \to \partial\Omega} \frac{u(x)}{\psi(d(x, \Omega))} = C_p$$

for some constant $C_p > 0$ depending only on p, and ψ is the inverse of $\varphi(t) = \int_0^t (-\log(\tau))^{-\frac{1}{p}}$.[16]

[16] Notice that $\log(t)$ is the primitive of t^q when $q = -1$.

• *Case $q \in (-\infty, -1)$ and $a(x) \equiv 1$: assume $\partial\Omega \in C^{2,\alpha}$ and $u \in C^{2,\alpha}(\Omega)$. Then there exists $\delta > 0$ sufficiently small such that*

$$\partial_\nu u(x) > 0 \quad \text{for each } x \in \Omega \setminus \Omega_\delta.$$

Moreover

$$\lim_{x \to \partial\Omega} \frac{u(x)}{(d(x,\Omega))^{\frac{p}{p-1-q}}} = C_{p,q}$$

for some explicit constant $C_{p,q} > 0$ depending only on p and q.

7. Further results

In this Section we furnish several results in different frameworks: the techniques mainly rely on perturbation arguments, based on fine estimates.

7.1. Superhomogeneous equations

In the superhomogeneous case $f(x,t) = g(t) = t^q$ with $q > 1$, quasiconcavity was already conjectured by Sacks (see [80, Remark 9]). As for the semilinear case [98, 104], we expect the solutions to be $\frac{p-1-q}{p}$-concave (i.e. $u^{-\frac{q-p+1}{p}}$ is convex) in Ω for any $q \in (p-1, p^*-1)$; notice that this is weaker than the log-concavity.

When $p = 2$ we see that the equation solved by $v = \varphi(u) = -u^{-\frac{q-1}{2}}$ (which we want to show being concave) is

$$-\Delta v = -\frac{1}{v}\left(\frac{q+1}{q-1}|\nabla v|^2 + \frac{q-1}{2}\right);$$

thus even in the semilinear case we see that the nonlinearity does not satisfy the basic assumption on the monotonicity of Theorem 4.9, i.e. it is not nonincreasing. Notice anyway that, being negative, it is harmonic concave by direct definition, and moreover $\lim_{t\to 0} \varphi'(t) = +\infty$ (see Corollary 4.4).

The strategy employed by [104] is the following: the author exploits the strong log-concavity of the first eigenfunction to show that for q near 1 the solutions are quasiconcave, while through a continuation argument it is shown to hold for each $q > 1$ (when the solution is a ground state). We give details below (see Remark 7.2). We refer also to [98, Lemma 4.8] where an evolutive argument has been used (see also [89, 106]).

We know by [104, Lemma 3] (see also [25, Proposition 4.3]) that when $p = 2$ and $q > 1$, q close to 1, then there exists a unique positive solution when Ω is bounded and convex. For a general $p > 2$, in [26, Corollary 1.4] they recently showed the uniqueness of the positive ground state when $q > p-1$, q close to $p-1$, and Ω is connected and $\partial\Omega \in C^{1,\alpha}$. In a similar setting, here we show concavity properties of a general solution of the equation, through a uniform convergence. This convergence partially answers also to a question raised in [26, pages 3545 and 3456].

Theorem 7.1. *Let $\Omega \subset \mathbb{R}^N$ $N \geq 2$, be open, bounded and Lipschitz, and let $p \in (1, +\infty)$. Consider*

$$\begin{cases} -\Delta_p u_q = u_q^q & \text{in } \Omega, \\ u_q > 0 & \text{in } \Omega, \\ u_q = 0 & \text{on } \partial\Omega, \end{cases}$$

with $q > p-1$, $q \to p-1$. Then $v_q := \frac{u_q}{\|u_q\|_\infty} \to u$ in $L^\infty(\Omega)$, where u is the first eigenfunction of the p-Laplacian in Ω, normalized with $\|u\|_\infty = 1$.

Assume now Ω convex. Then $\log(u)$ is concave and

$$\|\log(v_q) - \log(u)\|_\infty \to 0 \quad \text{as } q \to p-1$$

together with

$$\log(v_q) \to \log(u) \quad \text{in } L^\infty(\Omega_\delta)$$

for each $\delta > 0$. As a consequence:

- *if $N=2$, then for each $\varepsilon>0$ we have $\log(u_q)$ is ε-uniform concave in Ω_δ for $q\in(p-1,q_0(\varepsilon,\delta))$.*
- *if $p=2$, then $\log(u_q)$ is strongly concave in Ω_δ, for $q\in(p-1,q_0(\delta))$; in particular, the level sets of u_q in Ω_δ are strictly convex and u_q has a single (and nondegenerate) critical point in Ω_δ.*

Proof. By [121, Corollary 4.2] (see also [38, Corollary 3.7]) we have $u_q\in C(\overline{\Omega})$. Set

$$M_q:=\|u_q\|_\infty,\quad v_q:=\frac{u_q}{M_q}>0,$$

which solve

$$\begin{cases}-\Delta_p v_q=M_q^{q-p+1}v_q^q & \text{in } \Omega,\\ v_q=0 & \text{on } \partial\Omega.\end{cases}\tag{7.1}$$

Notice that $\|v_q\|_\infty=1$. We want to show that M_q^{q-p+1} is equibounded by a blow-up argument. Assume by contradiction that $M_q^{q-p+1}\to+\infty$; let us set

$$u_q^*(x):=\frac{1}{M_q}u_q(\varepsilon_q x+x_q),\quad x\in\Omega$$

where $\varepsilon_q:=M_q^{-\frac{q-p+1}{p}}\to 0$, $x_q\in\Omega$ such that $u_q(x_q)=\|u_q\|_\infty$. We clearly have $u_q^*(0)=1$, $\|u_q^*\|_\infty=1$. Moreover $u_q^*>0$ solve

$$\begin{cases}-\Delta_p u_q^*=(u_q^*)^q & \text{in } \Omega_q,\\ u_q^*=0 & \text{on } \partial\Omega_q,\end{cases}$$

where $\Omega_q:=\frac{1}{\varepsilon_q}(\Omega-x_q)$. By [121, Corollary 4.2] and Ascoli-Arzelà theorem we have that u_q^* is locally equibounded in $C^{0,\alpha}_{loc}(\Lambda)$ and converges in $L^\infty_{loc}(\Lambda)$ to a function $u^*\geq 0$ which satisfies

$$-\Delta_p u^*=(u^*)^{p-1}\quad \text{in } \Lambda$$

where $\Lambda=\mathbb{R}^N_+$ or $\Lambda=\mathbb{R}^N$, depending on the fact that x_q approaches or not $\partial\Omega$. Since $u^*(0)=1>0$, by the strong maximum principle [122, Theorem 5] we have that $u^*>0$. But this is in contradiction with [74, Lemma 2.1]. Thus M_q^{q-p+1} is equibounded.

Let us come back to (7.1). Arguing as before, one can show that, up to a subsequence, v_q converge in $L^\infty(\Omega)$ to some u, together with $M_q^{q-p+1}\to\lambda\in\mathbb{R}$, which thus satisfy

$$\begin{cases}-\Delta_p u=\lambda u^{p-1} & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega.\end{cases}$$

Being $\|u\|_\infty=1$, we have $u\not\equiv 0$, thus $v>0$ by the strong maximum principle. Hence u must coincide with the first eigenfunction, and λ with the first eigenvalue. By a classical topological argument, we see that the whole family v_q converges to u.

The ε-uniform concavity comes from Theorem 1.3 and Proposition 2.6. The case $p=2$ is already known, but let us give here some details: by the uniform estimates with respect to q (see also Remark 7.3 below) and regularity theory for the classical Laplacian [52, Theorem 2.30], we know that $\log(u_q)\to\log(u)$ in $C^2(\Omega_\delta)$. Moreover, by Theorem 1.1 we have the strong concavity of $\log(u)$, and thus the claim by Proposition 2.5. $\square$

Remark 7.2. As already mentioned, similarly to the semilinear case, we expect the solutions to be $\frac{p-1-q}{p}$-concave for any $q\in(p-1,p^*-1)$, which is weaker than the log-concavity. Anyway, to conclude as in [104], we would need the following three ingredients:

(i) the first eigenfunction u is strongly log-concave and $v_q\to u$ in $C^2_{loc}(\Omega)$;
(ii) the map $q\in(p-1,p^*-1)\mapsto u_q$ is well defined (that is, u_q is unique) and continuous (in C^2_{loc} topology, see also Remark 7.3 below);

(iii) an Hessian constant rank theorem in the spirit of [91] holds, which allows to obtain strict concavity (strongly far from the boundary) from concavity.

At this point the argument would run as follow:[17] fixed $q_0 \in (p-1, p^*-1)$, one preliminary shows that u_q is $\frac{p-1-q}{p}$-concave in some $\Omega \setminus \Omega_{2\delta}$, where $\delta > 0$ does not depend on $q \in (p-1, q_0)$ (see Remark 7.3 below and the argument in Remark 4.8). Then by (i) one shows that $\log(u_q)$ is strictly concave in Ω_δ for $q \in (p-1, q(\delta))$, thus u_q is strictly $\frac{p-1-q}{p}$-concave in Ω_δ. Then one defines $E := \{q \in (p-1, q_0) \mid u_q \text{ is } \frac{p-1-q}{p}\text{-concave in } \Omega_\delta\}$ which satisfies $(p-1, q(\delta)) \subset E$ (in particular, $E \neq \emptyset$). By contradiction $E \subsetneq (p-1, q_0)$, thus there exists a boundary point q^* of E in $(p-1, q_0)$; by using only the continuity of $u \mapsto u_q$ in L^∞_{loc} in (ii) one observes that $q^* \in E$. Then, thanks to (iii), u_{q^*} is shown to be strictly $\frac{p-1-q^*}{p}$-concave in Ω_δ, and strongly in $\Omega_{2\delta}$. By the absurd assumption there exist $q_n \in (p-1, q_0) \setminus E$, $q_n \to q^*$, from which one deduces by the C^2_{loc} continuity in (ii) that u_{q_n} is strongly $\frac{p-1-q_n}{p}$-concave in $\Omega_{2\delta}$ for large n; but by the choice of δ it is $\frac{p-1-q_n}{p}$-concave in the whole Ω_δ, and this is a contradiction, since $q_n \notin E$. By the arbitrariness of q_0, we have the claim. The concavity can then be further improved again by (iii).

Due to (i)–(iii) this recipe seems not the right way to proceed for the quasilinear framework (see also comments in [23] regarding (iii)).

Remark 7.3. In view of possible developments, we recall the following uniform estimates by [26, Theorems 1.1 and 2.5, Proposition 2.4]: let u_q be positive solutions of

$$\begin{cases} -\Delta_p u_q = \lambda u_q^q & \text{in } \Omega, \\ u_q = 0 & \text{on } \partial\Omega, \end{cases}$$

where $\lambda > 0$ and $q \in (p-1, p^*-1)$. Assume Ω to be open and bounded, and fix $q_0 \in (p-1, p^*-1)$. Then the following properties hold.

(i) There exists $C = C(N, p, q_0) > 0$ such that

$$\|u_q\|_{L^\infty(\Omega)} \leq C \left(\lambda^{\frac{N}{p(q+1)}} \|u_q\|_{L^{q+1}(\Omega)} \right)^{\frac{p(q+1)}{p(q+1)-(q-p+1)N}}$$

for each $q \in [p-1, q_0]$.

(ii) Assume Ω with $C^{1,\alpha}$ boundary, and let $\lambda = \lambda_q$, where

$$\lambda_q := \inf \left\{ \int_\Omega |\nabla u|^p \mid \int_\Omega |u|^{q+1} = 1 \right\},$$

and let u_q be a positive minimizer (i.e. a ground state). Then there exists $L, \beta, \delta, \mu_0, \mu_1$ positive and depending on $N, p, q_0, \alpha, \Omega$ such that

$$\|u_q\|_{C^{1,\beta}(\overline{\Omega})} \leq L, \qquad |\nabla u_q| \geq \mu_0 \quad \text{in } \Omega \setminus \Omega_\delta, \qquad u_q \geq \mu_1 \quad \text{in } \overline{\Omega_\delta}$$

for each $q \in [p-1, q_0]$. In particular, if $x_q \in \overline{\Omega}$ is a maximum point for u_q, then $x_q \in \Omega_\delta$ for each $q \in [p-1, q_0]$.

(iii) The set $Z_q := \{x \in \Omega \mid \nabla u_q(x) = 0\}$ is compact and with zero measure. Moreover, $u_q \in C^2(\Omega \setminus Z_q)$. Recall that, when $N = 2$ and $\partial\Omega \in C^2$, by Theorem 1.3 we have $\#Z_1 = 1$.

(iv) For $q > p-1$, $q \approx p-1$, the minimizer u_q is unique. Assume now that, for $q \in (p-1, \bar{q}) \subset (p-1, p^*-1)$, the minimizer is unique: then the map $q \in (p-1, \bar{q}) \mapsto u_q \in C^1(\overline{\Omega})$ is continuous. Indeed by [7] $q \mapsto \lambda_q$ is continuous, thus if $q_n \to q_\infty$ we have

$$\int_\Omega |\nabla u_\infty|^p = \lambda_{q_\infty} = \lim_n \lambda_{q_n} = \lim_n \int_\Omega |\nabla u_n|^p$$

that is, $u_n \to u_\infty$ in $W^{1,p}_0(\Omega)$; by the uniform estimates in (i)-(ii) and Ascoli-Arzelà theorem we obtain $u_n \to u$ in $C^1(\overline{\Omega})$.

[17] We thank Sunra Mosconi for some comments on a preliminary version of this argument.

We highlight that the uniform bound from below on $|\nabla u_q|$ could be used, together with a strict quasiconcavity on Ω_δ (if proved), to ensure that the critical point of u_q is unique and nondegenerate in the whole Ω, somehow extending Theorem 1.3 to the superhomogeneous case. We recall that in the semilinear case $p = 2$ [32] (see also [43]) the authors showed that every semistable solution has a unique critical point when $q \leq p-1$ and $a(x) \equiv 1$, but for $q > p-1$ every solution is unstable [84, Theorem 2]. When q is sufficiently close to the critical exponent $p^* - 1$, anyway, uniqueness of the critical point has been achieved by [58]. We refer also to [117, Theorem 6.1] for results in the quasilinear setting in presence of symmetric domains.

7.2. **Large p: towards strict quasiconcavity**

By [81, Theorem 1], as $p \to +\infty$ we know that solutions u_p of the torsion problem $-\Delta_p u_p = 1$ satisfy

$$\lim_{p\to+\infty} u_p = d(\cdot, \partial\Omega) \quad \text{in } L^\infty(\Omega) \tag{7.2}$$

which is a concave function; this is coherent with $\frac{p-1}{p} \to 1$. When Ω is strictly convex we have that the distance function is also strictly quasiconcave (see Proposition 2.1). Moreover, when $\partial\Omega \in C^2$ and it is strongly convex, then (7.2) can be improved [81, Theorem 2]. We further mention that some partial concavity results when $p = +\infty$ are contained in [76, Section 4].

For any p we already know that u_p is quasiconcave: exploiting the ideas of the previous Sections, the strict quasiconcavity and the uniform convergence (7.2) we obtain the following result (see [63, Definition 2.8]).

Proposition 7.4. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open and strictly convex. Let $\varepsilon > 0$. Then there exists $p_0 = p_0(\varepsilon) \in (1, +\infty)$ such that, for any $p \in [p_0, +\infty)$, the positive solution to the torsion problem*

$$\begin{cases} -\Delta_p u_p = 1 & \text{in } \Omega, \\ u_p = 0 & \text{on } \partial\Omega, \end{cases}$$

is ε-uniformly quasiconcave, *that is for some $\rho = \rho(\varepsilon, p) > 0$*

$$u_p\left(\tfrac{x+y}{2}\right) \geq \max\{u_p(x), u_p(y)\} + \rho \qquad \text{for each } x, y \in \Omega,\ |x-y| \geq \varepsilon.$$

Proof. It is sufficient to recall that u_p converge uniformly to $d(\cdot, \partial\Omega)$, which is strictly quasiconcave, and argue as in Proposition 2.6 (notice that a *quasiconcavity function* can be defined straightforwardly). □

We highlight again that, if one could show that the relation obtained in Proposition 7.4 holds for $\varepsilon = 0$, then uniqueness and nondegeneracy of the critical point would hold for large p in any dimension N, extending [23] in the case of strictly convex domains.

Remark 7.5. The behaviour as $p \to 1$ of the solutions of $-\Delta_p u_p = 1$ is quite more nasty, essentially blowing up at $+\infty$ or shrinking to 0 depending on the Cheeger constant of Ω, see [81, Section 3 and 4] and [31]; in particular, if Ω is a ball $B_R(0)$, then $u_p^{\frac{p-1}{p}} \to \frac{R}{N}$.

Regarding the eigenfunction problem, by [82, Remark 10] the solutions u_p of $-\Delta_p u_p = \lambda_p u_p$ as $p \to 1$ essentially converge in BV to the characteristic function of the Cheeger set; the convergence as $p \to +\infty$ has been instead investigated in [77].

7.3. **Fractional equations**

Very few is known regarding concavity results for fractional equations: some partial results are contained in [61] (see also [75] and references therein). Anyway, the only result, close to our framework and known to the authors, is contained in [94]: here, when $N = 2$, $p = 2$, $s = \frac{1}{2}$ and $f \equiv 1$, the author shows that the positive solution of the torsion problem

$$\begin{cases} \sqrt{-\Delta} u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \Omega^c, \end{cases}$$

on $\Omega \subset \mathbb{R}^2$, is concave in Ω (actually strictly concave, for some class of domains Ω). The advantage of this case is that the solution is exactly concave, thus there is no need of a transformation $\varphi(u)$, for which is not known if $\varphi(u)$ solves a different fractional PDE (since Leibniz rule does not hold for $(-\Delta)^s$); moreover, when $s = \frac{1}{2}$ the equation satisfied by the s-harmonic extension is simpler. On the other hand, the lack of a transformation (with singular derivative) brings to much more difficulties in handling the boundary.

The above result suggests that for $(-\Delta)^s u = 1$ the solution is $\frac{1}{2s}$-concave (which is coherent with $1 = (-\Delta)^s u \overset{s\to 0}{\to} u$, u constant, i.e. ∞-concave), or more likely $\min\{\frac{1}{2s}, 1\}$-concave. More generally, one may expect that the concavity properties of $(-\Delta)^s u_s = g(u_s)$ are better than the ones of $-\Delta u = g(u)$: in particular, one may expect that if u is α-concave, than u_s is α-concave as well. What we do in this Section is some partial results in this direction, through perturbation arguments.

Indeed, consider the p-fractional Laplacian

$$(-\Delta)^s_p u(x) := \int_{\mathbb{R}^N} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{N+sp}} dy \quad \text{for } x \in \mathbb{R}^N$$

where the integral is in the principal value sense. We highlight that the choice of the constant (in this case, equal to 1) influences the statements of the following results. Through a perturbation argument, we show the following.

Theorem 7.6. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded, with $\partial\Omega \in C^{1,1}$, and let $p \in (1, +\infty)$. Let $u_s \in W^{s,p}_0(\Omega)$ be positive solutions of*

$$\begin{cases} (-\Delta)^s_p u_s = \lambda_s u_s^{p-1} & \text{in } \Omega, \\ u_s = 0 & \text{on } \Omega^c, \end{cases}$$

normalized at $\|u_s\|_p = 1$. Then

$$\|u_s - u\|_\infty \to 0 \quad \text{as } s \to 1$$

where u is the positive solution of

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

normalized at $\|u\|_p = 1$. Assume now Ω convex. Then $\log(u)$ is concave and, for every $\delta > 0$, we have

$$\|\log(u_s) - \log(u)\|_{L^\infty(\Omega_\delta)} \to 0.$$

As a consequence:

- *if $N = 2$, then for each $\varepsilon > 0$ we have $\log(u_s)$ is ε-uniformly concave in Ω_δ for $s \in (s_0(\varepsilon, \delta), 1)$.*
- *if $p = 2$, then for each $\varepsilon > 0$ we have $\log(u_s)$ is ε-strongly concave in Ω_δ for $s \in (s_0(\varepsilon, \delta), 1)$.*

Proof. By the assumptions, $\|u_s\|_p \equiv 1$, while by [27, Theorem 1.2] we have that

$$(1 - s)\lambda_s \leq C$$

as $s \to 1$. By [27, Theorem 2.10] (see also [73, Corollary 4.2]) we know that u_s are in $L^\infty(\Omega)$, more precisely

$$\|u_s\|_\infty \leq \Big(C \frac{1}{(N - sp)^{p-1}} s(1 - s)\lambda_s \Big)^{\frac{N}{sp^2}} \|u_s\|_p \quad \text{if } p < N \text{ (thus } sp < N),$$

$$\|u_s\|_\infty \leq \big(C \mathrm{diam}(\Omega)^{N(s-\frac{1}{2})} s(1 - s)\lambda_s \big)^{\frac{2}{N}} \|u_s\|_p \quad \text{if } p = N \text{ (and } s \geq \tfrac{3}{4}),$$

$$\|u_s\|_\infty \leq \big(C\mathrm{diam}(\Omega)^{sp-N}(1-s)\lambda_s\big)^{\frac{1}{p}}\|u_s\|_p \quad \text{if } p > N \text{ (and } sp > N),$$

where $C = C(N, p) > 0$. In particular,

$$\|u_s\|_\infty \leq C \quad \text{for } s \geq \tfrac{3}{4}. \tag{7.3}$$

By [27, (2.27) in proof of Theorem 2.10] we also know that

$$[u_s]_{C^{0,s-\frac{N}{p}}(\overline{\Omega})} \leq \big(C(1-s)\lambda_s\big)^{\frac{1}{p}}\|u_s\|_p \quad \text{if } p > N \text{ (and } sp > N)$$

thus $\|u_s\|_{C^{0,s_0-\frac{N}{p}}(\overline{\Omega})} \leq C$ for $s \geq s_0$ ($s_0 > \frac{N}{p}$ however fixed). We need to deal with $p \leq N$.

We know [73, Theorem 1.1] that u_s are in some C^{0,α_s} for some $\alpha_s = \alpha_s(p, N) \in (0, s)$, and such that, set $f_s := \lambda_s u_s^{p-1}$,

$$\|u_s\|_{C^{0,\alpha_s}(\overline{\Omega})} \leq C_s \|f_s\|_\infty^{\frac{1}{p-1}}$$

for some $C_s = C_s(\Omega, p, N)$; from the proof we easily see that α_s and C_s are equibounded for $s \to 1$. On the other hand $\|f_s\|_\infty$ are not equibounded in s, thus we give a closer inspection of the proof: we know indeed that

$$K_1 := 0 \leq (-\Delta)^s_p u_s \leq \lambda_s \|u_s\|_\infty^{p-1} =: K_2^s.$$

The role of K_1 is played in [73, Theorem 5.2 and Lemma 5.3] (and used in [73, Theorem 5.4]), while the role of K_2^s is played in the proof of [73, Theorem 1.1] (implicitly also in [73, Theorem 5.4 and Corollary 5.5]) only to bound $\|u_s\|_\infty$: we can thus substitute this bound with the finer (7.3), which is uniform in s. Thus, for some $\alpha \in (0, 1)$ we have

$$\|u_s\|_{C^{0,\alpha}(\overline{\Omega})} \leq C$$

for each s large. By (3.1) we have

$$\|u_s - u\|_\infty \leq \big(\|u_s\|_{C^{0,\alpha_s}(\overline{\Omega})} + \|u\|_{C^{0,\alpha}(\overline{\Omega})}\big)^{1-\theta_p}\|u_s - u\|_p^{\theta_p}$$

where $\theta_p = \frac{\alpha}{\alpha + \frac{N}{p}}$, thus

$$\|u_s - u\|_\infty \leq C\|u_s - u\|_p^{\theta_p}.$$

By [27, Theorem 1.2] we have that there exists $s_k \to 1$ such that $u_{s_k} \to u$ in $L^p(\Omega)$, thus

$$\|u_{s_k} - u\|_\infty \to 0.$$

Actually we can say better: by [21, Theorem 5.1][18] we know that, for each $s_k \to 1$, being $(1-s_k)\lambda_{s_k} \to \lambda$, there exists $u_{s_{k_n}} \to \bar{u}$ in $L^p(\Omega)$, where $\bar{u} = u$ by uniqueness of the problem and the L^p-constraint; by topological arguments, we have $u_s \to u$ as $s \to 1$ in $L^p(\Omega)$, which implies (by the previous argument) the convergence in $L^\infty(\Omega)$. In particular, for each $\delta > 0$,

$$\|\log(u_s) - \log(u)\|_{L^\infty(\Omega_\delta)} \to 0 \quad \text{as } s \to 1.$$

We conclude by Theorem 1.3, Theorem 1.1 and Propositions 2.5 and 2.6. $\square$

Remark 7.7. Arguing as in the end of the proof of Proposition 7.1, we observe that a C^2 convergence $u_s \to u$ (if proved) would imply that u_s is actually strongly convex for $s \in (s_0(\delta), 1)$, and thus uniqueness and nondegeneracy of the critical point of u_s. We notice that a $C^{2,\beta}$ estimate, uniform in s, is given for $p = 2$ in [33, Lemma 4.4] for equations in $\mathbb{R}^N$; see also [27, Remark 4.1] for further comments.

[18] Notice the difference in the notation in that paper, where $(-\Delta)^s_p$ is substituted with $(1-s)(-\Delta)^p_s$.

Remark 7.8. Differently from Remark 3.7, a global convergence of the type $\|\log(u_s)-\log(u)\|_{L^\infty(\Omega)} \to 0$ is here not possible: indeed, considered $(-\Delta)^{s_1}u_1 = f(u_1)$ and $(-\Delta)^{s_2}u_2 = g(u_2)$ with suitable f and g, by regularity results [73, Theorem 4.4] and fractional Hopf boundary lemma [44, Theorem 1.5(2)], we have $0 < \frac{1}{C} \leq \frac{u_1}{d(\cdot,\Omega)^{s_1}}, \frac{u_2}{d(\cdot,\Omega)^{s_2}} \leq C$, thus $\frac{u_1}{u_2}, \frac{u_2}{u_1} \in L^\infty(\Omega)$ if $s_1 = s_2$; for the same reason, this does not happen if $s_1 \neq s_2$.

Remark 7.9. Similar arguments could be employed for p-subhomogeneous problems

$$(-\Delta)^s_p u_s = u_s^q$$

with $q < p-1$, by exploiting some estimate uniform in s and [21, Theorem 4.5]. Furthermore, in the case $q = 0$, one could exploit [123, Theorem 3.11] to deduce concavity properties for nonautonomous equations, i.e. $(-\Delta)^s_p = a(x)$, when a is close to a constant, in the spirit of Section 3. If one has a stability argument also for $s_k \to s^* \in (0,1)$, then one could use the result by [94] to say that, for $p = 2$ and $s < \frac{1}{2}$, close to $\frac{1}{2}$, the solutions of the torsion problem $(-\Delta)^s u = 1$, $\Omega \subset \mathbb{R}^2$ are ε-uniformly concave, at least for some class of smooth domains [94, Definition 2.1].

By exploiting the results in [50] and the argument of Section 7.1 we see that some result can be obtained also for the semilinear superhomogeneous case $p = 2$, $q > 1$.

Proposition 7.10. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 2$, be open, bounded, with $\partial\Omega \in C^2$. Let $s \in (0,1)$, $q \in (1, 2^*-1)$ and let $u_{s,q}$ be positive solutions of*

$$\begin{cases} (-\Delta)^s u_{s,q} = u_{s,q}^q & \text{in } \Omega, \\ u_{s,q} = 0 & \text{on } \partial\Omega. \end{cases}$$

Then,

$$\|u_{s,q} - u\|_{L^\infty(\Omega)} \to 0 \quad \text{as } s \to 0 \text{ and } q \to 1.$$

In particular, for each $\delta > 0$,

$$\|\log(u_{s,q}) - \log(u)\|_{L^\infty(\Omega_\delta)} \to 0 \quad \text{as } s \to 0 \text{ and } q \to 1.$$

As a consequence:

- *if $N = 2$, then for each $\varepsilon > 0$ we have $\log(u_{s,q})$ is ε-uniformly concave in Ω_δ for $q \in [p-1, q_0(\delta,\varepsilon)]$ and $s \in (s_0(\varepsilon,\delta,q_0), 1)$.*
- *if $p = 2$, then for each $\varepsilon > 0$ we have $\log(u_{s,q})$ is ε-strongly concave in Ω_δ for $q \in [p-1, q_0(\delta,\varepsilon)]$ and $s \in (s_0(\varepsilon,\delta,q_0), 1)$.*

Proof. By [50, Lemma 3.1] we have that $\frac{u_{q,s}}{\|u_{q,s}\|_\infty} \to u_s$ uniformly in $\overline{\Omega}$, as $q \to 1$. The claim comes by combining this convergence with the one in Theorem 7.6 and arguing as in its proof. $\square$

In light of Remark 2.7, we finally point out that it could be interesting to study the *fractional concavity* of solutions of fractional equations, where the notion has been introduced in [45].

7.4. Parabolic equations

We make some final comments on the parabolic case, showing that in this framework some results on the perturbed concavity of the eigenfunction equation can be deduced with a trick. Similar arguments could be set also for the power case $a(x)u^q$. We refer to [47, 85, 89, 97] for some results on the parabolic framework.

For the sake of simplicity, consider the case $p = 2$. Assume $\log(u_0)$ concave and consider

$$\begin{cases} \partial_\tau u - \Delta u = a(x)u & \text{in } \Omega \times (0,+\infty), \\ u > 0 & \text{in } \Omega \times (0,+\infty), \\ u = 0 & \text{on } \partial\Omega \times (0,+\infty), \\ u = u_0 & \text{on } \partial\Omega \times \{\tau = 0\}. \end{cases}$$

Setting $v := \log(u)$ we have

$$\begin{cases} \partial_\tau v - \Delta v = |\nabla v|^2 + a(x) & \text{in } \Omega \times (0, +\infty), \\ v = \log(u_0) & \text{on } \partial\Omega \times \{\tau = 0\}. \end{cases}$$

We implement now the following well known substitution

$$w(x,\tau) := e^{-\mu\tau} v(x,\tau)$$

for some $\mu > 0$. Thus w satisfies

$$\begin{cases} \partial_\tau w - \Delta w = e^{\mu\tau}|\nabla w|^2 + e^{-\mu\tau}a(x) - \mu w & \text{in } \Omega \times (0, +\infty), \\ w = \log(u_0) & \text{on } \partial\Omega \times \{\tau = 0\}. \end{cases}$$

Set

$$b(x,t,\xi,\tau) := e^{\mu\tau}|\xi|^2 + e^{-\mu\tau}a(x) - \mu t$$

where

$$\partial_t b(x,t,\xi,\tau) = -\mu < 0.$$

Let us fix $T > 0$ and consider the equation restricted to $[0,T]$. Similarly to Remark 5.5, $\mathcal{C}_w(x,y,\lambda,\tau)$ cannot be positive near the boundary of $\partial\Omega$, thus it has a maximum in $\Omega \times \Omega \times [0,1] \times [0,T]$, that we call $(x_0, y_0, \lambda_0, \tau_0)$. Notice that, being $w_0 = \log(u_0)$ concave by assumption, we can assume $\tau_0 \in (0,T]$. Arguing as in [29, Remark 2.7] we have

$$\mathcal{C}_w(x_0,y_0,\lambda_0,\tau_0) \leq \frac{1}{\mu}\mathcal{C}_{b(\cdot,w(\cdot),\xi_0,\tau_0)}(x_0,y_0,\lambda_0)$$

which implies

$$\mathcal{C}_{\log(u)}(x_0,y_0,\lambda_0,\tau_0) \leq \frac{1}{\mu}\mathcal{C}_{\tilde{b}(\cdot,w(\cdot),\xi_0,\tau_0)}(x_0,y_0,\lambda_0), \tag{7.4}$$

where

$$\tilde{b}(x,t,\xi,\tau) := e^{2\mu\tau}|\xi|^2 - \mu e^{\mu\tau} t + a(x).$$

The choice of μ is completely free, thus it can be chosen in such a way to be smaller than the error deduced from $\mathcal{C}_{b(\cdot,w(\cdot),\xi_0,\tau_0)}$ (for example, $\mu \sim \sqrt{\mathrm{osc}(a)}$, see Theorem 1.9). We leave the details to the interested reader.

We notice that, even if $u(\cdot,t) \to u^*$ as $t \to +\infty$, u^* solution of the stationary equation, differently from [89, 98, 106], we cannot pass (7.4) to the limit, because of the presence of the term $e^{\mu\tau_0}$, which may explode. Thus we cannot deduce from the parabolic framework an information on the perturbed concavity of $\log(u)$ in the elliptic framework.

Appendix A. Some facts on the p-Laplacian

In what follows, we recall some probably known results, but of which the authors was not able to find a proof. We will state the results in the case of a general $f = f(x,t)$, that is equation (5.1). See also Section 6.2 for additional results in the singular framework $f(x,t) = a(x)t^q$, $q < 0$.

A.1. Uniqueness

We state a uniqueness result when $\frac{f(x,t)}{t^{p-1}}$ is strictly decreasing (see also [66, Theorem 2.1]). When $f(x,t) \equiv g(t)$, we refer also to [22, Proposition 3.8] which essentially says that the same uniqueness holds if $\frac{g(t)}{t^{p-1}}$ is nonincreasing and decreasing only on a small region $[0,\delta]$; see also [112] for further comments. See also [13, 83] where the tool of hidden convexity has been used to deal with general Ω. Finally, we refer to Section 7.1 for the superhomogeneous case.

Lemma A.1 (Brezis-Oswald uniqueness). *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded, connected and satisfying the interior sphere condition, and let $p \in (1,+\infty)$. Assume that $f : \Omega \times (0,+\infty) \to \mathbb{R}$ satisfies*

- $t \mapsto \frac{f(x,t)}{t^{p-1}}$ *is nonincreasing,*
- f *is bounded on bounded sets,*

and let $u, v \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ *be two solutions of* (5.1). *Then*

$$v = ku \quad \text{for some } k \in (0, +\infty).$$

Assume moreover that

- $t \mapsto \frac{f(x,t)}{t^{p-1}}$ *is strictly decreasing.*

Then $u = v$.

Proof. First, we notice that $\Delta_p u, \Delta_p v \in L^\infty(\Omega)$ and that $\frac{u}{v}, \frac{v}{u} \in L^\infty(\Omega)$: indeed, being $u \in C^1(\overline{\Omega})$, by Hopf boundary lemma [122, Theorem 5] we have $\partial_\nu u, \partial_\nu v < 0$ on $\partial\Omega$, and thus by de l'Hopital rule we have, for any $x \in \partial\Omega$,

$$\limsup_{t \to 0} \frac{u(x+t\nu)}{v(x+t\nu)} \leq \limsup_{t \to 0} \frac{\partial_t u(x+t\nu)}{\partial_t v(x+t\nu)} = \limsup_{t \to 0} \frac{\nabla u(x+t\nu) \cdot \nu}{\nabla v(x+t\nu) \cdot \nu} = \frac{\partial_\nu u(x)}{\partial_\nu v(x)} < +\infty,$$

thus the claim. Hence we can apply [48, Lemma 2] and obtain

$$\int_\Omega \left(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}} \right) (u^p - v^p) = \int_\Omega \left(\frac{-\Delta_p u}{u^{p-1}} - \frac{-\Delta_p v}{v^{p-1}} \right) (u^p - v^p) \geq 0.$$

Since $t \mapsto \frac{f(x,t)}{t^{p-1}}$ is nonincreasing, we have

$$\frac{f(x,u)}{u^{p-1}} = \frac{f(x,v)}{v^{p-1}}; \tag{A.1}$$

notice that, if $t \mapsto \frac{f(x,t)}{t^{p-1}}$ is strictly decreasing, we have the second claim. By Picone's inequality [2] we have

$$|\nabla v|^{p-2} \nabla v \cdot \nabla \left(\tfrac{u^p}{v^{p-1}} \right) \leq |\nabla u|^p \tag{A.2}$$

where the equality is attained if and only if u and v are proportional. Integrating, we have

$$\int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\tfrac{u^p}{v^{p-1}} \right) \leq \int_\Omega |\nabla u|^p.$$

Notice that $\frac{u^p}{v^{p-1}} = (\frac{u}{v})^{p-1} u \in L^\infty(\Omega) \subset L^p(\Omega)$ and $\nabla \left(\frac{u^p}{v^{p-1}} \right) = p \left(\frac{u}{v} \right)^{p-1} \nabla u + (p-1) \left(\frac{u}{v} \right)^p \nabla v \in L^p(\Omega)$, thus $\frac{u^p}{v^{p-1}} \in W_0^{1,p}(\Omega)$ and by the equations and the above relation we obtain

$$\int_\Omega f(x,v) \frac{u^p}{v^{p-1}} = \int_\Omega |\nabla v|^{p-2} \nabla v \cdot \nabla \left(\tfrac{u^p}{v^{p-1}} \right) \leq \int_\Omega |\nabla u|^p = \int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla u = \int_\Omega f(x,u) u$$

that is

$$\int_\Omega \left(\frac{f(x,v)}{v^{p-1}} - \frac{f(x,u)}{u^{p-1}} \right) u^p \leq 0.$$

On the other hand, due to (A.1), the last is an equality, and so it must be (A.2), which implies the claim. □

A.2. **Comparison principle**

We show now a comparison principle for quasilinear equation in presence of a p-subhomogeneous function; notice that we are not requiring f itself to be decreasing. The proof is inspired by [111, Theorem 1.5]; notice moreover that the result was obtained, when f is a power, by [28, Theorem 4.1] in a more general setting through the use of the interesting tool of the hidden convexity. See also [42, Theorems 1.3 and 3.3], [23, Lemma 2.1].

Finally, notice that the uniqueness result part of Lemma A.1 can be deduced from this comparison principle.

Lemma A.2 (Comparison principle)**.** *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded, connected and satisfying the interior sphere condition, and let $p \in (1, +\infty)$. Assume that $f : \Omega \times (0, +\infty) \to \mathbb{R}$ satisfies*

- *$t \mapsto \frac{f(x,t)}{t^{p-1}}$ strictly decreasing,*
- *f is bounded on bounded sets,*

and let $u, v \in C^1(\overline{\Omega})$ be a subsolution and a supersolution, namely

$$-\Delta_p u - f(x,u) \leq -\Delta_p v - f(x,v) \quad \text{in } \Omega$$

with

$$u, v > 0 \ \text{ in } \Omega, \quad u \leq v \ \text{ on } \partial\Omega.$$

Then

$$u \leq v \quad \text{in } \Omega.$$

Proof. Set $w := (u^p - v^p)^+$, we need to prove that $w \equiv 0$. Consider $\Omega_+ := \mathrm{supp}(w) = \{u \geq v\}$. As in the proof of Lemma A.1 we have $\frac{u}{v}, \frac{v}{u} \in L^\infty(\Omega)$, thus we compute

$$\nabla \left(\tfrac{w}{u^{p-1}}\right) = \chi_{\Omega_+} \left(\nabla u - \nabla \left(\tfrac{v^p}{u^{p-1}}\right) + \delta(p-1)\nabla u\right);$$

this first tells us that $\frac{w}{u^{p-1}} \in W^{1,p}_0(\Omega)$. Moreover

$$|\nabla u|^{p-2}\nabla u \cdot \nabla \left(\tfrac{w}{u^{p-1}}\right) = \chi_{\Omega_+} \left(|\nabla u|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla \left(\tfrac{v^p}{u^{p-1}}\right)\right);$$

similarly

$$\nabla \left(\tfrac{w}{v^{p-1}}\right) = \chi_{\Omega_+} \left(-\nabla v + \nabla \left(\tfrac{u^p}{v^{p-1}}\right)\right);$$

thus $\frac{w}{v^{p-1}} \in W^{1,p}_0(\Omega)$ and

$$|\nabla v|^{p-2}\nabla v \cdot \nabla \left(\tfrac{w}{v^{p-1}}\right) = \chi_{\Omega_+} \left(-|\nabla v|^p + |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\tfrac{u^p}{v^{p-1}}\right)\right).$$

Therefore

$$\begin{aligned}
&|\nabla u|^{p-2}\nabla u \cdot \nabla \left(\tfrac{w}{u^{p-1}}\right) - |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\tfrac{w}{v^{p-1}}\right) \\
&\quad = \chi_{\Omega_+} \left(|\nabla v|^p - |\nabla u|^{p-2}\nabla u \cdot \nabla \left(\tfrac{v^p}{u^{p-1}}\right)\right) + \chi_{\Omega_+} \left(|\nabla u|^p - |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\tfrac{u^p}{v^{p-1}}\right)\right) \geq 0
\end{aligned}$$

by Picone's inequality (A.2). In particular, being $\frac{w}{u^{p-1}}, \frac{w}{v^{p-1}}$ nonnegative test functions,

$$\int_\Omega \left(f(x,u)\frac{w}{u^{p-1}} - f(x,v)\frac{w}{v^{p-1}}\right) \geq \int_{\Omega_+} \left(|\nabla u|^{p-2}\nabla u \cdot \nabla \left(\tfrac{w}{u^{p-1}}\right) - |\nabla v|^{p-2}\nabla v \cdot \nabla \left(\tfrac{w}{v^{p-1}}\right)\right) \geq 0.$$

On the other hand, by the monotonicity $\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}} \leq 0$ on Ω_+, thus

$$\left(\frac{f(x,u)}{u^{p-1}} - \frac{f(x,v)}{v^{p-1}}\right) w \equiv 0 \quad \text{on } \Omega.$$

Finally, by the strict monotonicity, we gain $w \equiv 0$, that is the claim. □

As a consequence we obtain a uniform bound from below. See also Remark 7.3 for the superhomogeneous case.

Corollary A.3. *Let $\Omega \subset \mathbb{R}^N$, $N \geq 1$, be open, bounded, connected and satisfying the interior sphere condition, and let $p \in (1, +\infty)$. Assume that $f_n, f_0 : \Omega \times (0, +\infty) \to \mathbb{R}$ satisfy*

- *$t \mapsto \frac{f_0(x,t)}{t^{p-1}}$ strictly decreasing,*
- *f_0 is bounded on bounded sets,*
- *$f_n(x,t) \geq f_0(x,t)$.*

Let $u_n \in W^{1,p}_0(\Omega) \cap C^1(\overline{\Omega})$ be positive solutions of $-\Delta u_n = f_n(x,t)$, $-\Delta u_0 = f_0(x,t)$ with Dirichlet boundary conditions. Then, for each $\delta > 0$, there exists $C = C(\delta) > 0$ such that

$$\inf_{\Omega_\delta} u_n \geq C > 0.$$

Acknowledgments

The authors thank Lorenzo Brasco, Umberto Guarnotta, Luigi Montoro, Berardino Sciunzi and Sunra Mosconi for having made us aware of some references cited in Section 6.2 and Appendix A.

References

[1] A. Acker, L. E. Payne, G. Philippin, *On the convexity of level lines of the fundamental mode in the clamped membrane problem, and the existence of convex solutions in a related free boundary problem*, Z. Angew. Math. Phys. **32** (1981), 683–694. 3, 4

[2] W. Allegretto, Y. X. Huang, *A Picone's identity for the p-Laplacian and applications*, Nonlinear Anal. **32** (1998), 819–830. 47

[3] N. M. Almousa, J. Assettini, M. Gallo, M. Squassina, *Concavity properties for quasilinear equations and optimality remarks*, Differential Integral Equations **37** (2024), 1–26. 2, 3, 5

[4] N. Almousa, C. Bucur, R. Cornale, M. Squassina, *Concavity principles for nonautonomous elliptic equations and applications*, Asymptot. Anal. **135** (2023), 509-524. 4, 20, 31, 35

[5] O. Alvarez, J.-M. Lasry, P.-L. Lions, *Convex viscosity solutions and state constraints*, J. Math. Pures Appl. **17** (1997), 265–288. 5

[6] A. L. Amadori, F. Gladiali, *Bifurcation and symmetry breaking for the Henon equation*, Adv. Differential Equations **19** (2014), 755–782. 6

[7] G. Anello, F. Faraci, A. Iannizzotto, *On a problem of Huang concerning best constants in Sobolev embeddings*, Ann. Mat. Pura Appl. **194** (2015), 767–779. 41

[8] C. A. Antonini, *Smooth approximation of Lipschitz domains, weak curvatures and isocapacitary estimates*, Calc. Var. Partial Differential Equations **63** (2024), 91(1–34). 9

[9] M. Avriel, W. E. Diewert, S. Schaible, I. Zang, "Generalized Concavity", SIAM, Philadelphia, 2010. 10, 13

[10] A. Avila, F. Brock, *A unified approach to symmetry for semilinear equations associated to the Laplacian in* $\mathbb{R}^N$, J. Math. Anal. Appl. **488** (2020), 124087(1–28). 6

[11] J. M. Ball, A. Zarnescu, *Partial regularity and smooth topology-preserving approximations of rough domains*, Calc. Var. Partial Differential Equations **56** (2017), 13(1–32). 9, 10

[12] R. F. Basener, *Nonlinear Cauchy-Riemann equations and q–pseudoconvexity*, Duke Math. J. **43** (1976), 203–213. 3

[13] M. Belloni, B. Kawohl, *A direct uniqueness proof for equations involving the p-Laplace operator*, Manuscripta Math. **109** (2002), 229–231. 6, 46

[14] M. S. Berger, "Nonlinearity and Functional Analysis", Lectures on Nonlinear Problems in Mathematical Analysis, Academic Press Inc., New York, 1977. 27

[15] S. Berhanu, F. Gladiali. G. Porru, *Qualitative properties of solutions to elliptic singular problems*, J. Inequal. Appl. **3** (1999), 313–330. 3, 9, 13, 16, 37

[16] B. Bian, P. Guan, *A microscopic convexity principle for nonlinear partial differential equations*, Invent. Math. **177** (2009), 307–335. 3

[17] M. Bianchini, P. Salani, *Power concavity for solutions of nonlinear elliptic problems in convex domains*, in "Geometric properties for parabolic and elliptic PDE's", eds. R. Magnanini, S. Sakaguchi, A. Alvino, Springer INdAM Series **2** (2013), 35–48. 4

[18] P. Blanc, M. Parviainen, J. Rossi, *A bridge between convexity and quasiconvexity*, arXiv:2301.10573 (2023). 12, 13

[19] Z. Błocki, *Smooth exhaustion functions in convex domains*, Proc. Amer. Math. Soc. **125** (1997), 477–484. 10

[20] L. Boccardo, L. Orsina, *Semilinear elliptic equations with singular nonlinearities*, Calc. Var. Partial Differential Equations **37** (2010), 363–380. 36, 38

[21] J. F. Bonder, A. Salort, *Stability of solutions for nonlocal problems*, Nonlinear Anal. **200** (2020), 112080(1–13). 44, 45

[22] W. Borrelli, S. Mosconi, M. Squassina, *Concavity properties for solutions to p-Laplace equations with concave nonlinearities*, Adv. Calc. Var. **17** (2022), 79–97. 5, 6, 8, 21, 27, 28, 31, 34, 36, 38, 46

[23] W. Borrelli, S. Mosconi, M. Squassina, *Uniqueness of the critical point for solutions of some p-Laplace equations in the plane*, Rend. Lincei Mat. Appl. **34** (2023), 61–88. 3, 5, 41, 42, 47

[24] H. J. Brascamp, E. H. Lieb, *On extensions of the Brunn-Minkowski and Prékopa-Leindler theorems, including inequalities for log concave functions, and with an application to the diffusion equation*, J. Funct. Anal. **22** (1976), 366–389. 3, 4

[25] L. Brasco, G. Franzina, *An overview on constrained critical points of Dirichlet integrals*, Rend. Semin. Mat. Univ. Politec. Torino **78** (2020), 7–50. 39

[26] L. Brasco, E. Lindgren, *Uniqueness of extremals for some sharp Poincaré-Sobolev constants*, Trans. Amer. Math. Soc. **376** (2023), 3541–3584. 39, 41

[27] L. Brasco, E. Parini, M. Squassina, *Stability of variational eigenvalues for the fractional p–Laplacian*, Discrete Contin. Dyn. Syst. **16** (2016), 1813–1845. 43, 44

[28] L. Brasco, F. Prinari, A. C. Zagati, *A comparison principle for the Lane–Emden equation and applications to geometric estimates*, Nonlinear Anal. **220** (2022), 112847(1–41). 47

[29] C. Bucur, M. Squassina, *Approximate convexity principles and applications to PDEs in convex domains*, Nonlinear Anal. **192** (2020), 111661(1–21). 4, 8, 11, 20, 21, 35, 46

[30] S.-S. Byun, D. K. Palagachev, P. Shin, *Global continuity of solutions to quasilinear equations with Morrey data*, C. R. Math. Acad. Sci. **353** (2015), 717–721. 15, 22, 24, 34

[31] H. Bueno, G. Ercole, S. Macedo, *Asymptotic behavior of the p-torsion functions as p goes to* 1, Arch. Math. **107** (2016), 63–72. 42

[32] X. Cabré, S. Chanillo, *Stable solutions of semilinear elliptic problems in convex domains*, Selecta Math. (N.S.) **4** (1998), 1–10. 42

[33] X. Cabré, Y. Sire, *Nonlinear equations for fractional Laplacians, I: regularity, maximum principles, and Hamiltonian estimates*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), no. 1, 23–53. 44

[34] L. A. Caffarelli, A. Friedmann, *Convexity of solutions of semilinear elliptic equations*, Duke Math. J. **52** (1985), 431–456. 3, 5, 18, 27

[35] L. A. Caffarelli, J. Spruck, *Convexity properties of solutions to some classical variational problems*, Comm. Partial Differential Equations **7** (1982), 1337–1379. 20

[36] A. Canino, F. Esposito, B. Sciunzi, *On the Höpf boundary lemma for singular semilinear elliptic equations*, J. Differential Equations **266** (2019), 5488-5499. 38

[37] A. Canino, B. Sciunzi, A. Trombetta, *Existence and uniqueness for p-Laplace equations involving singular nonlinearities*, Nonlinear Differ. Equ. Appl. **26** (2016), 8(1–18). 36, 37

[38] A. Canino, M. Degiovanni, *A variational approach to a class of singular semilinear elliptic equations*, J. Convex Anal. **11** (2004), 147–162. 28, 36, 38, 40

[39] A. Chau, B. Weinkove, *Concavity of solutions to semilinear equations in dimension two*, Bull. Lond. Math. Soc. **55** (2022), 706–716. 2, 19

[40] G. Ciraolo, M. Cozzi, M. Perugini, L. Pollastro, *A quantitative version of the Gidas-Ni-Nirenberg Theorem*, arXiv:2308.00409 (2023). 4

[41] G. Crasta, I. Fragalà, *The Brunn–Minkowski inequality for the principal eigenvalue of fully nonlinear homogeneous elliptic operators*, Adv. Math. **359** (2020), 106855(1–24). 5

[42] L. Damascelli, B. Sciunzi, *Regularity, monotonicity and symmetry of positive solutions of m-Laplace equations*, J. Differential Equations **206** (2004), 483–515. 2, 47

[43] F. De Regibus, M. Grossi, D. Mukherjee, *Uniqueness of the critical point for semi-stable solutions in* $\mathbb{R}^2$, Calc. Var. Partial Differential Equations **60** (2021), 25(1–13). 42

[44] L. M. Del Pezzo, A. Quaas, *A Hopf's lemma and a strong minimum principle for the fractional p-Laplacian*, J. Differential Equations **263** (2017), 765–778. 45

[45] L. M. Del Pezzo, A. Quaas, J. D. Rossi, *Fractional convexity*, Math. Ann. **381** (2022), 1687–1719. 45

[46] M. C. Delfour, J.-P. Zolésio, *Shape analysis via oriented distance functions*, J. Funct. Anal. **123** (1994), 129–201. 10

[47] J. I. Díaz, B. Kawohl, *On convexity and starshapedness of level sets for some nonlinear elliptic and parabolic problems on convex rings*, J. Math. Anal. Appl. **177** (1993), 263–286. 45

[48] J. I. Díaz, J. E. Saa, *Existence et unicité de solutions positives pour certaines équations elliptiques quasilinéaires*, C. R. Acad. Sci. Paris (1987), 521–524. 47

[49] E. DiBenedetto, $C^{1+\alpha}$ *local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), 827–850. 24

[50] A. Dieb, I. Ianni, A. Saldaña, *Uniqueness and nondegeneracy for Dirichlet fractional problems in bounded domains via asymptotic methods*, Nonlinear Anal. **236** (2023), 113354(1–21). 45

[51] P. Doktor, *Approximation of domains with Lipschitzian boundary*, Cas. Pestovani Mat. **101** (1976), 237–255. 9, 10
[52] X. Fernández-Real, X. Ros-Oton, "Regularity Theory for Elliptic PDE", Zurich Lectures in Advanced Mathematics, EMS Press, 2022. 33, 40
[53] R. L. Foote, *Regularity of the distance function*, Proc. Amer. Math. Soc. **92** (1984), 153–155. 9, 10
[54] M. Giaquinta, E. Giusti, *On the regularity of the minima of variational integrals*, Acta Math. **148** (1982), 31–46. 23
[55] B. Gidas, W.-M. Ni, L. Nirenberg, *Symmetry and related properties via the maximum principle*, Commun. Math. Phys. **68** (1979), 209–243. 2
[56] D. Gilbarg, N. S. Trudinger, "Elliptic Partial Differential Equations of Second Order", Classics in Mathematics, Springer-Verlag, Berlin Heidelberg, 2001. 10
[57] F. Gladiali, *Proprietà qualitative e non esistenza di soluzioni positive per alcuni problemi ellittici non lineari*, Ph.D. Thesis, Sapienza Università di Roma (2002). 36, 38
[58] F. Gladiali, M. Grossi, *Strict convexity of level sets of solutions of some nonlinear elliptic equations*, Proc. Roy. Soc. Edinburgh Sect. A **134** (2004), 363–373. 42
[59] F. Gladiali, M. Grossi, *On the number of critical points of solutions of semilinear equations in* $\mathbb{R}^2$, Amer. J. Math. **144** (2022), 1221-1240. 3
[60] J. M. Gomes, *Sufficient conditions for the convexity of the level sets of ground-state solutions*, Arch. Math. **88** (2007), 269–278. 4
[61] A. Greco, *Fractional convexity maximum principle*, utexas.edu (2014). 42
[62] A. Greco, G. Porru, 1993, *Convexity of solutions to some elliptic partial differential equations*, SIAM J. Math. Anal. **24** (1993), 833–839. 20, 21, 31, 36
[63] G. Grelier, M. Raja, *On uniformly convex functions*, J. Math. Anal. Appl. **505** (2022), 125442(1–25). 12, 42
[64] P. Grisvard, "Elliptic Problems in Nonsmooth Domains", Classics in Applied Mathematics, SIAM, USA, 2011. 9, 10
[65] U. Guarnotta, *Existence results for singular convective elliptic problems*, Ph.D. Thesis, Università degli Studi di Palermo (2021). 38
[66] M. Guedda, L. Veron, *Quasilinear elliptic equations involving critical Sobolev exponents*, Nonlinear Anal. **13** (1989), 879–902. 15, 22, 46
[67] D. D. Hai, *On a class of singular p-Laplacian boundary value problems*, J. Math. Anal. Appl. **383** (2011), 619–626. 38
[68] F. Hamel, N. Nadirashvili, Y. Sire, *Convexity of level sets for elliptic problems in convex domains or convex rings: two counterexamples*, Amer. J. Math. **138** (2016), 499–527. 3
[69] A. Henrot, C. Nitsch, P. Salani, C. Trombetti, *Optimal concavity of the torsion function*, J. Optim. Theory Appl. **178** (2018), 26–35. 2, 3
[70] A. Henrot, M. Pierre, "Shape Variation and Optimization", Tracts in Mathematics **28**, EMS, Germany, 2018. 9, 10
[71] L. Hörmander, "Notions of Convexity", Modern Birkhäuser Classics, Birkhäuser, Boston, 2007. 9, 10, 12
[72] D. H. Hyers, S. M. Ulam, *Approximately convex functions*, Proc. Amer. Math. Soc. **3** (1952), 821–828. 8, 21, 31
[73] A. Iannizzotto, S. Mosconi, M. Squassina, *Global Hölder regularity for the fractional p-Laplacian*, Rev. Mat. Iberoam. **32** (2016), 1353–1392. 43, 44, 45
[74] L. Iturriaga, S. Lorca, J. Sánchez, *Existence and multiplicity results for the p-Laplacian with a p-gradient term*, NoDEA Nonlinear Differential Equations Appl. **15** (2008), 729–743. 40
[75] S. Jarohs, T. Kulczucki, P. Salani, *Starshapedness of the superlevel sets of solutions to equations involving the fractional Laplacian in starshaped rings*, Math. Nachr. **292** (2019), 1008–1021. 42
[76] P. Juutinen, *Concavity maximum principle for viscosity solutions of singular equations*, NoDEA Nonlinear Differential Equations Appl. **17** (2010), 601–618. 21, 42
[77] P. Juutinen, P. Lindqvist, J. J. Manfredi, *The ∞-eigenvalue problem*, Arch. Ration. Mech. Anal. **148** (1999), 89–105. 42
[78] J. Kadlec, *On the regularity of the solution of the Poisson problem on a domain with boundary locally similar to the boundary of a convex open set*, Czechoslovak Math. J. **14** (1964), 386–393 (in Russian). 22
[79] B. Kawohl, "Rearrangements and convexity of level sets in PDE", Lecture Notes in Math. **1150**, Springer-Verlag Heidelberg, 1985. 20, 21

[80] B. Kawohl, *When are solutions to nonlinear elliptic boundary value problems convex?*, Comm. Partial Differential Equations **10** (1985), 1213–1225. 5, 27, 39

[81] B. Kawohl, *On a family of torsional creep problems* J. Reine Angew. Math. **410** (1990), 1–22. 5, 42

[82] B. Kawohl, V. Fridman, *Isoperimetric estimates for the first eigenvalue of the p-Laplace operator and the Cheeger constant*, Comment. Math. Univ. Carolin. **44** (2003), 659–667. 42

[83] B. Kawohl, P. Lindqvist, *Positive eigenfunctions for the p-Laplace operator revisited*, Analysis (Berlin) **26** (2006), 545–550. 6, 46

[84] J. Karátson, P. L. Simon, *On the stability properties of nonnegative solutions of semilinear problems with convex or concave nonlinearity*, J. Comput. Appl. Math. **131** (2001), 497–501. 42

[85] G. Keady, *The persistence of logconcavity for positive solutions of the one dimensional heat equation*, J. Aust. Math. Soc. **48** (1990), 246-263. 45

[86] G. Keady, A. McNabb, *The elastic torsion problem: solutions in convex domains*, New Zealand J. Math. **22** (1993), 43–64. 2, 19

[87] A. U. Kennington, *An improved convexity maximum principle and some applications*, Ph.D. Thesis at University of Adelaide (1984). (Bull. Aust. Math. Soc. **31** (1985), 159-160). 2, 3, 4

[88] A. U. Kennington, *Power concavity and boundary value problems*, Indiana Univ. Math. J. **34** (1985), 687–704. 1, 3, 4, 5, 6, 8, 10, 11, 20, 34, 36

[89] A. U. Kennington, *Convexity of level curves for an initial value problem*, J. Math. Anal. Appl. **133** (1988), 324–330. 5, 10, 39, 45, 46

[90] N. J. Korevaar, *Convex solutions to nonlinear elliptic and parabolic boundary value problems*, Indiana Univ. Math. J. **32** (1983), 603–614. 18, 19, 20, 36

[91] N. J. Korevaar, J. L. Lewis, *Convex solutions of certain elliptic equations have constant rank Hessians*, Arch. Ration. Mech. Anal. **97** (1987), 19–32. 3, 41

[92] S. G. Krantz, H. R. Parks, *Distance to C^k hypersurfaces*, J. Differential Equations **40** (1981), 116–120. 10

[93] S. G. Krantz, H. R. Parks, “The Geometry of Domains in Space”, Birkhäuser Advanced Texts, Springer-Science+Business Media, New York, 1999. 10

[94] T. Kulczycki, *On concavity of solutions of the Dirichlet problem for the equation $(-\Delta)^{1/2}\varphi = 1$ in convex planar regions*, J. Eur. Math. Soc. **19** (2017), 1361–1420. 43, 45

[95] O. A. Ladyzhenskaya, N. N. Ural’tseva, “Linear and Quasilinear Elliptic Equations”, Academic Press Inc. **46**, New York, 1968. 15, 22, 23, 24, 27

[96] A. Lê, *On the local Hölder continuity of the inverse of the p-Laplace operator*, Proc. Amer. Math. Soc. **135** (2007), 3553–3560. 13

[97] K.-A. Lee, *Power concavity on nonlinear parabolic flows*, Comm. Pure Appl. Math. **58** (2004), 1529–1543. 45

[98] K. Lee, J. L. Vázquez, *Parabolic approach to nonlinear elliptic eigenvalue problems*, Adv. Math. **219** (2008), 2006–2028. 4, 6, 39, 46

[99] M. Lewicka, Y. Peres, *Which domains have two-sided supporting unit spheres at every boundary point?*, Expo. Math. **38** (2020), 548–558. 9

[100] G. M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. **117** (1985), 329–352. 10

[101] G. M. Lieberman, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), 1203–1219. 15, 22, 24, 28

[102] C.-S. Lin, *Uniqueness of least energy solutions to a semilinear elliptic equation in $\mathbb{R}^2$*, Manuscripta Math. **84** (1994), 13–19. 3, 5, 39, 40

[103] P. O. Lindberg, *Power convex functions*, in “Generalized Concavity in Optimization and Economics”, eds. S. Schaible, W. T. Ziemba, Academic Press Inc., London, 1981. 10, 13

[104] P. Lindqvist, *A note on the nonlinear Rayleigh quotient*, Potential Anal. **2** (1993), 199–218. 3, 5, 39, 40

[105] P. Lindqvist, “Notes on the p-Laplace equation”, Report of University of Jyväskylä, Department of Mathematics and Statistics, 2017. 14

[106] P.-L. Lions, *Two geometrical properties of solutions of semilinear problems*, Appl. Anal. **12** (1981), 267–272. 31, 39, 46

[107] X.-N. Ma, S. Shi, Y. Ye, *The convexity estimates for the solutions of two elliptic equations*, Comm. Partial Differential Equations **37** (2012), 2116–2137. 4

[108] L. G. Makar-Limanov, *Solution of Dirichlet’s problem for the equation $\Delta u = -1$ in a convex region*, Mat. Zametki **9** (1971), 89–92. 3

[109] V. Maz'ya, "Sobolev Spaces, with Applications to Elliptic Partial Differential Equations", Comprehensive Studies in Mathematics **342**, Springer-Verlag Berlin Heidelberg, 2011. 10

[110] S. Merchána, L. Montoro, I. Peral, B. Sciunzi, *Existence and qualitative properties of solutions to a quasilinear elliptic equation involving the Hardy-Leray potential*, Ann. Inst. H. Poincaré Anal. Non Linéaire **31** (2014), 1–22. 2

[111] L. Montoro, B. Sciunzi, A. Trombetta, *A comparison principle for a doubly singular quasilinear anisotropic problem*, Commun. Contemp. Math. (2024), 2350060(1–18). 38, 47

[112] S. Mosconi, *A non-smooth Brezis-Oswald uniqueness result*, Open Math. **21** (2023), 20220594(1–28). 46

[113] S. Mosconi, G. Riey, M. Squassina, *Concave solutions to Finsler p-Laplace type equations*, arXiv:2308.05069 (2024). 18, 19, 20, 21, 31

[114] T. Ng, K. Nikodem, *On approximately convex functions*, Proc. Amer. Math. Soc. **118** (1993), 103–108. 21, 31

[115] R. .T. Rockafellar, "Convex Analysis", Princeton Landmarks in Mathematics and Physics **30**, Princeton University Press, 1970. 34, 37

[116] S. Sakaguchi, *Concavity properties of solutions to some degenerate quasilinear elliptic Dirichlet problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 **14** (1987), 403–421. 1, 5, 6, 8, 18, 21, 22, 28, 38

[117] B. Sciunzi, *Some results on the qualitative properties of positive solutions of quasilinear elliptic equations*, NoDEA Nonlinear Differential Equations Appl. **14** (2007), 315–334. 42

[118] I. M. Singer, B. Wong, S.-T. Yau, S. S.-T. Yau, *An estimate of the gap of the first two eigenvalues in the Schrödinger operator*, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 **12** (1985), 319–333. 3, 4

[119] S. Steinerberger, *On concavity of solutions of the nonlinear Poisson equation*, Arch. Ration. Mech. Anal. **244** (2022), 209–224. 2, 19

[120] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), 126–150. 24

[121] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747. 15, 24, 28, 40

[122] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), 191–202. 22, 33, 38, 40, 47

[123] M. Warma, *Local Lipschitz continuity of the inverse of the fractional p-Laplacian, Hölder type continuity and continuous dependence of solutions to associated parabolic equations on bounded domains*, Nonlinear Anal. **135** (2015), 129–157. 45

[124] T. Weth, *On the lack of directional quasiconcavity of the fundamental mode in the clamped membrane problem*, Arch. Math. **97** (2011), 365–372. 3

(M. Gallo, M. Squassina)
Dipartimento di Matematica e Fisica
Università Cattolica del Sacro Cuore
Italy, Brescia, BS, Via della Garzetta 48, 25133
Email address: marco.gallo1@unicatt.it
Email address: marco.squassina@unicatt.it