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On Explosive Solutions for a Class of Quasi-linear Elliptic Equations

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Abstract

We study existence, uniqueness, multiplicity and symmetry of large solutions for a class of quasi-linear elliptic equations. Furthermore, we characterize the boundary blow-up rate of solutions, including the case where the contribution of boundary curvature appears.

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1 Introduction and results

The study of explosive solutions of elliptic equations goes back to 1916 by Bieberbach [6] for the problem $\Delta u = e^u$ on a bounded two dimensional domain, arising in Riemannian geometry as related to exponential metrics with constant Gaussian negative curvature. The result was then extended to three dimensional domains by Rademacher [36] in 1943. Large solutions of more general elliptic equations $\Delta u = f(u)$ in smooth bounded domains Ω of $\mathbb{R}^N$ were originally studied by Keller [26] and Osserman [33] around 1957, and subsequently refined in a series of more recent contributions, see [1, 2, 3, 8, 9, 10, 12, 15, 22, 29, 32, 34] and references therein. The aim of this paper is to study existence, uniqueness, symmetry as well as asymptotic behavior on $\partial\Omega$ for the quasi-linear problem

$$\begin{cases} \operatorname{div}(a(u)Du) = \frac{a'(u)}{2}|Du|^2 + f(u) & \text{in } \Omega, \\ u(x) \to +\infty \quad \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{1.1}$$

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where Ω is a bounded smooth domain of $\mathbb{R}^N$, $N \geq 1$, and $\mathrm{d}(x, \partial\Omega)$ is the distance of x from the boundary of Ω. Here and in the following $a : \mathbb{R} \to \mathbb{R}^+$ is a C^1 function such that there exists $\nu > 0$ with $a(s) \geq \nu$ for any $s \in \mathbb{R}$, and $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function. In problem (1.1), the terms depending upon a are formally associated with the functional $\int_\Omega a(u)|Du|^2$ and the problem can be thought as related to the study of blow-up solutions in presence of a Riemannian metric tensor depending upon the unknown u itself, see e.g. [37, 40] for more details. We shall cover the situations where a and f have an exponential, polynomial or logarithmic type growth at infinity. In the semi-linear case $a \equiv 1$, typical situations where the exponential nonlinearity appears is the Liouville [30] equation $\Delta u = 4e^{2u}$ in $\Omega \subset \mathbb{R}^2$, while for a typical polynomial growth one can think to the Loewner-Nirenberg [31] equation $\Delta u = 3u^5$ in $\Omega \subset \mathbb{R}^3$. Logarithmic type nonlinearities usually appear in theories of quantum gravity [41] and in particular in the framework of nonlinear Schrödinger equations [5]. The function a can be regarded as responsible for the diffusion effects while, on the contrary, f can be considered as playing the rǒle of an external source. Roughly speaking, in some sense, a is competing with f for the existence and nonexistence of solutions to (1.1) and the asymptotic behavior of $a(s)$ and $f(s)$ as $s \to +\infty$ determines the blow-up rate of $u(x)$ as x approaches the boundary of Ω. For the literature on these type of quasi-linear operators in frameworks different from that of large solutions, we refer the reader to [38] and the reference therein. In order to give precise characterization of existence and explosion rate, we shall convert the quasi-linear problem (1.1) into a corresponding semi-linear problem through a change of variable procedure involving the globally defined Cauchy problem

$$g' = \frac{1}{\sqrt{a \circ g}}, \qquad g(0) = 0. \tag{1.2}$$

The precise knowledge of the asymptotic behavior of the solution g of (1.2) as $s \to +\infty$ depending of the asymptotics of the function a will be crucial in studying the qualitative properties of the solutions to (1.1). We shall obtain for (1.1) existence, nonexistence, uniqueness and multiplicity results in arbitrary smooth bounded domains, uniqueness and symmetry results when the problem is set in the ball and, finally, results about the blow-up rate of the solution with or without the second order contribution of the local curvature of the boundary $\partial\Omega$. For instance, if $a(s) \sim a_\infty s^k$ as $s \to +\infty$ and $f(s) \sim f_\infty s^p$ as $s \to +\infty$ with $p > 2k+3$, then a solution to (1.1) exists and any solution satisfies

$$u(x) = \frac{\Gamma}{(\mathrm{d}(x, \partial\Omega))^{\frac{2}{p-k-1}}}(1 + o(1)), \qquad \Gamma = \Big[\frac{p-k-1}{\sqrt{2(p+1)}} \frac{\sqrt{f_\infty}}{\sqrt{a_\infty}}\Big]^{\frac{2}{k+1-p}},$$

as x approaches $\partial\Omega$. If instead $k + 1 < p \leq 2k + 3$, then we have

$$u(x) = \Gamma \frac{1}{(\mathrm{d}(x, \partial\Omega))^{\frac{2}{p-k-1}}}(1 + o(1)) + \Gamma' \frac{\mathcal{H}(\sigma(x))}{(\mathrm{d}(x, \partial\Omega))^{\frac{3+k-p}{p-k-1}}}(1 + o(1)), \quad \Gamma' = \frac{2(N-1)}{p+k+3}\Gamma,$$

for x approaching $\partial\Omega$, being $\sigma(x)$ the orthogonal projection on $\partial\Omega$ of a $x \in \Omega$ and denoting $\mathcal{H}$ the mean curvature of the boundary $\partial\Omega$.

In this paper we shall restrict the attention on the study of explosive solutions in smooth and bounded domains. Concerning the study of large solutions of quasi-linear equations (including non-degenerate and non-autonomous problems) on the entire space, a vast recent literature currently exists on the subject. We refer the reader to the contributions [13, 14, 16, 17, 15, 18, 19, 20, 35] of (in alphabetical order) Dupaigne, Farina, Filippucci, Pucci, Rigoli and Serrin and the references therein.

Concerning the existence of solutions to (1.1), we have the following

Theorem 1.1 (Existence of solutions) *The following statements hold:*

1. *Assume that there exist* $k > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{s^k} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}} = f_\infty. \tag{1.3}$$
Then (1.1) *admits a solution.*

2. *Assume that there exist* $k > 0$, $p > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{s^k} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{s^p} = f_\infty. \tag{1.4}$$
Then (1.1) *admits a solution if and only if* $p > k + 1$.

3. *Assume that there exist* $k > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{s^k} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{(\log s)^\beta} = f_\infty. \tag{1.5}$$
Then (1.1) *admits no solution.*

4. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{e^{2\gamma s}} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}} = f_\infty. \tag{1.6}$$
Then (1.1) *admits a solution if and only if* $\beta > \gamma$.

5. *Assume that there exist* $\gamma > 0$, $p > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{e^{2\gamma s}} = a_\infty, \quad \lim_{s\to+\infty}\frac{f(s)}{s^p} = f_\infty.$$
Then (1.1) *admits no solution.*

6. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{e^{2\gamma s}} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{(\log s)^\beta} = f_\infty.$$
Then (1.1) *admits no solution.*

7. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{(\log s)^{2\gamma}} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}} = f_\infty. \tag{1.7}$$
Then (1.1) *admits a solution.*

8. *Assume that there exist* $\gamma > 0$, $p > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*
$$\lim_{s\to+\infty}\frac{a(s)}{(\log s)^{2\gamma}} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{s^p} = f_\infty. \tag{1.8}$$
Then (1.1) *admits a solution if and only if* $p > 1$.

9. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{a(s)}{(\log s)^{2\gamma}} = a_\infty, \qquad \lim_{s\to+\infty} \frac{f(s)}{(\log s)^{\beta}} = f_\infty.$$

Then (1.1) *admits no solution.*

Concerning the uniqueness of solutions, we have the following

Theorem 1.2 (Uniqueness of solutions) *Suppose that*

$$2f'(s)a(s) - f(s)a'(s) \geq 0 \quad \text{if } s \geq 0, \quad f(s) = 0 \quad \text{if } s \leq 0, \quad f(s) > 0 \quad \text{if } s > 0, \tag{1.9}$$

and that

$$\lim_{s\to+\infty} \frac{[2f'(s)a(s) - f(s)a'(s)]g^{-1}(s)}{2(a(s))^{\frac{3}{2}} f(s)} > 1, \tag{1.10}$$

where g is as defined in (1.2)*. Then* (1.1) *admits a unique solution, which is positive. Moreover, assume that a and f satisfy one of the existence conditions of Theorem 1.1 and* $\partial\Omega$ *is of class* C^3 *and its mean curvature is nonnegative. Then if* (1.9) *is satisfied and if*

$$\text{there exists } R > 0 \text{ such that:} \qquad \Big(\frac{f(g(s))}{\sqrt{a(g(s))}}\Big)^{\frac{1}{2}} \text{ is convex in } (R, +\infty), \tag{1.11}$$

then (1.1) *admits a unique solution, which is positive.*

Now consider problem (1.1) *set in the unit ball* $B_1(0)$

$$\begin{cases} \operatorname{div}(a(u)Du) = \frac{a'(u)}{2}|Du|^2 + f(u) & \text{in } B_1(0), \\ u(x) \to +\infty \quad \text{as } \mathrm{d}(x, \partial B_1(0)) \to 0. \end{cases} \tag{1.12}$$

Let λ_1 *be the first eigenvalue of* $-\Delta$ *in* $B_1(0)$ *with Dirichlet boundary conditions. Assume a and f satisfy one of the existence conditions of Theorem 1.1 and, in addition, that*

$$2f'(s)a(s) - f(s)a'(s) + 2\lambda_1 a^2(s) \geq 0, \qquad \text{for all } s \in \mathbb{R}. \tag{1.13}$$

Then (1.12) *admits a unique solution.*

Concerning the multiplicity of solutions, we have the following

Theorem 1.3 (Nonuniqueness of solutions) *Assume that the functions a and f satisfy one of the existence conditions of Theorem 1.1. Let* Ω *be bounded, convex,* C^2, $f(0) = 0$ *and*

$$\text{there exists } R \geq 0 \text{ such that:} \qquad f|_{(R,+\infty)} > 0, \qquad (2f'a - fa')|_{(R,+\infty)} \geq 0. \tag{1.14}$$

Assume that there exists $1 < q < \frac{N+2}{N-2}$ *if* $N \geq 3$, $q > 1$ *if* $N = 1, 2$ *such that*

$$0 < \lim_{s\to-\infty} \frac{f(s)}{\sqrt{a(s)}|g^{-1}(s)|^q} < +\infty.$$

Then (1.1) *admits two solutions, one positive and one sign-changing. In particular, if there exist* $k > 0$ *and* $p_-, p_+ > 1$ *such that*

$$0 < \lim_{s\to+\infty} \frac{a'(s)}{s^{k-1}} < +\infty, \qquad a(-s) = a(s), \;\; \textit{for all } s \in \mathbb{R}, \tag{1.15}$$

$$0 < \lim_{s\to-\infty} \frac{f(s)}{|s|^{p_-}} < +\infty, \qquad k+1 < p_- < \frac{k}{2} + \frac{k+2}{2}\frac{N+2}{N-2} \;\; \textit{for } N \geq 3, \tag{1.16}$$

$$0 < \lim_{s\to-\infty} \frac{f(s)}{|s|^{p_-}} < +\infty, \qquad p_- > k+1 \;\; \textit{for } N = 1, 2, \tag{1.17}$$

$$0 < \lim_{s\to+\infty} \frac{f'(s)}{s^{p_+-1}} < +\infty, \qquad p_+ > k+1 \;\; \textit{for } N \geq 1, \tag{1.18}$$

then (1.1) *admits two solutions, one positive and one sign-changing.*

Concerning the symmetry of solutions to (1.12), we have the following

Theorem 1.4 (Symmetry of solutions) *Let* a *and* f *be of class* $C^2(\mathbb{R})$. *Then the following statements hold:*

1. *Assume that there exist* $k > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{a'(s)}{s^{k-1}} = ka_\infty, \quad \lim_{s\to+\infty} \frac{a''(s)}{s^{k-2}} = k(k-1)a_\infty, \tag{1.19}$$

$$\lim_{s\to+\infty} \frac{f''(s)}{e^{2\beta s}} = 4\beta^2 f_\infty, \tag{1.20}$$

   (only the right limit in (1.19) *for* $k > 1$*). Then any solution to* (1.12) *is radially symmetric and increasing.*

2. *Assume that there exist* $k > 0$, $p > 1$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that* (1.19) *hold and*

$$\lim_{s\to+\infty} \frac{f''(s)}{s^{p-2}} = p(p-1)f_\infty. \tag{1.21}$$

   Then, if $p > k+1$, *any solution to* (1.12) *is radially symmetric and increasing.*

3. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that* (1.20) *holds and*

$$\lim_{s\to+\infty} \frac{a''(s)}{e^{2\gamma s}} = 4\gamma^2 a_\infty.$$

   Then, if $\beta > \gamma$, *any solution to* (1.12) *is radially symmetric and increasing.*

4. *Assume that there exist* $\gamma > 0$, $\beta > 0$, $a_\infty > 0$ *and* $f_\infty > 0$ *such that* (1.20) *holds and*

$$\lim_{s\to+\infty} \frac{a'(s)s}{(\log s)^{2\gamma-1}} = 2\gamma a_\infty, \quad \lim_{s\to+\infty} \frac{a''(s)s^2}{(\log s)^{2\gamma-1}} = -2\gamma a_\infty. \tag{1.22}$$

   Then, any solution to (1.12) *is radially symmetric and increasing.*

5. *Assume that* (1.21) *and* (1.22) *hold. Then, if* $p > 1$, *any solution to* (1.12) *is radially symmetric and increasing.*

Concerning the blow-up behavior of solution, we have two results. The first is the following

Theorem 1.5 (Boundary behavior I) *Let Ω be a bounded domain of $\mathbb{R}^N$ satisfying an inner and an outer sphere condition on $\partial\Omega$. Let η denote the unique solution to problem*

$$\eta' = -\sqrt{2F \circ g \circ \eta}, \qquad \lim_{t\to 0^+} \eta(t) = +\infty. \tag{1.23}$$

Then any solution $u \in C^2(\Omega)$ to (1.1) *satisfies*

$$u(x) = g \circ \eta(\mathrm{d}(x, \partial\Omega)) + o(1),$$

whenever $\mathrm{d}(x, \partial\Omega)$ *goes to zero, provided that one of the following situations occurs:*

1. *Conditions* (1.3) *hold.*
2. *Conditions* (1.4) *hold with $p > 2k + 3$.*
3. *Conditions* (1.6) *with $\beta > 2\gamma$.*
4. *Conditions* (1.7) *hold.*
5. *Conditions* (1.8) *hold with $p > 3$.*

In general, in addition to the blow-up term $g \circ \eta(\mathrm{d}(x, \partial\Omega))$, the expansion of a large solution u could contain other blow-up terms, one of them typically depends upon the local mean curvature of the boundary. We will study this in a particular, but meaningful, situation.
For $p > k + 1$, in the framework of (1.4), let us now introduce the positive constants

$$\Gamma := \left[\frac{p-k-1}{\sqrt{2(p+1)}} \frac{\sqrt{f_\infty}}{\sqrt{a_\infty}} \right]^{\frac{2}{k+1-p}} \qquad \text{and} \qquad \Gamma' := \frac{2(N-1)}{p+k+3}\Gamma. \tag{1.24}$$

When a and f behave like powers at infinity, we have the following characterization

Theorem 1.6 (Boundary behavior II) *Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^4 and assume that* (1.4) *hold with $p > k + 1$. If conditons* (1.14) *hold with $R = 0$ and η is as in* (1.23)*, let us set*

$$J(t) := \frac{N-1}{2} \int_0^t \frac{\int_0^{\eta(s)} \sqrt{2F(g(\sigma))}d\sigma}{F(g(\eta(s)))} ds, \qquad t > 0,$$
$$\mathbb{T}(x) := \frac{(\mathrm{d}(x, \partial\Omega))^{\frac{k+2}{k+1-p}}}{\min\{u^{k/2}(x)), (\mathrm{d}(x, \partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x, \partial\Omega)))^{k/(k+1-p)}\}}, \qquad x \in \Omega,$$

where $\sigma(x)$ denotes the projection on $\partial\Omega$ of a $x \in \Omega$ and $\mathcal{H}$ is the mean curvature of $\partial\Omega$. Then there exists a positive constant L such that

$$|u(x) - g \circ \eta(\mathrm{d}(x, \partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x, \partial\Omega)))| \leq L\mathbb{T}(x)o(\mathrm{d}(x, \partial\Omega)),$$

whenever $\mathrm{d}(x, \partial\Omega)$ *goes to zero. Furthermore, the following facts hold:*

1. *If $p > 2k + 3$ (even if* (1.14) *do not hold), then*

$$u(x) = \frac{\Gamma}{(\mathrm{d}(x, \partial\Omega))^{\frac{2}{p-k-1}}}(1 + o(1)),$$

whenever x approaches $\partial\Omega$.

2. *If* (1.14) *hold with* $R = 0$ *and* $k + 1 < p \leq 2k + 3$*, then*

$$u(x) = \Gamma \frac{1}{(\mathrm{d}(x, \partial\Omega))^{\frac{2}{p-k-1}}}(1 + o(1)) + \Gamma' \frac{\mathcal{H}(\sigma(x))}{(\mathrm{d}(x, \partial\Omega))^{\frac{3+k-p}{p-k-1}}}(1 + o(1)),$$

whenever x *approaches* $\partial\Omega$*.*

The proofs of these assertions is contained in Sections 3-8.

2 Some remarks

Some remarks are now in order on the results stated in the previous section.

Remark 2.1 (Derivatives blow-up) According to a result due to Bandle and Marcus [3, see Section 3, Theorem 3.1] for general semi-linear equations, not only the solution u blows up along the boundary but also the modulus of the gradient $|Du|$ explodes. Hence, concerning problem (1.1), a result in the spirit of Theorem 1.5 for the gradient could be stated too, under suitable assumptions on the asymptotic behavior of a and f yielding to

$$\lim_{\mathrm{d}(x,\partial\Omega)\to 0} \frac{a(u)|Du(x)|^2}{2F \circ g \circ \eta(\mathrm{d}(x, \partial\Omega))} = 1,$$

under suitable assumptions on the domain. In the case (1.4) this turns into

$$|Du(x)| = \frac{\Gamma_\flat}{(\mathrm{d}(x, \partial\Omega))^{\frac{p+1-k}{p-k-1}}}(1 + o(1)), \qquad \text{for } \mathrm{d}(x, \partial\Omega) \to 0,$$

where $\Gamma_\flat := \frac{2\Gamma}{p-k-1}$, by exploiting the information provided in (1.77).

Remark 2.2 (Sign condition on a**)** Assume that a satisfies $a'(s)s \geq 0$ for every $s \in \mathbb{R}$. Then a nonnegative solution u of (1.1) satisfies the inequality $\mathrm{div}(a(u)Du) \geq f(u)$ in Ω. In this case the problem become simpler. We will not assume sign conditions on a.

Remark 2.3 (Nonnegative solutions) Assuming that $f(s) = 0$ for every $s \leq 0$, any solution to (1.31) is nonnegative (and hence any solution to (1.1) is nonnegative). In fact, since $g(0) = 0$ and $g' > 0$, it is $g(s) \leq 0$ for all $s \leq 0$ and therefore

$$h(s) = \frac{f(g(s))}{\sqrt{a(g(s))}} = 0, \qquad \text{for every } s \leq 0. \tag{1.25}$$

Let $v : \Omega \to \mathbb{R}$ be a classical solution of $\Delta v = h(v)$ such that $v(x) \to +\infty$ as x approaches $\partial\Omega$ and define the open bounded set $\Omega_- = \{x \in \Omega : v(x) < 0\}$. We have

$$\partial\Omega_- \subseteq \{x \in \Omega : v(x) = 0\}. \tag{1.26}$$

In fact, if $x \in \partial\Omega_- = \overline{\Omega_-} \setminus \Omega_-$, there is a sequence $(\xi_j) \subset \Omega$ with $\xi_j \to x$ and $v(\xi_j) < 0$. It follows that $x \in \Omega$, otherwise $v(\xi_j) \to +\infty$ if $x \in \partial\Omega$. In addition, $v(x) \leq 0$. Since $x \notin \Omega_-$, we also have $v(x) \geq 0$, proving the claim. In view of (1.25), v is harmonic in Ω_-. Assume by contradiction that $\Omega_- \neq \emptyset$ and let $\xi \in \Omega_-$. By (1.26) and the maximum principle in Ω_-

$$v(\xi) \geq \min_{\Omega_-} v \geq \min_{\bar{\Omega}_-} v = \min_{\partial\Omega_-} v = 0,$$

yielding a contradiction. Then $v \geq 0$ and for the solution u of (1.1), $u = g(v) \geq 0$.

Remark 2.4 (Negative large solutions) In analogy with the study of positively blowing up solutions, it is possible to formulate existence and nonexistence results for the problem

$$\begin{cases} \operatorname{div}(a(u)Du) = \frac{a'(u)}{2}|Du|^2 + f(u) & \text{in } \Omega, \\ u(x) \to -\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{1.27}$$

by assuming that a is an even function, that there exists $r \in \mathbb{R}$ such that $f(r) < 0$ and $f(s) \leq 0$ for all $s < r$ and by prescribing suitable asymptotic conditions on a and f as $s \to -\infty$. Furthermore, by arguing as in Remark 2.3, it is readily seen that if $f(s) = 0$ for every $s \geq 0$, any solution to (1.27) is nonpositive. See Remark 3.1 in Section 3 for more details on how to detect solutions to (1.27) when a is even by reducing the problem to a related one with positive blow-up.

Remark 2.5 (Lower bound of solutions) Under assumption (1.9) and (1.10) problem (1.1) has a unique positive solution u in Ω and it is possible to estimate the minimum of u in Ω in terms of the minimum of the unique radial solution z of (1.1) in a ball B such that $|B| = |\Omega|$, yielding

$$\min_{x\in\Omega} u(x) \geq \min_{x\in B} z(x). \tag{1.28}$$

To prove this, let us consider the associated semi-linear problem (1.31). Quoting a result of [23] (see Theorem 3.1), we get

$$\min_{x\in\Omega} v(x) \geq \min_{x\in B} z_\sharp(x)$$

where $z_\sharp(x)$ is the unique solution of (1.31) in a ball B such that $|B| = |\Omega|$. The monotonicity of g then yields (1.28) via Lemma 3.1. Moreover, in some cases, the unique radial solution $z_\sharp$ of (1.31) in a ball B is explicitly known and this provides an estimate on the minimum of $z_\sharp$ (hence of z) in terms of $|\Omega|$.

Remark 2.6 (Convexity of sublevel sets in strictly convex domains) Assume the domain Ω is strictly convex and that for a solution u to (1.1) we have:

$$u > 0, \quad 2f'(s)a(s) - f(s)a'(s) > 0 \text{ for } s > 0 \quad s \mapsto \frac{\sqrt{a(g(s))}}{f(g(s))} \quad \text{is convex on } (0, +\infty). \tag{1.29}$$

Then $g^{-1}(u)$ is strictly convex for $N = 2$. The same occurs in higher dimensions provided that the Gauss curvature of $\partial\Omega$ is strictly positive (see [39], for example, for the definition of Gauss curvature of a surface). In these cases, furthermore, the sublevel sets of u are strictly convex. In fact, we have $u = g(v)$, for some $v \in C^2(\Omega)$, $v > 0$, which satisfies $\Delta v = h(v)$, see Lemma 3.1, where h is defined as in (1.31). Furthermore, by (1.29), we have that $h > 0$, h strictly increasing and $1/h$ convex. Let us consider the Concavity function

$$C(v, x, y) := v(\frac{x+y}{2}) - \frac{1}{2}v(x) - \frac{1}{2}v(y)$$

defined in $\Omega \times \Omega$ as introduced in [27] to study the convexity of the level sets of solutions of some semi-linear equations. We are then in position to apply a result [25, Theorem 3.13], by Kawohl, which implies that the Concavity function cannot attain a positive maximum in $\Omega \times \Omega$. Moreover by [25, Lemma 3.11], the Concavity function is negative in a neighborhood of $\partial(\Omega \times \Omega)$ so that $C(v, x, y) \leq 0$ in $\Omega \times \Omega$ and hence v is convex in Ω. If $N = 2$, from a result of Caffarelli and Friedman [7, Corollary 1.3], v needs to be strictly convex in Ω. In higher dimensions, assuming that the Gauss curvature of Ω is strictly positive, it is possible to find some points close to the boundary $\partial\Omega$ where the Hessian matrix of v has full rank. Then, from a result due to Korevaar and Lewis [28, Theorem 1] we would get that v is strictly convex. In these cases, from the strict monotonicity of g, we get that the (closed) sublevel sets of u are strictly convex.

Remark 2.7 (A case of uniqueness) Assume that $p > k + 1$ and

$$f(s) = \begin{cases} s^p & \text{if } s \geq 0, \\ 0 & \text{if } s < 0, \end{cases} \qquad a(s) = 1 + |s|^k, \quad s \in \mathbb{R}. \tag{1.30}$$

Then condition (1.9) is satisfied. Moreover, using Lemma 3.2 with $a_\infty = 1$, we have

$$\begin{aligned} &\lim_{s\to+\infty} \frac{[2f'(s)a(s) - f(s)a'(s)]g^{-1}(s)}{2(a(s))^{\frac{3}{2}} f(s)} \\ &= \lim_{s\to+\infty} \frac{2ps^{p-1}(1+s^k) - s^p(ks^{k-1})}{2(1+s^k)^{\frac{3}{2}} s^p} \frac{g^{-1}(s)}{s^{\frac{k+2}{2}}} s^{\frac{k+2}{2}} \\ &= \lim_{s\to+\infty} \frac{(2p-k)s^{p+k-1}(1+o(1))s^{\frac{k+2}{2}}}{2s^{p+\frac{3}{2}k}(1+o(1))} \frac{1}{g_\infty^{\frac{k+2}{2}}} = \frac{2p-k}{k+2} > 1, \end{aligned}$$

since $p > k + 1$, so that (1.10) is fulfilled. In particular

$$\begin{cases} \operatorname{div}((1+u^k)Du) - \frac{k}{2}u^{k-1}|Du|^2 = u^p & \text{in } \Omega, \\ u(x) \to +\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases}$$

admits a unique nonnegative solution in every bounded smooth domain Ω.

Remark 2.8 Let us observe that, under the existence conditions of Theorem (1.1), condition (1.11) is always satisfied. This is proved, for instance, in Section 6 where we assume that a and g are of class C^2. Then, if Ω is of class C^3 and has positive mean curvature on $\partial\Omega$ the solution of (1.1) is unique if the function h is nondecreasing.

Remark 2.9 Various results appeared in the recent literature about existence and qualitative properties of large solutions for the m-Laplacian equation $\Delta_m u = f(u)$ with $m > 1$ on a smooth bounded Ω, see for example [23]. On this basis, using a suitable modification of the change of variable Cauchy problem (1.2) and of the Keller-Ossermann condition (1.33), many of the properties stated in our results might be extended to cover the study of blow-up solutions of

$$\operatorname{div}(a(u)|Du|^{m-2}Du) = \frac{a'(u)}{m}|Du|^m + f(u) \quad \text{in } \Omega.$$

Remark 2.10 The results of the paper can be extended, with slight adaptations, to the quasi-linear elliptic problem $\operatorname{div}(a(u)Du) = \beta a'(u)|Du|^2 + f(u)$, for $\beta < 1$, by arguing on the Cauchy problem $g'(s) = (a(g(s)))^{\beta-1}$, $g(0) = 0$. In fact, setting $u = g(v)$, v is a solution to $\Delta v = h(v)$, with $h(s) := f(g(s))(a(g(s))^{-\beta}$. For $\beta = 1/2$, this problem reduces precisely to the one investigated in this paper. For any $\beta \neq 1/2$ the problem is not variational.

3 Existence of solutions

Assume that $a : \mathbb{R} \to \mathbb{R}$ is a function of class C^1 such that there exists $\nu > 0$ with $a(s) \geq \nu$, for every $s \in \mathbb{R}$. The function $g : \mathbb{R} \to \mathbb{R}$, defined in (1.2), is smooth and strictly increasing. Then, it is possible to associate to problem (1.1) the semi-linear problem

$$\begin{cases} \Delta v = h(v) & \text{in } \Omega, \\ v(x) \to +\infty & \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{1.31}$$

where we let $h(s) := \frac{f(g(s))}{\sqrt{a(g(s))}}$. More precisely, we have the following

Lemma 3.1 *If $v \in C^2(\Omega)$ is a classical solution to problem* (1.31)*, then $u = g(v)$ is a classical solution to problem* (1.1)*. Vice versa, if $u \in C^2(\Omega)$ is a classical solution to problem* (1.1)*, then $v = g^{-1}(u)$ is a classical solution to problem* (1.31)*.*

Proof. Observe first that the solution g to the Cauchy problem (1.2) is globally defined, of class C^2, strictly increasing (thus invertible with inverse g^{-1}) and

$$\lim_{s\to\pm\infty} g(s) = \pm\infty, \qquad \lim_{s\to\pm\infty} g^{-1}(s) = \pm\infty. \tag{1.32}$$

Now, if $v \in C^2(\Omega)$ is a classical solution to (1.31), from the regularity of g it is $u = g(v) \in C^2(\Omega)$. Furthermore, from (1.32), it is $u(x) \to +\infty$ as $\mathrm{d}(x, \partial\Omega) \to 0$. In addition $Dv = a^{1/2}(u)Du$ and $\Delta v = a^{1/2}(u)\Delta u + \frac{1}{2}a'(u)a^{-1/2}(u)|Du|^2$, so that from (1.31) it is $a(u)\Delta u + \frac{1}{2}a'(u)|Du|^2 - f(u) = 0$ which easily yields the assertion. Vice versa, if u is a classical solution to (1.1), then $v = g^{-1}(u)$ is of class C^2 and by (1.32), it is $v(x) \to +\infty$ as $\mathrm{d}(x, \partial\Omega) \to 0$. Moreover, it follows that $\Delta v = a^{1/2}(u)\Delta u + \frac{1}{2}a'(u)a^{-1/2}(u)|Du|^2 = f(u)a^{-1/2}(u) = f(g(v))a^{-1/2}(g(v))$, concluding the proof.

On the basis of Lemma 3.1, we try to establish existence of solutions of problem (1.1) using existence condition known in the literature for semi-linear problems like (1.31).

Hereafter we let $f : \mathbb{R} \to \mathbb{R}$ be a C^1 function. Consider the following condition:

E. There exists $r \in \mathbb{R}$ such that $f(r) > 0$ and $f(s) \geq 0$ for all $s > r$ and

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F \circ g}} < +\infty, \qquad F(t) := \int_r^t f(\tau)d\tau. \tag{1.33}$$

Essentially, **E** depends upon the asymptotic behavior of the function F and g.

Proposition 3.1 *Let Ω be any smooth bounded domain in $\mathbb{R}^N$. Then problem* (1.1) *admits a solution if and only if* **E** *holds.*

Proposition 3.1 readily follows by combining Lemma 3.1 with the assertion of [12, Theorem 1.3], where the authors proved the equivalence between the Keller-Osserman condition, the sharpened Keller-Osserman condition and the existence of blow-up solutions in arbitrary bounded domains without any monotonicity assumption on the nonlinearity.

We shall now investigate the asymptotic behavior of g as $s \to +\infty$ according to the cases when a behaves like a polynomial, an exponential function or a logarithmic function. In turn, in these situation, we discuss the validity of condition (1.33).

3.1 Polynomial growth

Assume that there exists $a_\infty > 0$ such that

$$\lim_{s\to+\infty} \frac{a(s)}{s^k} = a_\infty. \tag{1.34}$$

In [24, Lemma 2.1], we proved the following

Lemma 3.2 *Assume that condition* (1.34) *holds. Then, we have*

$$\lim_{s\to+\infty}\frac{g(s)}{s^{\frac{2}{k+2}}}=g_\infty,\qquad \lim_{t\to+\infty}\frac{g^{-1}(t)}{t^{\frac{k+2}{2}}}=g_\infty^{-\frac{k+2}{2}},\tag{1.35}$$

where $g_\infty=(\frac{k+2}{2}\frac{1}{\sqrt{a_\infty}})^{\frac{2}{k+2}}$.

We can now formulate the following existence result.

Proposition 3.2 (f with exponential growth) *Assume that* (1.34) *holds and that there exist* $\beta>0$ *and* $f_\infty>0$ *such that*

$$\lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}}=f_\infty.\tag{1.36}$$

Then (1.1) *always admits a solution in any smooth domain* Ω.

Proof. In light of assumption (1.36), there exists $r>0$ such that $f(t)>0$ for all $t\ge r$. Since

$$\lim_{s\to+\infty}\frac{F(s)}{e^{2\beta s}}=\frac{f_\infty}{2\beta},$$

on account of Lemma 3.2, for any $\varepsilon>0$ we obtain

$$\begin{aligned}\lim_{s\to+\infty}\frac{F(g(s))}{e^{(2\beta g_\infty-2\varepsilon)s^{\frac{2}{k+2}}}}&=\lim_{s\to+\infty}\frac{f_\infty e^{2\beta g_\infty s^{\frac{2}{2+k}}(1+o(1))}(1+o(1))}{2\beta e^{(2\beta g_\infty-2\varepsilon)s^{\frac{2}{k+2}}}}\\&=\frac{f_\infty}{2\beta}\lim_{s\to+\infty}e^{2s^{\frac{2}{2+k}}(\varepsilon+o(1))}(1+o(1))=+\infty.\end{aligned}$$

In particular, having fixed ε such that $\varepsilon<\beta g_\infty$, there exists $R=R(\varepsilon)>0$ such that

$$\sqrt{F(g(s))}\ge e^{(\beta g_\infty-\varepsilon)s^{\frac{2}{k+2}}},\qquad \text{for every } s\ge R.$$

Therefore,

$$\int_{g^{-1}(r)}^{+\infty}\frac{1}{\sqrt{F(g(s))}}ds\le\int_{g^{-1}(r)}^{R}\frac{1}{\sqrt{F(g(s))}}ds+\int_R^{+\infty}\frac{1}{e^{(\beta g_\infty-\varepsilon)s^{2/(k+2)}}}ds<+\infty.$$

The assertion then follows by Proposition 3.1.

Proposition 3.3 (f with polynomial growth) *Assume that* (1.34) *holds and that there exist* $p>1$ *and* $f_\infty>0$ *such that*

$$\lim_{s\to+\infty}\frac{f(s)}{s^p}=f_\infty.\tag{1.37}$$

Then (1.1) *admits a solution in any smooth domain* Ω *if and only if* $p>k+1$.

Proof. In light of (1.37), there exists $r>0$ such that $f(t)>0$ for all $t\ge r$. Since

$$\lim_{s\to+\infty}\frac{F(s)}{s^{p+1}}=\frac{f_\infty}{p+1},$$

on account of Lemma 3.2, we get

$$\lim_{s\to+\infty}\frac{F(g(s))}{s^{\frac{2(p+1)}{k+2}}}=\lim_{s\to+\infty}\frac{f_\infty(g(s))^{p+1}(1+o(1))}{(p+1)s^{\frac{2(p+1)}{k+2}}}$$
$$=\frac{f_\infty}{p+1}\lim_{s\to+\infty}\frac{(g_\infty s^{\frac{2}{k+2}}(1+o(1)))^{p+1}(1+o(1))}{s^{\frac{2(p+1)}{k+2}}}=\frac{f_\infty g_\infty^{p+1}}{p+1}.$$

In particular, there exists $R>0$ such that

$$\frac{\sqrt{f_\infty}g_\infty^{\frac{p+1}{2}}}{\sqrt{2p+2}}s^{\frac{p+1}{k+2}}\le\sqrt{F(g(s))}\le\frac{\sqrt{2f_\infty}g_\infty^{\frac{p+1}{2}}}{\sqrt{p+1}}s^{\frac{p+1}{k+2}},\qquad\text{for every } s\ge R.$$

Then, if $p>k+1$, we have

$$\int_{g^{-1}(r)}^{+\infty}\frac{1}{\sqrt{F(g(s))}}ds\le\int_{g^{-1}(r)}^{R}\frac{1}{\sqrt{F(g(s))}}ds+\frac{\sqrt{2p+2}}{\sqrt{f_\infty}g_\infty^{\frac{p+1}{2}}}\int_R^{+\infty}s^{-\frac{p+1}{k+2}}ds<+\infty,$$

so that **E** holds true. On the contrary, assuming that $p\le k+1$, for every $r\in\mathbb{R}$ with $f(r)>0$ and $f(t)\ge 0$ for all $t>r$, we obtain

$$\int_{g^{-1}(r)}^{+\infty}\frac{1}{\sqrt{F(g(s))}}ds\ge\frac{\sqrt{p+1}}{\sqrt{2f_\infty}g_\infty^{\frac{p+1}{2}}}\int_{\max\{R,g^{-1}(r)\}}^{+\infty}s^{-\frac{p+1}{k+2}}ds=+\infty.$$

The assertion then follows by Proposition 3.1.

We also have the following

Proposition 3.4 (**f with logarithmic growth**) *Assume that* (1.34) *holds and that there exist* $\beta>0$ *and* $f_\infty>0$ *such that*

$$\lim_{s\to+\infty}\frac{f(s)}{(\log s)^\beta}=f_\infty. \tag{1.38}$$

Then, in any smooth domain Ω*, problem* (1.1) *admits no solution.*

Proof. The proof proceeds just like in the proof of Proposition 3.3 observing that for any $k>0$ there exists $p_0<k+1$ such that $\sqrt{F(g(s))}\le s^{\frac{p_0+1}{k+2}}$ for s large.

3.2 Exponential growth

Assume now that there exist $\gamma>0$ and $a_\infty>0$ such that

$$\lim_{s\to+\infty}\frac{a(s)}{e^{2\gamma s}}=a_\infty. \tag{1.39}$$

Then we have the following

Lemma 3.3 *Assume that condition* (1.39) *holds. Then, we have*

$$\lim_{s\to+\infty}\frac{g(s)}{\log s}=\frac{1}{\gamma},\qquad\lim_{t\to+\infty}\frac{g^{-1}(t)}{e^{\gamma t}}=\frac{\sqrt{a_\infty}}{\gamma}. \tag{1.40}$$

Proof. From the definition of g, we have $g'(s)a^{1/2}(g(s)) = 1$. Integrating on $[0, s]$ yields

$$s = \int_0^s g'(\sigma)a^{1/2}(g(\sigma))d\sigma = \int_0^{g(s)} a^{1/2}(\sigma)d\sigma.$$

In turn, we reach

$$\begin{aligned}
\lim_{s\to+\infty} \frac{g(s)}{\frac{1}{\gamma}\log s} &= \gamma \lim_{s\to+\infty} sg'(s) = \gamma \lim_{s\to+\infty} \frac{s}{\sqrt{a(g(s))}} \\
&= \gamma \lim_{s\to+\infty} \frac{s}{\sqrt{a_\infty e^{2\gamma g(s)}(1+o(1))}} = \frac{\gamma}{\sqrt{a_\infty}} \lim_{s\to+\infty} \frac{\int_0^{g(s)} a^{1/2}(\sigma)d\sigma}{e^{\gamma g(s)}} \\
&= \frac{\gamma}{\sqrt{a_\infty}} \lim_{s\to+\infty} \frac{a^{1/2}(g(s))g'(s)}{\gamma e^{\gamma g(s)} g'(s)} = \frac{1}{\sqrt{a_\infty}} \lim_{s\to+\infty} \frac{a^{1/2}(g(s))}{e^{\gamma g(s)}} = 1,
\end{aligned}$$

in light of condition (1.39). Furthermore, taking into account the above computations,

$$\lim_{t\to+\infty} \frac{g^{-1}(t)}{e^{\gamma t}} = \lim_{s\to+\infty} \frac{s}{e^{\gamma g(s)}} = \frac{\sqrt{a_\infty}}{\gamma},$$

concluding the proof.

We can now formulate the following existence result.

Proposition 3.5 (f with exponential growth) *Assume that* (1.39) *holds and that there exist* $\beta > 0$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{f(s)}{e^{2\beta s}} = f_\infty. \tag{1.41}$$

Then (1.1) *admits a solution in any smooth domain* Ω *if and only if* $\beta > \gamma$.

Proof. In light of (1.41), there exists $r > 0$ such that $f(t) > 0$ for all $t \geq r$. Furthermore, taking into account Lemma 3.3 and that

$$\lim_{s\to+\infty} \frac{F(s)}{e^{2\beta s}} = \frac{f_\infty}{2\beta},$$

we have in turn, for any $\varepsilon > 0$,

$$\begin{aligned}
\lim_{s\to+\infty} \frac{F(g(s))}{s^{\frac{2\beta}{\gamma}-2\varepsilon}} &= \lim_{s\to+\infty} \frac{f_\infty}{2\beta} \frac{e^{2\beta g(s)}(1+o(1))}{s^{\frac{2\beta}{\gamma}-2\varepsilon}} \\
&= \frac{f_\infty}{2\beta} \lim_{s\to+\infty} \frac{e^{\frac{2\beta}{\gamma}(1+o(1))\log s}(1+o(1))}{s^{\frac{2\beta}{\gamma}-2\varepsilon}} \\
&= \frac{f_\infty}{2\beta} \lim_{s\to+\infty} \frac{s^{\frac{2\beta}{\gamma}} s^{o(1)}}{s^{\frac{2\beta}{\gamma}-2\varepsilon}} = \frac{f_\infty}{2\beta} \lim_{s\to+\infty} s^{2\varepsilon+o(1)} = +\infty.
\end{aligned}$$

Consider the case $\beta > \gamma$. We can fix ε in such a way that $0 < \varepsilon < \frac{\beta}{\gamma} - 1$. Corresponding to this choice of ε there exists $R = R(\varepsilon) > 0$ large enough that

$$F(g(s)) \geq s^{\frac{2\beta}{\gamma}-2\varepsilon}, \qquad \text{for every } s \geq R,$$

yielding in turn,

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \le \int_{g^{-1}(r)}^{R} \frac{1}{\sqrt{F(g(s))}} ds + \int_{R}^{+\infty} \frac{1}{s^{\frac{\beta}{\gamma}-\varepsilon}} ds < +\infty,$$

so that **E** is fulfilled. Assume now that $\beta < \gamma$. Given $\varepsilon > 0$, by arguing as above, we obtain

$$\lim_{s\to+\infty} \frac{F(g(s))}{s^{\frac{2\beta}{\gamma}+2\varepsilon}} = \lim_{s\to+\infty} \frac{f_\infty}{2\beta} \frac{s^{\frac{2\beta}{\gamma}} s^{o(1)}}{s^{\frac{2\beta}{\gamma}+2\varepsilon}} = \frac{f_\infty}{2\beta} \lim_{s\to+\infty} \frac{s^{o(1)}}{s^{2\varepsilon}} = 0.$$

Hence, fixed $\bar{\varepsilon} > 0$ sufficiently small that $\frac{\beta}{\gamma} + \bar{\varepsilon} < 1$, there exists $R = R(\bar{\varepsilon}) > 0$ such that

$$F(g(s)) \le s^{\frac{2\beta}{\gamma}+2\bar{\varepsilon}}, \qquad \text{for every } s \ge R.$$

Thus, for every $r \in \mathbb{R}$ with $f(r) > 0$ and $f(t) \ge 0$ for all $t > r$, we obtain

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \ge \int_{\max\{R, g^{-1}(r)\}}^{+\infty} \frac{1}{s^{\frac{\beta}{\gamma}+\bar{\varepsilon}}} ds = +\infty.$$

The assertion then follows by Proposition 3.1.

Next, we formulate the following existence result.

Proposition 3.6 (*f* **with polynomial growth**) *Assume that* (1.39) *holds and that there exist* $p > 1$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{f(s)}{s^p} = f_\infty. \tag{1.42}$$

Then, in any smooth domain Ω*, problem* (1.1) *admits no solution.*

Proof. Since

$$\lim_{s\to+\infty} \frac{F(s)}{s^{p+1}} = \frac{f_\infty}{p+1},$$

there exist $\beta < \gamma$ and $R > 0$ such that $F(s) \le e^{2\beta s}$ for every $s \ge R$. Fixed now $\bar{\varepsilon} > 0$ so small that $\frac{\beta}{\gamma}(1 + \bar{\varepsilon}) < 1$, taking into account Lemma 3.3, there exists $R = R(\bar{\varepsilon}) > 0$ such that

$$\sqrt{F(g(s))} \le e^{\beta g(s)} = e^{\beta \frac{g(s)}{\log s} \log s} \le s^{\frac{\beta}{\gamma}(1+\bar{\varepsilon})},$$

for all $s \ge R$. Then, for every $r \in \mathbb{R}$ with $f(r) > 0$ and $f(t) \ge 0$ for all $t > r$, we have

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \ge \int_{\max\{R, g^{-1}(r)\}}^{+\infty} \frac{1}{s^{\frac{\beta}{\gamma}(1+\bar{\varepsilon})}} ds = +\infty.$$

The assertion then follows by Proposition 3.1.

We also have the following

Proposition 3.7 (*f* **with logarithmic growth**) *Assume that* (1.39) *holds and that there exist* $\beta > 0$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{f(s)}{(\log s)^\beta} = f_\infty. \tag{1.43}$$

Then, in any smooth domain Ω*, problem* (1.1) *admits no solution.*

Proof. The proof proceeds as for Proposition 3.6 since $F(s) \le e^{2\beta s}$ with $\beta < \gamma$ for s large.

3.3 Logarithmic growth

Assume now that there exist $\gamma > 0$ and $a_\infty > 0$ such that

$$\lim_{s\to+\infty} \frac{a(s)}{(\log s)^{2\gamma}} = a_\infty. \tag{1.44}$$

Then we have the following

Lemma 3.4 *Assume that condition* (1.44) *holds. Then, we have*

$$\lim_{s\to+\infty} \frac{g(s)}{s^{1-\varepsilon}} = +\infty, \qquad \lim_{s\to+\infty} \frac{g(s)}{s} = 0. \tag{1.45}$$

for every $\varepsilon \in (0,1)$. *In particular, for every* $\varepsilon \in (0,1)$ *there exists* $R = R(\varepsilon) > 0$ *such that*

$$s^{1-\varepsilon} \le g(s) \le s, \quad \textit{for every } s \ge R. \tag{1.46}$$

Proof. We have, for every $\varepsilon \in (0,1)$

$$\begin{aligned}
\lim_{s\to+\infty} \frac{g(s)}{s^{\frac{1}{1+\varepsilon}}} &= \lim_{s\to+\infty} \Big(\frac{g^{1+\varepsilon}(s)}{s}\Big)^{\frac{1}{1+\varepsilon}} = \Big(\lim_{s\to+\infty} \frac{g^{1+\varepsilon}(s)}{\int_0^{g(s)} \sqrt{a(\sigma)}d\sigma}\Big)^{\frac{1}{1+\varepsilon}} \\
&= \Big(\lim_{s\to+\infty} \frac{(1+\varepsilon)g^{\varepsilon}(s)g'(s)}{\sqrt{a(g(s))}g'(s)}\Big)^{\frac{1}{1+\varepsilon}} \\
&= (1+\varepsilon)^{\frac{1}{1+\varepsilon}}\Big(\lim_{s\to+\infty} \frac{g^{\varepsilon}(s)}{(\log g(s))^{\gamma}} \frac{(\log g(s))^{\gamma}}{\sqrt{a(g(s))}}\Big)^{\frac{1}{1+\varepsilon}} = +\infty.
\end{aligned}$$

Furthermore, we have

$$\lim_{s\to+\infty} \frac{g(s)}{s} = \lim_{s\to+\infty} \frac{g(s)}{\int_0^{g(s)} a^{1/2}(\sigma)d\sigma} = \lim_{s\to+\infty} \frac{1}{a^{1/2}(g(s))} = 0,$$

which concludes the proof of the lemma.

We can now formulate the following existence result.

Proposition 3.8 (f with exponential growth) *Assume that* (1.44) *holds and that there exist* $\beta > 0$ *and* $f_\infty > 0$ *such that*

$$\lim_{s\to+\infty} \frac{f(s)}{e^{2\beta s}} = f_\infty. \tag{1.47}$$

Then (1.1) *admits a solution in any smooth domain* Ω.

Proof. By (1.47), there exists $r > 0$ such that $f(t) > 0$ for all $t \ge r$. By virtue of Lemma 3.4, for every $\varepsilon \in (0,1)$ there exists $R = R(\varepsilon) > 0$ such that

$$F(g(s)) = \frac{f_\infty}{2\beta} e^{2\beta g(s)}(1+o(1)) \ge \frac{f_\infty}{4\beta} e^{2\beta g(s)} \ge \frac{f_\infty}{4\beta} e^{2\beta s^{1-\varepsilon}},$$

for every $s \ge R$. In turn, for any $\varepsilon \in (0,1)$, we conclude

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \le \int_{g^{-1}(r)}^{R} \frac{1}{\sqrt{F(g(s))}} ds + \frac{2\sqrt{\beta}}{\sqrt{f_\infty}} \int_R^{+\infty} \frac{1}{e^{\beta s^{1-\varepsilon}}} ds < +\infty,$$

concluding the proof of **E**. The assertion then follows by Propositions 3.1.

Next we state the following existence result.

Proposition 3.9 (f with polynomial growth) *Assume that* (1.44) *holds and that there exist $p > 0$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty} \frac{f(s)}{s^p} = f_\infty. \tag{1.48}$$

Then (1.1) *admits a solution in any smooth domain Ω if and only if $p > 1$.*

Proof. By (1.48), there exists $r > 0$ such that $f(t) > 0$ for all $t \geq r$. By virtue of Lemma 3.4, for every $\varepsilon \in (0, 1)$ there exists $R = R(\varepsilon) > 0$ such that

$$F(g(s)) = \frac{f_\infty}{p+1} g^{p+1}(s)(1 + o(1)) \geq \frac{f_\infty}{2p+2} g^{p+1}(s) \geq \frac{f_\infty}{2p+2} s^{(p+1)(1-\varepsilon)},$$

for every $s \geq R$. In turn, if $p > 1$, fixed $\varepsilon \in (0, 1)$ with $\frac{(p+1)(1-\varepsilon)}{2} > 1$, we conclude

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \leq \int_{g^{-1}(r)}^{R} \frac{1}{\sqrt{F(g(s))}} ds + \frac{\sqrt{2p+2}}{\sqrt{f_\infty}} \int_R^{+\infty} s^{-\frac{(p+1)(1-\varepsilon)}{2}} ds < +\infty.$$

If $p \leq 1$, exploiting again Lemma 3.4, we can find R such that

$$F(g(s)) = \frac{f_\infty}{p+1} g^{p+1}(s)(1 + o(1)) \leq \frac{2f_\infty}{p+1} s^{p+1} \qquad \text{for all } s \geq R$$

yielding, as $\frac{p+1}{2} \leq 1$, for every $r \in \mathbb{R}$ with $f(r) > 0$ and $f(t) \geq 0$ for $t > r$,

$$\int_{g^{-1}(r)}^{+\infty} \frac{1}{\sqrt{F(g(s))}} ds \geq \frac{\sqrt{p+1}}{\sqrt{2f_\infty}} \int_{\max\{R, g^{-1}(r)\}}^{+\infty} s^{-\frac{p+1}{2}} ds = +\infty,$$

concluding the proof of condition **E**. The assertion then follows by Proposition 3.1.

Finally, we have the following

Proposition 3.10 (f with logarithmic growth) *Assume that* (1.44) *holds and that there exist $\beta > 0$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty} \frac{f(s)}{(\log s)^\beta} = f_\infty. \tag{1.49}$$

Then, in any smooth domain Ω, problem (1.1) *admits no solution.*

Proof. Taking into account Lemma 3.4, the proof proceeds as for Proposition 3.9 since there exists $p_0 < 1$ such that $F(s) \leq s^{p_0+1}$ for every s large.

Remark 3.1 (Negative large solutions II) Assume that a is even and consider the following condition:

E-. There exists $r \in \mathbb{R}$ such that $f(r) < 0$ and $f(s) \leq 0$ for all $s < r$ and

$$\int_{-\infty}^{g^{-1}(r)} \frac{1}{\sqrt{F \circ g}} < +\infty, \qquad F(t) := \int_r^t f(\tau) d\tau. \tag{1.50}$$

Then problem (1.27) has a solution if and only if **E-** holds. In fact, being a even, it is readily seen that g is odd, and letting

$$f_0(s) := -f(-s) \quad \text{for all } s \in \mathbb{R}, \qquad F_0(t) = \int_{-r}^t f_0(\tau) d\tau,$$

two changes of variable yield

$$\int_{g^{-1}(-r)}^{+\infty} \frac{1}{\sqrt{F_0 \circ g}} = \int_{-\infty}^{g^{-1}(r)} \frac{1}{\sqrt{F \circ g}}.$$

Therefore **E-** holds for problem (1.27) if and only if **E** holds for the problem

$$\begin{cases} \operatorname{div}(a(w)Dw) - \frac{a'(w)}{2}|Dw|^2 = f_0(w) & \text{in } \Omega, \\ w(x) \to +\infty \quad \text{as } \mathrm{d}(x, \partial\Omega) \to 0, \end{cases} \tag{1.51}$$

in which case (1.51) admits a solution. Then $u := -w$ is a solution to (1.27).

3.4 Proof of Theorem 1.1

The assertions of Theorem 1.1 follows from a combination of Propositions 3.2, 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9 and 3.10.

4 Uniqueness of solutions

Concerning the uniqueness of solutions to problems (1.1) and (1.12), we have the following

Proposition 4.1 *If* (1.9) *and* (1.10) *are satisfied then problem* (1.1) *admits a unique solution which is nonnegative. If, else, the conditions which guarantee the existence of solutions are fulfilled, if $\partial\Omega$ is of class C^3 and its mean curvature is nonnegative and* (1.9) *and* (1.11) *are satisfied, then problem* (1.1) *admits a unique solution which is nonnegative. Finally, if the conditions which guarantee the existence of solutions are fulfilled and* (1.13) *is satisfied then problem* (1.12) *admits a unique solution.*

Proof. According to [21, Theorem 1] problem (1.31) has a unique non-negative solution in a smooth bounded domain Ω if $h(0) = 0$ and $h'(s) \geq 0$ for any $s \geq 0$ and if there exist $m > 1$ and $t_0 > 0$ such that $\frac{h(t)}{t^m}$ is increasing for $t \geq t_0$. The second hypothesis (which guarantees the existence of the solution) is equivalent to requiring that $\lim_{t\to+\infty} \frac{th'(t)}{h(t)} > 1$. Recalling that $h(s) = f(g(s))a^{-1/2}(g(s))$, we have

$$h'(s) = \frac{2f'(g(s))a(g(s)) - f(g(s))a'(g(s))}{2a^2(g(s))}, \qquad \text{for every } s \in \mathbb{R}. \tag{1.52}$$

In turn, the uniqueness conditions of [21] turn into (1.9) and (1.10) which readily yields the desired conclusion since g is a bijection.

Now we quote a result of Costin, Dupaigne and Goubet [11, Theorem 1.3], which says that, under smoothness assumption on $\partial\Omega$ and positivity of its mean curvature, if h is nondecreasing and $\sqrt{h}$ is asymptotically convex then the solution of problem (1.31) is unique. Conditions (1.9) and (1.11) allow us to use their result and provide the uniqueness of the solution u of problem (1.1) via the transformation (1.2).

In the case of $\Omega = B_1(0)$, according to [10, Corollary 1.4], the uniqueness of large solutions of $\Delta v = h(v)$ in $B_1(0)$ is guaranteed provided that the existence conditions are satisfied and the map $\{s \mapsto h(s) + \lambda_1 s\}$ is nondecreasing on $\mathbb{R}$. The uniqueness condition turns into

$$2f'(g(s))a(g(s)) - f(g(s))a'(g(s)) + 2\lambda_1 a^2(g(s)) \geq 0, \qquad \text{for every } s \in \mathbb{R}.$$

which readily yields the desired conclusion since g is a bijection.

4.1 Proof of Theorem 1.2

The assertion of Theorem 1.2 follows by Proposition 4.1.

5 Nonuniqueness of solutions

In this section we discuss the existence of two distinct solutions to (1.1) under suitable assumptions on a and f. By virtue of [1, Theorem 1], we have the following

Proposition 5.1 *Let Ω be bounded, convex and C^2. Assume that condition* **E** *holds, $f(0) = 0$ and, for some $R > 0$ large,*

$$f|_{(R,+\infty)} > 0, \qquad \Big(\frac{f'}{f} - \frac{a'}{2a}\Big)|_{(R,+\infty)} \geq 0, \tag{1.53}$$

and there exists $1 < q < \frac{N+2}{N-2}$ if $N \geq 3$ and $q > 1$ for $N = 1, 2$ such that

$$0 < \lim_{s\to-\infty} \frac{f(s)}{\sqrt{a(s)}|g^{-1}(s)|^q} < +\infty. \tag{1.54}$$

Then problem (1.1) *admits at least two distinct solutions, one positive and one sign-changing.*

Proof. The function $h(s) = f(g(s))\sqrt{a^{-1/2}(g(s))}$ is smooth and $h(0) = 0$. Recalling that $g(s) \to +\infty$ as $s \to +\infty$, by virtue of (1.53) there exists $R > 0$ sufficiently large that

$$h(s) > 0, \qquad \frac{f'(g(s))}{f(g(s))} \geq \frac{a'(g(s))}{2a(g(s))}, \qquad \text{for any } s \geq R,$$

namely $h|_{(R,+\infty)} > 0$ and $h'|_{(R,+\infty)} \geq 0$. Finally, due to (1.54) and (1.32), we get

$$\lim_{s\to-\infty} \frac{h(s)}{|s|^q} = \lim_{s\to-\infty} \frac{f(g(s))}{\sqrt{a(g(s))}|s|^q} = \lim_{s\to-\infty} \frac{f(s)}{\sqrt{a(s)}|g^{-1}(s)|^q} \in (0, +\infty).$$

Whence, in light of [1, Theorem 1] we find two distinct large solutions $v_1 > 0$ and v_2 (with $v_2^- \neq 0$ and $v_2^+ \neq 0$) to the problem $\Delta v = h(v)$. In turn, via Lemma 3.1, $u_1 = g(v_1) > 0$ and $u_2 = g(v_2)$ (with $u_2^\pm = (g(v_2))^\pm = g(v_2^\pm) \neq 0$) are two distinct explosive solutions of problem (1.1). This concludes the proof.

Proposition 5.2 *Consider assumptions* (1.15)-(1.18) *in Proposition 5.1.Then* (1.1) *admits two solutions, one positive and one sign-changing in any C^2 convex and bounded domain.*

Proof. It suffices to verify that under condition (1.15)-(1.18), assumptions (1.53)-(1.54) of Proposition 5.1 are fulfilled. Let us observe first, that assumptions (1.53) and (1.54) imply (1.4), so that condition **E** holds. In light of (1.18) it is readily seen that the left condition in (1.53) is satisfied, for some $R > 0$ large enough. Moreover, by combining (1.15) and (1.18) and recalling that $p_+ > k + 1 > k/2$ it follows that also the right condition in (1.53) is satisfied, up to enlarging R. Concerning (1.54), recalling Lemma 3.2, (1.16)-(1.17) and the fact that g^{-1} is odd, choosing

$$1 < q := \frac{2p_- - k}{k+2} < \frac{N+2}{N-2}, \quad N \geq 3, \quad q > 1, \quad N = 1, 2,$$

we have

$$\lim_{s\to-\infty} \frac{f(s)}{\sqrt{a(s)}|g^{-1}(s)|^q} = \lim_{s\to-\infty} \frac{f(s)}{|s|^{p_-}} \lim_{s\to-\infty} \frac{|s|^{k/2}}{\sqrt{a(s)}} \lim_{s\to-\infty} \frac{|s|^{\frac{q(k+2)}{2}}}{|g^{-1}(s)|^q} \in (0, +\infty),$$

concluding the proof.

5.1 Proof of Theorem 1.3

The assertion of Theorem 1.3 follows by Propositions 5.1 and 5.2.

6 Symmetry of solutions

Concerning a first condition for the symmetry for the solutions of problem (1.1) in the ball $B_1(0)$, we have the following

Proposition 6.1 *Assume that the conditions which guarantee the existence of solutions are fulfilled and that there exists $\rho \in \mathbb{R}$ such that*

$$2f'(s)a(s) - f(s)a'(s) + 2\rho a^2(s) \geq 0, \qquad \textit{for all } s \in \mathbb{R}. \tag{1.55}$$

Then any solution to problem (1.12) *is radially symmetric and increasing.*

Proof. According to [10, Corollary 1.7] the symmetry of large solutions of $\Delta v = h(v)$ in $B_1(0)$ is guaranteed provided that the existence conditions are satisfied the map $\{s \mapsto h(s) + \rho s\}$ is nondecreasing on $\mathbb{R}$ for some $\rho \in \mathbb{R}$. Then, the assertion follows arguing as in the proof of Proposition 4.1.

Remark 6.1 Considering the same framework (1.30) of Remark 2.7, condition (1.55) is fulfilled for every choice of $\rho \geq 0$, and hence large solutions in $B_1(0)$ are radially symmetric and increasing.

Next, we would like to get the radial symmetry of the solutions to (1.12) in the unit ball under a merely asymptotic condition on the data a and f (as opposed to the global condition imposed in (1.55)) by using [34, Theorem 1.1]. Throughout the rest of this section we shall assume that $f \in C^2(\mathbb{R})$ and $a \in C^2(\mathbb{R})$. By direct computation, from (1.52), there holds

$$\begin{aligned} h''(s) = &\frac{1}{2}a^{-7/2}(g(s))\Big\{2f''(g(s))a^2(g(s)) - 3f'(g(s))a'(g(s))a(g(s)) \\ &- f(g(s))a''(g(s))a(g(s)) + 2f(g(s))(a'(g(s)))^2\Big\}, \qquad \text{for every } s \in \mathbb{R}. \end{aligned} \tag{1.56}$$

In [34, Theorem 1.1] one of the main assumption is that the function h is asymptotically convex, namely there exists $R > 0$ such that $h|_{(R,+\infty)}$ is convex. On account of formula (1.56), a sufficient condition for this to be the case is that

$$\liminf_{s\to+\infty} \Big\{2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2\Big\} > 0. \tag{1.57}$$

Hence this condition only depends on the asymptotic behavior of a and f and their first and second derivatives. We shall now discuss the various situations, as already done for the study of existence of solutions.

6.1 Polynomial growth

Assume that there exists $a_\infty > 0$ such that

$$\lim_{s\to+\infty} \frac{a'(s)}{s^{k-1}} = a_\infty k, \qquad \lim_{s\to+\infty} \frac{a''(s)}{s^{k-2}} = a_\infty k(k-1). \tag{1.58}$$

First observe that condition (1.58) implies (1.34), and that for $k > 1$ only the right limit in (1.58) is needed. We can now formulate the following symmetry results in the various situation where we have already established existence of large solutions.

Proposition 6.2 (**f with exponential growth**) *Assume that condition* (1.58) *holds and that there exist $\beta > 0$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty} \frac{f''(s)}{e^{2\beta s}} = 4\beta^2 f_\infty. \tag{1.59}$$

Then any solution to problem (1.1) *in $B_1(0)$ is radially symmetric and increasing.*

Proof. Condition (1.59) implies that

$$\lim_{s\to+\infty} \frac{f(s)}{e^{2\beta s}} = f_\infty, \qquad \lim_{s\to+\infty} \frac{f'(s)}{e^{2\beta s}} = 2\beta f_\infty.$$

Concerning (1.57), we have

$$\begin{aligned}
&2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2\\
&= 8\beta^2 f_\infty e^{2\beta s}(1+o(1))a_\infty^2 s^{2k}(1+o(1)) - 6\beta f_\infty e^{2\beta s}(1+o(1))a_\infty^2 k s^k s^{k-1}(1+o(1))\\
&- f_\infty e^{2\beta s}(1+o(1))a_\infty^2 k(k-1)s^{k-2}s^k(1+o(1)) + 2f_\infty e^{2\beta s}(1+o(1))a_\infty^2 k^2 s^{2k-2}(1+o(1))\\
&= f_\infty a_\infty^2 e^{2\beta s}s^{2k-2}\Big[8\beta^2 s^2(1+o(1)) - 6\beta k s(1+o(1)) - k(k-1) + 2k^2 + o(1)\Big]\\
&= f_\infty a_\infty^2 e^{2\beta s}s^{2k}[8\beta^2 + o(1)] > 0, \quad \text{for all } s > 0 \text{ large},
\end{aligned}$$

concluding the proof.

Proposition 6.3 (**f with polynomial growth**) *Assume that condition* (1.58) *holds and that there exist $p > 1$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty} \frac{f''(s)}{s^{p-2}} = f_\infty p(p-1). \tag{1.60}$$

Then, if $p > k+1$, any solution to (1.1) *in $B_1(0)$ is radially symmetric and increasing.*

Proof. First observe that condition (1.60) implies that $\lim_{s\to+\infty} \frac{f'(s)}{s^{p-1}} = f_\infty p$ so that (1.37) holds. Concerning (1.57), we have

$$\begin{aligned}
&2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2\\
&= 2p(p-1)f_\infty s^{p-2}a_\infty^2 s^{2k}(1+o(1)) - 3f_\infty p s^{p-1}a_\infty^2 k s^k s^{k-1}(1+o(1))\\
&- f_\infty s^p a_\infty^2 k(k-1)s^{k-2}s^k(1+o(1)) + 2f_\infty s^p a_\infty^2 k^2 s^{2k-2}(1+o(1))\\
&= f_\infty a_\infty^2 s^{p+2k-2}\Big[2p(p-1) - 3pk - k(k-1) + 2k^2 + o(1)\Big]\\
&= f_\infty a_\infty^2 s^{p+2k-2}\Big[2p^2 - (2+3k)p + k^2 + k + o(1)\Big]\\
&= f_\infty a_\infty^2 s^{p+2k-2}\Big[(p-k-1)(2p-k) + o(1)\Big] > 0
\end{aligned}$$

for all $s > 0$ large being $p > k+1$, concluding the proof.

6.2 Exponential growth

Assume that there exist $\gamma > 0$ and $a_\infty > 0$ such that

$$\lim_{s\to+\infty} \frac{a''(s)}{e^{2\gamma s}} = 4\gamma^2 a_\infty. \tag{1.61}$$

Then, we have the following

Proposition 6.4 (**f with exponential growth**) *Assume that conditions* (1.59) *and* (1.61) *hold. Then, if $\beta > \gamma$, any solution to* (1.1) *in $B_1(0)$ is radially symmetric and increasing.*

Proof. First observe that condition (1.61) implies

$$\lim_{s\to+\infty} \frac{a'(s)}{e^{2\gamma s}} = 2\gamma a_\infty, \qquad \lim_{s\to+\infty} \frac{a(s)}{e^{2\gamma s}} = a_\infty.$$

Concerning (1.57), we have

$$\begin{aligned}
&2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2 \\
&= 8\beta^2 f_\infty e^{2\beta s} a_\infty^2 e^{4\gamma s}(1 + o(1)) - 6\beta f_\infty e^{2\beta s} 2\gamma a_\infty^2 e^{2\gamma s} e^{2\gamma s}(1 + o(1)) \\
&- f_\infty e^{2\beta s} 4\gamma^2 a_\infty^2 e^{4\gamma s}(1 + o(1)) + 2f_\infty e^{2\beta s} 4a_\infty^2 \gamma^2 e^{4\gamma s}(1 + o(1)) \\
&= 4f_\infty a_\infty^2 e^{2(\beta+2\gamma)s}[2\beta^2 - 3\gamma\beta + \gamma^2 + o(1)] > 0
\end{aligned}$$

for all $s > 0$ large if $2\beta^2 - 3\gamma\beta + \gamma^2 = (\beta - \gamma)(2\beta - \gamma) > 0$. Since $\beta > \gamma$, we conclude.

6.3 Logarithmic growth

Assume that there exist $\gamma > 0$ and $a_\infty > 0$ such that

$$\lim_{s\to+\infty} \frac{a'(s)s}{(\log s)^{2\gamma-1}} = 2\gamma a_\infty, \qquad \lim_{s\to+\infty} \frac{a''(s)s^2}{(\log s)^{2\gamma-1}} = -2\gamma a_\infty. \tag{1.62}$$

Then, we have the following

Proposition 6.5 (**f with exponential growth**) *Assume that* (1.59) *and* (1.62) *hold for $\beta > 0$ and $f_\infty > 0$. Then any solution to* (1.1) *in $B_1(0)$ is radially symmetric and increasing.*

Proof. First observe that (1.62) implies (1.44). Then, concerning (1.57), we have

$$\begin{aligned}
&2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2 \\
&= 8\beta^2 f_\infty e^{2\beta s} a_\infty^2 (\log s)^{4\gamma}(1 + o(1)) - 12\beta f_\infty e^{2\beta s} \gamma a_\infty^2 \frac{(\log s)^{2\gamma-1}}{s} (\log s)^{2\gamma}(1 + o(1)) \\
&+ 2f_\infty e^{2\beta s} \gamma a_\infty^2 \frac{(\log s)^{2\gamma-1}}{s^2} (\log s)^{2\gamma}(1 + o(1)) + 8f_\infty e^{2\beta s} \gamma^2 a_\infty^2 \frac{(\log s)^{4\gamma-2}}{s^2}(1 + o(1)) \\
&= f_\infty a_\infty^2 e^{2\beta s}(\log s)^{4\gamma}\Big[8\beta^2 - \frac{12\beta\gamma}{s\log s} + \frac{2\gamma}{s^2\log s} + \frac{8\gamma^2}{s^2(\log s)^2} + o(1)\Big] > 0
\end{aligned}$$

for all $s > 0$ large, completing the proof.

Then, we have the following

Proposition 6.6 (**f with polynomial growth**) *Assume that* (1.60) *and* (1.62) *hold. Then, if $p > 1$, any solution to* (1.1) *in $B_1(0)$ is radially symmetric and increasing.*

Proof. Concerning (1.57), we have

$$
\begin{aligned}
&2f''(s)a^2(s) - 3f'(s)a'(s)a(s) - f(s)a''(s)a(s) + 2f(s)(a'(s))^2\\
&= 2f_\infty p(p-1)s^{p-2}a_\infty^2(\log s)^{4\gamma}(1+o(1)) - 6f_\infty p s^{p-1}\gamma a_\infty^2\frac{(\log s)^{2\gamma-1}}{s}(\log s)^{2\gamma}(1+o(1))\\
&+ 2f_\infty s^p\gamma a_\infty^2\frac{(\log s)^{2\gamma-1}}{s^2}(\log s)^{2\gamma}(1+o(1)) + 8f_\infty s^p\gamma^2a_\infty^2\frac{(\log s)^{4\gamma-2}}{s^2}(1+o(1))\\
&= f_\infty a_\infty^2 s^{p-2}(\log s)^{4\gamma}\Big[2p(p-1) - \frac{6p\gamma}{\log s} + \frac{2\gamma}{\log s} + \frac{8\gamma^2}{(\log s)^2} + o(1)\Big] > 0
\end{aligned}
$$

for all $s > 0$ large, completing the proof.

6.4 Proof of Theorem 1.4

The assertion of Theorem 1.4 follows by Propositions 6.2, 6.3, 6.4, 6.5 and 6.6.

7 Blow-up rate of solutions, I

Let Ω be a bounded domain of $\mathbb{R}^N$ which satisfies an inner and an outer sphere condition at each point of the boundary $\partial\Omega$. Consider the following condition

$$
\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt = 0, \tag{1.63}
$$

which merely depends upon the asymptotic behavior of F and a. Then, assuming that condition **E** holds, by directly applying [10, Theorem 1.10] to the semi-linear problem (1.31), if η denotes the unique solution to

$$
\eta' = -\sqrt{2F\circ g\circ\eta}, \qquad \lim_{t\to 0^+}\eta(t) = +\infty,
$$

it follows that any blow-up solution $v \in C^2(\Omega)$ to (1.31) satisfies

$$
v(x) = \eta(\mathrm{d}(x,\partial\Omega)) + o(1), \qquad \text{as } \mathrm{d}(x,\partial\Omega)\to 0 \tag{1.64}
$$

if and only if (1.63) holds. By virtue of (1.2) and the asymptotic behavior of $a(s)$ for s large (i.e. (1.34), (1.39) or (1.44)), it is readily verified that there exists a positive constant L such that

$$
|g(\tau_2) - g(\tau_1)| \le L\frac{|\tau_2-\tau_1|}{a^{\frac12}(\min\{g(\tau_1), g(\tau_2)\})}, \qquad \text{for every } \tau_1,\tau_2 > 0 \text{ large.} \tag{1.65}
$$

Therefore, under assumption (1.63), any blow-up solution $u \in C^2(\Omega)$ to the quasi-linear problem (1.1) satisfies

$$
|u(x) - g(\eta(\mathrm{d}(x,\partial\Omega)))| = |g(v(x)) - g(\eta(\mathrm{d}(x,\partial\Omega)))| \le L\frac{|v(x)-\eta(\mathrm{d}(x,\partial\Omega))|}{a^{\frac12}(\min\{u(x), g(\eta(\mathrm{d}(x,\partial\Omega)))\})}
$$

as $\mathrm{d}(x,\partial\Omega)\to 0$, namely due to (1.64)

$$
u(x) = g\circ\eta(\mathrm{d}(x,\partial\Omega)) + \frac{1}{a^{\frac12}(\min\{u(x), g(\eta(\mathrm{d}(x,\partial\Omega)))\})}o(1), \qquad \text{as } x\to x_0\in\partial\Omega. \tag{1.66}
$$

Moreover, in case (1.34) holds, then Lemma 3.2 implies there exists a positive constant C such that

$$a^{\frac{1}{2}}(\min\{u(x), g(\eta(\mathrm{d}(x,\partial\Omega)))\}) \geq C\min\{u(x)^{\frac{k}{2}}, \eta^{\frac{k}{k+2}}(\mathrm{d}(x,\partial\Omega))\},$$

as $\mathrm{d}(x,\partial\Omega) \to 0$, while, if (1.39) is satisfied, from Lemma 3.3 we have that there exists a positive constant C such that

$$a^{\frac{1}{2}}(\min\{u(x), g(\eta(\mathrm{d}(x,\partial\Omega)))\}) \geq C\min\{e^{\gamma u(x)}, e^{\gamma g(\eta(\mathrm{d}(x,\partial\Omega)))}\} \geq \min\{e^{\gamma u(x)}, \eta^{\frac{1}{\alpha}}(\mathrm{d}(x,\partial\Omega))\}$$

as $\mathrm{d}(x,\partial\Omega) \to 0$, for any $\alpha > 1$. Using (1.66) we get Theorem 1.5 once (1.63) is satisfied.

7.1 Polynomial growth

We have the following

Proposition 7.1 *Assume that there exist $k > 0$, $a_\infty > 0$, $p > k+1$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty}\frac{a(s)}{s^k} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{s^p} = f_\infty.$$

Then condition (1.63) *is fulfilled if and only if $p > 2k+3$. When $p < 2k+3$ it holds*

$$\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt = +\infty.$$

Proof. We denote by C a positive constant and by C' a constant without any sign restriction, which may vary from one place to another. By the proof of Proposition 3.3, we learn that there exists a constant $R > 0$ such that

$$Cs^{\frac{p+1}{k+2}} \leq \sqrt{F(g(s))} \leq 2Cs^{\frac{p+1}{k+2}}, \qquad \text{for every } s \geq R. \tag{1.67}$$

In turn, for every $t \geq R$, we obtain

$$Ct^{\frac{k+p+3}{k+2}} + C' \leq \int_0^t\sqrt{F(g(s))}ds \leq C + \int_R^t\sqrt{F(g(s))}ds \leq C' + Ct^{\frac{k+p+3}{k+2}}, \tag{1.68}$$

Furthermore, by (1.67)-(1.68), we get

$$\begin{aligned}
&\limsup_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt\\
&\leq C\limsup_{u\to+\infty}u^{\frac{p+1}{k+2}}\int_u^{+\infty}\frac{C' + Ct^{\frac{k+p+3}{k+2}}}{t^{\frac{3p+3}{k+2}}}dt\\
&\leq C\limsup_{u\to+\infty}u^{\frac{p+1}{k+2}}\int_u^{+\infty}t^{\frac{k-2p}{k+2}}dt = C\limsup_{u\to+\infty}u^{\frac{2k+3-p}{k+2}} = 0,
\end{aligned}$$

yielding (1.63). Assume now instead that $k+1<p\le 2k+3$, we have

$$\begin{aligned}
&\liminf_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt\\
&\ge C\liminf_{u\to+\infty}u^{\frac{p+1}{k+2}}\int_u^{+\infty}\frac{C'+Ct^{\frac{k+p+3}{k+2}}}{t^{\frac{3p+3}{k+2}}}dt\\
&\ge C\liminf_{u\to+\infty}u^{\frac{p+1}{k+2}}\int_u^{+\infty}(C't^{-\frac{3p+3}{k+2}}+Ct^{\frac{k-2p}{k+2}})dt\\
&=\liminf_{u\to+\infty}(C'u^{\frac{k-2p}{k+2}}+Cu^{\frac{2k+3-p}{k+2}})=\begin{cases}+\infty, & \text{for } p<2k+3\\ C, & \text{for } p=2k+3,\end{cases}
\end{aligned}$$

being $\frac{k-2p}{k+2}<-1$. This concludes the proof.

We have the following

Proposition 7.2 *Assume that there exist $k>0$, $a_\infty>0$, $\beta>0$ and $f_\infty>0$ such that*

$$\lim_{s\to+\infty}\frac{a(s)}{s^k}=a_\infty,\qquad \lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}}=f_\infty.$$

Then condition (1.63) *is always fulfilled.*

Proof. We denote by C a positive constant and by C' a constant without any sign restriction, which may vary from one one place to another. By the proof of Proposition 3.2, we learn that for any fixed ε there exists a constant $R=R(\varepsilon)>0$ such that

$$\sqrt{F(g(s))}\ge e^{(\beta g_\infty-\varepsilon)s^{\frac{2}{k+2}}},\qquad \text{for every } s\ge R. \tag{1.69}$$

Moreover, for every $\varepsilon>0$ fixed we have

$$\lim_{s\to+\infty}\frac{F(g(s))}{e^{(2\beta g_\infty+2\varepsilon)s^{\frac{2}{k+2}}}}=\frac{f_\infty}{2\beta}\lim_{s\to+\infty}\frac{e^{2\beta g_\infty s^{\frac{2}{k+2}}(1+o(1))}}{e^{(2\beta g_\infty+2\varepsilon)s^{\frac{2}{k+2}}}}=\frac{f_\infty}{2\beta}\lim_{s\to+\infty}\frac{e^{2\beta g_\infty o(1)s^{\frac{2}{k+2}}}}{e^{2\varepsilon s^{\frac{2}{k+2}}}}=0.$$

In turn, increasing R if needed, we have

$$e^{(\beta g_\infty-\varepsilon)s^{\frac{2}{k+2}}}\le\sqrt{F(g(s))}\le e^{(\beta g_\infty+\varepsilon)s^{\frac{2}{k+2}}},\qquad \text{for every } s\ge R. \tag{1.70}$$

Choose now $\bar\varepsilon>0$ such that $7\bar\varepsilon<\beta g_\infty$ and let $\bar R\ge R$ be such that

$$w^{k/2}\le e^{\bar\varepsilon w},\qquad \text{for every } w\ge \bar R^{\frac{2}{k+2}}.$$

Then, in light of inequality (1.70), we obtain

$$\begin{aligned}
\int_0^t\sqrt{F(g(s))}ds&=\int_0^{\bar R}\sqrt{F(g(s))}ds+\int_{\bar R}^t\sqrt{F(g(s))}ds\le C+\int_{\bar R}^t e^{(\beta g_\infty+\bar\varepsilon)s^{\frac{2}{k+2}}}ds\\
&=C+C\int_{\bar R^{\frac{2}{k+2}}}^{t^{\frac{2}{k+2}}}e^{(\beta g_\infty+\bar\varepsilon)w}w^{k/2}dw\le C+C\int_{\bar R^{\frac{2}{k+2}}}^{t^{\frac{2}{k+2}}}e^{(\beta g_\infty+2\bar\varepsilon)w}dw\\
&=C'+Ce^{(\beta g_\infty+2\bar\varepsilon)t^{\frac{2}{k+2}}}\le Ce^{(\beta g_\infty+2\bar\varepsilon)t^{\frac{2}{k+2}}},
\end{aligned}$$

for every t large. Therefore,

$$
\begin{aligned}
&\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt \\
&\leq C\lim_{u\to+\infty}e^{(\beta g_\infty+\bar{\varepsilon})u^{\frac{2}{k+2}}}\int_u^{+\infty}\frac{e^{(\beta g_\infty+2\bar{\varepsilon})t^{\frac{2}{k+2}}}}{e^{(3\beta g_\infty-3\bar{\varepsilon})t^{\frac{2}{k+2}}}}dt \\
&= C\lim_{u\to+\infty}e^{(\beta g_\infty+\bar{\varepsilon})u^{\frac{2}{k+2}}}\int_{u^{\frac{2}{k+2}}}^{+\infty}e^{(-2\beta g_\infty+5\bar{\varepsilon})w}w^{k/2}dw \\
&\leq C\lim_{s\to+\infty}e^{(\beta g_\infty+\bar{\varepsilon})s}\int_s^{+\infty}e^{(-2\beta g_\infty+6\bar{\varepsilon})w}dw \\
&\leq C\lim_{s\to+\infty}e^{(\beta g_\infty+\bar{\varepsilon})s}e^{(-2\beta g_\infty+6\bar{\varepsilon})s}=C\lim_{s\to+\infty}e^{-(\beta g_\infty-7\bar{\varepsilon})s}=0,
\end{aligned}
$$

yielding (1.63). This concludes the proof.

7.2 Exponential growth

Proposition 7.3 *Assume that there exist $\gamma>0$, $a_\infty>0$, $\beta>\gamma$ and $f_\infty>0$ such that*

$$
\lim_{s\to+\infty}\frac{a(s)}{e^{2\gamma s}}=a_\infty,\qquad \lim_{s\to+\infty}\frac{f(s)}{e^{2\beta s}}=f_\infty.
$$

Then condition (1.63) *is fulfilled provided that $\beta>2\gamma$. When $\beta<2\gamma$ it holds*

$$
\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt=+\infty.
$$

Proof. We denote by C a positive constant and by C' a constant without any sign restriction, which may vary from one place to another. By the proof of Proposition 3.5, we learn that for every $\varepsilon>0$ there exists a positive value $R=R(\varepsilon)$ large enough that $F(g(s))\geq s^{2\beta/\gamma-2\varepsilon}$ for every $s\geq R$. Furthermore, enlarging R if needed, we have $F(g(s))\leq s^{2\beta/\gamma+2\varepsilon}$, for every $s\geq R$. Assume that $\beta>2\gamma$. Choosing now $\bar{\varepsilon}>0$ so small that $2-\frac{\beta}{\gamma}+5\bar{\varepsilon}<0$, we have

$$
s^{\frac{2\beta}{\gamma}-2\bar{\varepsilon}}\leq F(g(s))\leq s^{\frac{2\beta}{\gamma}+2\bar{\varepsilon}},\qquad \text{for every } s\geq R. \tag{1.71}
$$

In turn, for every $t\geq R$, we obtain

$$
Ct^{1+\frac{\beta}{\gamma}-\bar{\varepsilon}}+C'\leq\int_0^t\sqrt{F(g(s))}ds\leq C+\int_R^t\sqrt{F(g(s))}ds\leq C'+Ct^{1+\frac{\beta}{\gamma}+\bar{\varepsilon}}\leq Ct^{1+\frac{\beta}{\gamma}+\bar{\varepsilon}}. \tag{1.72}
$$

In turn, by (1.71)-(1.72), we get

$$
\begin{aligned}
&\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt \\
&\leq C\lim_{u\to+\infty}u^{\frac{\beta}{\gamma}+\bar{\varepsilon}}\int_u^{+\infty}\frac{t^{1+\frac{\beta}{\gamma}+\bar{\varepsilon}}}{t^{3\beta/\gamma-3\bar{\varepsilon}}}dt \\
&\leq C\lim_{u\to+\infty}u^{\frac{\beta}{\gamma}+\bar{\varepsilon}}\int_u^{+\infty}t^{1-\frac{2\beta}{\gamma}+4\bar{\varepsilon}}dt=C\lim_{u\to+\infty}u^{2-\frac{\beta}{\gamma}+5\bar{\varepsilon}}=0,
\end{aligned}
$$

yielding (1.63). Assume now instead that $\beta < 2\gamma$. Choosing now $\bar{\varepsilon} > 0$ so small that $2 - \frac{\beta}{\gamma} - 5\bar{\varepsilon} > 0$, we have

$$\begin{aligned}\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt &\geq \lim_{u\to+\infty}u^{\frac{\beta}{\gamma}-\bar{\varepsilon}}\int_u^{+\infty}\frac{C'+Ct^{1+\frac{\beta}{\gamma}-\bar{\varepsilon}}}{t^{3\beta/\gamma+3\bar{\varepsilon}}}dt\\ &\geq \lim_{u\to+\infty}u^{\frac{\beta}{\gamma}-\bar{\varepsilon}}\int_u^{+\infty}C't^{-\frac{3\beta}{\gamma}-3\bar{\varepsilon}}+Ct^{1-\frac{2\beta}{\gamma}-4\bar{\varepsilon}}dt\\ &= \lim_{u\to+\infty}(C'u^{1-\frac{2\beta}{\gamma}-4\bar{\varepsilon}}+Cu^{2-\frac{\beta}{\gamma}-5\bar{\varepsilon}})=+\infty,\end{aligned}$$

concluding the proof since $1 - \frac{2\beta}{\gamma} - 4\bar{\varepsilon} < 0$ and $2 - \frac{\beta}{\gamma} - 5\bar{\varepsilon} > 0$.

7.3 Logarithmic growth

We have the following

Proposition 7.4 *Assume that there exist $\gamma > 0$, $a_\infty > 0$, $p > 1$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty}\frac{a(s)}{(\log s)^{2\gamma}} = a_\infty, \qquad \lim_{s\to+\infty}\frac{f(s)}{s^p} = f_\infty.$$

Then condition (1.63) *is fulfilled provided that $p > 3$. When $1 < p < 3$ it holds*

$$\lim_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt = +\infty.$$

Proof. We denote by C a positive constant and by C' a constant without any sign restriction, which may vary from one place to another. Taking into account Lemma 3.4, for every $\varepsilon \in (0, 1)$ there exists $R = R(\varepsilon) > 0$ such that

$$Cs^{(p+1)(1-\varepsilon)} \leq F(g(s)) \leq Cs^{p+1}, \qquad \text{for every } s \geq R.$$

So for every $\varepsilon \in (0, 1)$ and for all $t \geq R$, we obtain

$$\begin{aligned}Ct^{\frac{p+1}{2}(1-\varepsilon)+1}+C' \leq \int_0^t\sqrt{F(g(s))}ds &= \int_0^R\sqrt{F(g(s))}ds+\int_R^t\sqrt{F(g(s))}ds\\ &\leq C + C\int_{R_{\bar{\varepsilon}}}^t s^{\frac{p+1}{2}}ds \leq Ct^{\frac{p+3}{2}}.\end{aligned}$$

Assume that $p > 3$ and let $\bar{\varepsilon} > 0$ with $p - 3 - 3\bar{\varepsilon}(p + 1) > 0$. Whence, we get

$$\begin{aligned}&\limsup_{u\to+\infty}\sqrt{F(g(u))}\int_u^{+\infty}\frac{\int_0^t\sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt\\ &\quad\leq C\limsup_{u\to+\infty}u^{\frac{p+1}{2}}\int_u^{+\infty}\frac{t^{\frac{p+3}{2}}}{t^{\frac{3p+3}{2}(1-\varepsilon)}}dt\\ &\quad= C\limsup_{u\to+\infty}u^{\frac{p+1}{2}}\int_u^{+\infty}t^{-p+\frac{3\bar{\varepsilon}}{2}(p+1)}dt\\ &\quad= C\limsup_{u\to+\infty}u^{-\frac{p-3-3\bar{\varepsilon}(p+1)}{2}} = 0.\end{aligned}$$

On the contrary, if $1 < p < 3$, fix $\bar{\varepsilon}$ so small that $\frac{3-p}{2} - \bar{\varepsilon}(p+1) > 0$. In turn, we deduce

$$\begin{aligned}
\liminf_{u\to+\infty} \sqrt{F(g(u))} \int_u^{+\infty} \frac{\int_0^t \sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt \\
&\geq C \liminf_{u\to+\infty} u^{\frac{p+1}{2}(1-\bar{\varepsilon})} \int_u^{+\infty} \frac{t^{\frac{p+1}{2}(1-\bar{\varepsilon})+1} + C'}{t^{\frac{3p+3}{2}}}dt \\
&= \lim_{u\to+\infty} u^{\frac{p+1}{2}(1-\bar{\varepsilon})}(Cu^{1-p-\bar{\varepsilon}\frac{p+1}{2}} + C'u^{1-\frac{3}{2}(p+1)}) \\
&= \lim_{u\to+\infty} (Cu^{\frac{3-p}{2}-\bar{\varepsilon}(p+1)} + C'u^{-p-\bar{\varepsilon}\frac{p+1}{2}}) = +\infty.
\end{aligned}$$

This concludes the proof.

We have the following

Proposition 7.5 *Assume that there exist $\gamma > 0$, $a_\infty > 0$, $\beta > 0$ and $f_\infty > 0$ such that*

$$\lim_{s\to+\infty} \frac{a(s)}{(\log s)^{2\gamma}} = a_\infty, \qquad \lim_{s\to+\infty} \frac{f(s)}{e^{2\beta s}} = f_\infty.$$

Then condition (1.63) *is fulfilled.*

Proof. We denote by C a positive constant and by C' a constant without any sign restriction, which may vary from one place to another. For every $\varepsilon \in (0,1)$ there exists $R = R(\varepsilon) > 0$ such that $Ce^{2\beta s^{1-\varepsilon}} \leq F(g(s)) \leq Ce^{2\beta s}$, for every $s \geq R$. Observe that

$$\lim_{u\to+\infty} \int_u^{+\infty} \frac{\int_0^t \sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt = \lim_{u\to+\infty} \int_1^{+\infty} \frac{\int_0^t \sqrt{F(g(s))}ds}{F^{3/2}(g(t))}\chi_{(u,+\infty)}dt = 0.$$

In fact, the integrand belongs to $L^1(1,+\infty)$ since, for R big enough and all t large, we have

$$\frac{\int_0^t \sqrt{F(g(s))}ds}{(F(g(t)))^{3/2}} \leq \frac{C + (t-R)\sqrt{F(g(t))}}{(F(g(t)))^{3/2}} \leq C\frac{t}{F(g(t))} \leq C\frac{t}{e^{2\beta t^{1-\varepsilon}}}.$$

Then, by virtue of l'Hŏpital rule, we get

$$\begin{aligned}
\lim_{u\to+\infty} \sqrt{F(g(u))} \int_u^{+\infty} \frac{\int_0^t \sqrt{F(g(s))}ds}{F^{3/2}(g(t))}dt \\
&= \lim_{u\to+\infty} \frac{-F^{-3/2}(g(u)) \int_0^u \sqrt{F(g(s))}ds}{-\frac{1}{2}F^{-3/2}(g(u))f(g(u))g'(u)} \\
&= 2 \lim_{u\to+\infty} \frac{\int_0^u \sqrt{F(g(s))}ds}{f(g(u))g'(u)} = 0.
\end{aligned}$$

We thus only need to justify this last limit. Observe that, if $\varepsilon \in (0,1)$, from $s^{1-\varepsilon} \le g(s)$ for s large enough (see Lemma 3.4), we get $g^{-1}(s) \le s^{1/(1-\varepsilon)}$. Then, for R large enough, we have

$$\begin{aligned}
0 \le \limsup_{u\to+\infty} \frac{\int_0^u \sqrt{F(g(s))}ds}{f(g(u))g'(u)} &\le \limsup_{u\to+\infty} \frac{\sqrt{a(g(u))}(C + (u-R)\sqrt{F(g(u))})}{f(g(u))} \\
&\le C \limsup_{u\to+\infty} \frac{u\sqrt{a(g(u))}\sqrt{F(g(u))}}{f(g(u))} \\
&= C \limsup_{t\to+\infty} \frac{g^{-1}(t)\sqrt{a(t)}\sqrt{F(t)}}{f(t)} \\
&\le C \limsup_{t\to+\infty} \frac{t^{\frac{1}{1-\varepsilon}}\sqrt{a(t)}\sqrt{F(t)}}{f(t)} \\
&= \frac{\sqrt{f_\infty a_\infty}}{\sqrt{2\beta}} \lim_{t\to+\infty} \frac{t^{\frac{1}{1-\varepsilon}}(\log t)^\gamma e^{\beta t}}{f_\infty e^{2\beta t}} = 0.
\end{aligned}$$

This concludes the proof.

7.4 Proof of Theorem 1.5

The assertion of Theorem 1.5 follows by Propositions 7.1, 7.2, 7.3, 7.4, 7.5.

8 Blow-up rate of solutions, II

Suppose that (1.14) hold with $R = 0$ and that the existence conditions of Theorem 1.1 are satisfied. As before, η denotes the unique solution to (1.23). We consider the following notations, where $F(t) = \int_0^t f(\sigma)d\sigma$ and

$$\psi(t) := \int_t^{+\infty} \frac{ds}{\sqrt{2F(g(s))}}, \qquad \Lambda(t) := \frac{\int_0^t \sqrt{2F(g(s))}ds}{F(g(t))}, \quad t > 0,$$

$$J(t) := \frac{N-1}{2}\int_0^t \Lambda(\eta(s))ds, \qquad B(t) := \frac{f(g(t))g'(t)}{\sqrt{2F(g(t))}}, \quad t > 0.$$

Notice that our condition **E** holds for every choice of $r > 0$ and

$$\psi(t) < +\infty, \quad \forall t > 0 \qquad \Longleftrightarrow \qquad \int_{g^{-1}(r)}^{+\infty} \frac{ds}{\sqrt{F_r(g(s))}} < +\infty, \quad \forall r > 0,$$

where $F_r(t) = \int_r^t f(\sigma)d\sigma$, justifying the finiteness of $\psi(t)$ at each $t > 0$.
In the following, we shall denote by $\sigma(x)$ the orthogonal projection on the boundary $\partial\Omega$ of a given point $x \in \Omega$. Moreover, we shall indicate by $\mathcal{H} : \partial\Omega \to \mathbb{R}$ the mean curvature of $\partial\Omega$ (see [39] for a definition of mean curvature). In particular, the function $x \mapsto \mathcal{H}(\sigma(x))$ is well defined on Ω. We can state the following

Proposition 8.1 *Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^4 and assume that* (1.14) *hold with $R = 0$ and that one of the existence conditions of Theorem 1.1 is satisfied. Let us set*

$$\mathbb{T}(x) := \frac{\eta(\mathrm{d}(x,\partial\Omega))}{a^{\frac{1}{2}}(\min\{u(x), g(\eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega))))\})}, \qquad x \in \Omega,$$

where $\sigma(x)$ denotes the projection on $\partial\Omega$ of a $x \in \Omega$ and $\mathcal{H}$ is the mean curvature of $\partial\Omega$. Then there exists a positive constant L such that

$$|u(x) - g \circ \eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))| \leq L\mathbb{T}(x)o(\mathrm{d}(x,\partial\Omega)),$$

whenever $\mathrm{d}(x,\partial\Omega)$ *goes to zero if the following conditions hold*

$$\liminf_{t\to+\infty} \frac{\psi(\nu t)}{\psi(t)} > 1, \quad \textit{for all } \nu \in (0,1), \tag{1.73}$$

$$\lim_{\delta\to 0} \frac{B(\eta(\delta(1+o(1))))}{B(\eta(\delta))} = 1, \tag{1.74}$$

$$\limsup_{t\to+\infty} B(t)\Lambda(t) < +\infty. \tag{1.75}$$

Proof. Assuming that Ω is a domain of class C^4, by using the main result of [4] due to Bandle and Marcus, if the problem $\Delta v = h(v)$ with $h(s) = f(g(s))a^{-1/2}(g(s))$ positive and nondecreasing on $(0,+\infty)$, satisfying the Keller-Osserman condition and (1.73), (1.74) and (1.75) then it follows

$$|v(x) - \eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))| \leq \eta(\mathrm{d}(x,\partial\Omega))o(\mathrm{d}(x,\partial\Omega)), \tag{1.76}$$

provided that $\mathrm{d}(x,\partial\Omega)$ goes to zero. The proof then follows from (1.65) and (1.76).

In the particular case where (1.4) is satisfied with $p > k+1$, we have the following

Proposition 8.2 *Let Ω be a bounded domain of $\mathbb{R}^N$ which satisfies an inner and an outer sphere condition at each point of the boundary $\partial\Omega$. Therefore, if* (1.4) *hold with $p > 2k+3$, any solution $u \in C^2(\Omega)$ to* (1.1) *satisfies*

$$u(x) = \frac{\Gamma}{(\mathrm{d}(x,\partial\Omega))^{\frac{2}{p-k-1}}}(1+o(1)), \qquad \Gamma := \left[\frac{p-k-1}{\sqrt{2(p+1)}}\frac{\sqrt{f_\infty}}{\sqrt{a_\infty}}\right]^{\frac{2}{k+1-p}} > 0,$$

whenever x approaches the boundary $\partial\Omega$.

Proof. Let η be the unique solution to $\eta' = -\sqrt{2F\circ g\circ\eta}$, $\lim_{t\to 0^+}\eta(t) = +\infty$. Let us prove that

$$\lim_{t\to 0^+} \frac{\eta(t)}{t^{\frac{k+2}{k+1-p}}} = \Gamma_0, \qquad \Gamma_0 = \left[\frac{p-k-1}{k+2}\frac{\sqrt{2f_\infty}}{\sqrt{p+1}}\left(\frac{k+2}{2\sqrt{a_\infty}}\right)^{\frac{p+1}{k+2}}\right]^{\frac{k+2}{k+1-p}}.$$

For any $t > 0$ sufficiently close to 0 we have $2F(g(\eta(t))) > 0$ and, from $\frac{\eta'}{\sqrt{2F\circ g\circ\eta}} = -1$,

$$\int_{\eta(t)}^{+\infty} \frac{d\xi}{\sqrt{2F(g(\xi))}} = \int_t^0 \frac{\eta'(s)}{\sqrt{2F(g(\eta(s)))}} = t.$$

Furthermore, from (1.4) and (1.35), we have

$$\lim_{t\to 0^+} \frac{\sqrt{2F(g(\eta(t)))}}{\eta(t)^{\frac{p+1}{k+2}}} = \lim_{s\to+\infty} \frac{\sqrt{2F(g(s))}}{s^{\frac{p+1}{k+2}}} = \frac{\sqrt{2f_\infty}g_\infty^{\frac{p+1}{2}}}{\sqrt{p+1}},$$

where g_∞ was introduced in Lemma 3.2. Whence, recalling that $p > k+1$, we get

$$\lim_{t\to0^+}\frac{\eta(t)}{t^{\frac{k+2}{k+1-p}}} = \lim_{t\to0^+}\left[\frac{\eta(t)^{\frac{k+1-p}{k+2}}}{t}\right]^{\frac{k+2}{k+1-p}} = \lim_{t\to0^+}\left[\frac{\eta(t)^{\frac{k+1-p}{k+2}}}{\int_{\eta(t)}^{+\infty}\frac{d\xi}{\sqrt{2F(g(\xi))}}}\right]^{\frac{k+2}{k+1-p}} \tag{1.77}$$

$$= \lim_{t\to0^+}\left[\frac{\frac{k+1-p}{k+2}\eta(t)^{\frac{-p-1}{k+2}}\eta'(t)}{-\frac{1}{\sqrt{2F(g(\eta(t)))}}\eta'(t)}\right]^{\frac{k+2}{k+1-p}}$$

$$= \lim_{t\to0^+}\left[\frac{\frac{p-k-1}{k+2}\sqrt{2F(g(\eta(t)))}}{\eta(t)^{\frac{p+1}{k+2}}}\right]^{\frac{k+2}{k+1-p}} = \Gamma_0.$$

Taking into account Lemma 3.2, we thus obtain

$$g\circ\eta(\mathrm{d}(x,\partial\Omega)) = g_\infty\Gamma_0^{\frac{2}{k+2}}(\mathrm{d}(x,\partial\Omega))^{\frac{2}{k+1-p}}(1+o(1)), \quad \text{as } \mathrm{d}(x,\partial\Omega)\to0.$$

Since $p > 2k+3$ by virtue of (1.66) it holds $u(x) = g\circ\eta(\mathrm{d}(x,\partial\Omega)) + o(1)$ as x approaches the boundary $\partial\Omega$. Combining these equations we get the assertion.

Proposition 8.3 *Let Ω be a bounded domain of $\mathbb{R}^N$ of class C^4, assume that* (1.4) *hold with $p > k+1$ and that* (1.14) *are satisfied with $R = 0$. Then the following facts hold*

1. *There exists a positive constant L such that*

$$|u(x) - g\circ\eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))| \le L\mathbb{T}(x)o(\mathrm{d}(x,\partial\Omega)),$$

whenever $\mathrm{d}(x,\partial\Omega)$ *goes to zero, where*

$$\mathbb{T}(x) := \frac{(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}}{\min\{u^{k/2}(x)), (\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))^{k/(k+1-p)}\}}, \qquad x\in\Omega,$$

where $\sigma(x)$ denotes the projection on $\partial\Omega$ of a $x\in\Omega$ and $\mathcal{H}$ is the mean curvature of $\partial\Omega$.

2. *If $k+3 < p \le 2k+3$, then*

$$u(x) = \Gamma\frac{1}{(\mathrm{d}(x,\partial\Omega))^{\frac{2}{p-k-1}}}(1+o(1)) + \Gamma'\mathcal{H}(\sigma(x))(\mathrm{d}(x,\partial\Omega))^{\frac{p-k-3}{p-k-1}}(1+o(1)),$$

whenever x approaches $\partial\Omega$, where Γ and Γ' are as defined in (1.24).

3. *If $p \le k+3$, then*

$$u(x) = \Gamma\frac{1}{(\mathrm{d}(x,\partial\Omega))^{\frac{2}{p-k-1}}}(1+o(1)) + \Gamma'\frac{\mathcal{H}(\sigma(x))}{(\mathrm{d}(x,\partial\Omega))^{\frac{3+k-p}{p-k-1}}}(1+o(1)),$$

whenever x approaches $\partial\Omega$.

Proof. We want to apply Proposition 8.1. Let us check that condition (1.73) holds. From the validity of the Keller-Osserman condition and the definition of ψ, we have $\psi(\nu t) \to 0$ as $t \to +\infty$, for every $\nu \in (0,1]$. Given $\nu \in (0,1)$, using (1.4) and (1.35), we have

$$\begin{aligned}
\lim_{t\to+\infty} \frac{\psi(\nu t)}{\psi(t)} &= \lim_{t\to+\infty} \frac{\int_{\nu t}^{+\infty} \frac{ds}{\sqrt{2F(g(s))}}}{\int_{t}^{+\infty} \frac{ds}{\sqrt{2F(g(s))}}} = \nu \sqrt{\lim_{t\to+\infty} \frac{F(g(t))}{F(g(\nu t))}} \\
&= \nu \sqrt{\lim_{t\to+\infty} \frac{F(g(t))}{(g(t))^{p+1}} \frac{(g(\nu t))^{p+1}}{F(g(\nu t))} \Big[\frac{g(t)}{t^{\frac{2}{k+2}}}\Big]^{p+1} \Big[\frac{(\nu t)^{\frac{2}{k+2}}}{g(\nu t)}\Big]^{p+1} \nu^{-\frac{2(p+1)}{k+2}}} \\
&= \nu^{\frac{k+1-p}{k+2}} \sqrt{\lim_{t\to+\infty} \frac{F(g(t))}{(g(t))^{p+1}} \lim_{t\to+\infty} \frac{(g(\nu t))^{p+1}}{F(g(\nu t))} \lim_{t\to+\infty} \Big[\frac{g(t)}{t^{\frac{2}{k+2}}}\Big]^{p+1} \lim_{t\to+\infty} \Big[\frac{(\nu t)^{\frac{2}{k+2}}}{g(\nu t)}\Big]^{p+1}} \\
&= \frac{1}{\nu^{\frac{p-k-1}{k+2}}} > 1, \qquad \text{being } \nu < 1 \text{ and } p > k+1.
\end{aligned}$$

Let us now check that condition (1.74) is fulfilled. Observe first that, using (1.4), (1.35) and (1.77), we have

$$\begin{aligned}
&\lim_{\delta\to0^+} \frac{f(g(\eta(\delta(1+o(1)))))}{f(g(\eta(\delta)))} \\
&= \lim_{\delta\to0^+} \frac{f(g(\eta(\delta(1+o(1)))))}{(g(\eta(\delta(1+o(1)))))^p} \cdot \lim_{\delta\to0^+} \frac{(g(\eta(\delta)))^p}{f(g(\eta(\delta)))} \cdot \\
&\quad \cdot \lim_{\delta\to0^+} \Big[\frac{g(\eta(\delta(1+o(1))))}{(\eta(\delta(1+o(1))))^{\frac{2}{k+2}}}\Big]^p \cdot \lim_{\delta\to0^+} \Big[\frac{(\eta(\delta))^{\frac{2}{k+2}}}{g(\eta(\delta))}\Big]^p \cdot \lim_{\delta\to0^+} \Big[\frac{\eta(\delta(1+o(1)))}{\eta(\delta)}\Big]^{\frac{2p}{k+2}} \\
&= \lim_{\delta\to0^+} \Big[\frac{\eta(\delta(1+o(1)))}{\eta(\delta)}\Big]^{\frac{2p}{k+2}} \\
&= \Big[\lim_{\delta\to0^+} \frac{\eta(\delta(1+o(1)))}{(\delta(1+o(1)))^{\frac{k+2}{k+1-p}}} \lim_{\delta\to0^+} \frac{\delta^{\frac{k+2}{k+1-p}}}{\eta(\delta)} \lim_{\delta\to0^+} \frac{(\delta(1+o(1)))^{\frac{k+2}{k+1-p}}}{\delta^{\frac{k+2}{k+1-p}}}\Big]^{\frac{2p}{k+2}} = 1.
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\Big(\lim_{\delta\to0^+} \frac{g'(\eta(\delta(1+o(1))))}{g'(\eta(\delta))}\Big)^2 &= \lim_{\delta\to0^+} \frac{a(g(\eta(\delta)))}{a(g(\eta(\delta(1+o(1)))))} \\
&= \lim_{\delta\to0^+} \frac{a(g(\eta(\delta)))}{(g(\eta(\delta)))^k} \cdot \lim_{\delta\to0^+} \frac{(g(\eta(\delta(1+o(1)))))^k}{a(g(\eta(\delta(1+o(1)))))} \cdot \lim_{\delta\to0^+} \Big[\frac{g(\eta(\delta))}{(\eta(\delta))^{\frac{2}{k+2}}}\Big]^k \cdot \\
&\quad \cdot \lim_{\delta\to0^+} \Big[\frac{(\eta(\delta(1+o(1))))^{\frac{2}{k+2}}}{g(\eta(\delta(1+o(1))))}\Big]^k \cdot \lim_{\delta\to0^+} \Big[\frac{\eta(\delta)}{\delta^{\frac{k+2}{k+1-p}}}\Big]^{\frac{2k}{k+2}} \cdot \\
&\quad \cdot \lim_{\delta\to0^+} \Big[\frac{(\delta(1+o(1)))^{\frac{k+2}{k+1-p}}}{\eta(\delta(1+o(1)))}\Big]^{\frac{2k}{k+2}} \cdot \lim_{\delta\to0^+} \frac{\delta^{\frac{2k}{k+1-p}}}{(\delta(1+o(1)))^{\frac{2k}{k+1-p}}} = 1.
\end{aligned}$$

Arguing in a similar fashion, there holds

$$\lim_{\delta\to0^+} \frac{F(\eta(\delta))}{F(\eta(\delta(1+o(1))))} = 1.$$

Therefore, collecting the above conclusions, from the definition of B it follows that

$$\lim_{\delta\to 0^+}\frac{B(\eta(\delta(1+o(1))))}{B(\eta(\delta))}=\lim_{\delta\to 0^+}\frac{f(g(\eta(\delta(1+o(1)))))}{f(g(\eta(\delta)))}\cdot$$
$$\cdot\lim_{\delta\to 0^+}\frac{g'(\eta(\delta(1+o(1))))}{g'(\eta(\delta))}\cdot\sqrt{\lim_{\delta\to 0^+}\frac{F(\eta(\delta))}{F(\eta(\delta(1+o(1))))}}=1,$$

as desired. For what concerns the quantity $B(w)\Lambda(w)$ we have, using (1.4) again,

$$\begin{aligned}
\lim_{w\to+\infty}B(w)\Lambda(w)&=\lim_{w\to+\infty}\frac{f(g(w))g'(w)}{\sqrt{2}}\frac{\int_0^w\sqrt{2F(g(s))}ds}{(F(g(w)))^{3/2}}\\
&=\lim_{w\to+\infty}\frac{f_\infty(g(w))^p}{\sqrt{2}\sqrt{a(g(w))}}\frac{\sqrt{(p+1)^3}}{\sqrt{f_\infty^3}}\frac{\int_0^w\sqrt{2F(g(s))}ds}{(g(w))^{\frac{3(p+1)}{2}}}\\
&=\frac{\sqrt{(p+1)^3}}{\sqrt{2}\sqrt{a_\infty}\sqrt{f_\infty}}\lim_{w\to+\infty}\frac{\int_0^w\sqrt{2F(g(s))}ds}{(g(w))^{\frac{p+3+k}{2}}}\\
&=\frac{\sqrt{(p+1)^3}}{\sqrt{2}\sqrt{a_\infty}\sqrt{f_\infty}}\frac{2}{(p+k+3)}\lim_{w\to+\infty}\frac{\sqrt{2F(g(w))}}{(g(w))^{\frac{p+1+k}{2}}g'(w)}\\
&=\frac{2\sqrt{(p+1)^3}}{\sqrt{a_\infty}\sqrt{f_\infty}(p+k+3)}\lim_{w\to+\infty}\frac{\sqrt{f_\infty}\sqrt{a_\infty}}{\sqrt{p+1}}\frac{(g(w))^{\frac{p+1}{2}}}{(g(w))^{\frac{p+1+k}{2}}(g(w))^{-\frac{k}{2}}}\\
&=\frac{2(p+1)}{p+k+3}
\end{aligned}\tag{1.78}$$

so that condition (1.75) follows from (1.78). In turn, from Proposition 8.1, again on account of Lemma 3.2 and by (1.77), up to possibly enlarging L we obtain

$$\begin{aligned}
&|u(x)-g\circ\eta(\mathrm{d}(x,\partial\Omega)-\mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))|\\
&\leq L\frac{(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}o(\mathrm{d}(x,\partial\Omega))}{\min\{u^{k/2}(x)),(\mathrm{d}(x,\partial\Omega)-\mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))^{k/(k+1-p)}\}},
\end{aligned}$$

provided that $\mathrm{d}(x,\partial\Omega)$ goes to zero. This proves (1) and the first assertion of Theorem 1.6. Let us now come to the proof of assertions (2) and (3). First we want to estimate the function $J(t)$ near 0. To this end, observe that the function $\Lambda(t)$ is defined for $t>0$ and that, from (1.4) it follows

$$\begin{aligned}
\lim_{t\to 0^+}\Lambda(\eta(t))&=\lim_{w\to+\infty}\Lambda(w)=\lim_{w\to+\infty}\frac{\int_0^w\sqrt{2F(g(s))}ds}{F(g(w))}\\
&=\sqrt{2}\lim_{w\to+\infty}\frac{\sqrt{F(g(w))}}{f(g(w))g'(w)}=\sqrt{2}\lim_{w\to+\infty}\frac{\left(\frac{f_\infty}{p+1}\right)^{\frac{1}{2}}(g(w))^{\frac{p+1}{2}}\sqrt{a(g(w))}}{f_\infty\,(g(w))^p}\\
&=\frac{\sqrt{2}}{\sqrt{f_\infty}\sqrt{p+1}}\lim_{w\to+\infty}\sqrt{a_\infty}(g(w))^{\frac{p+1}{2}-p+\frac{k}{2}}=\frac{\sqrt{2}\sqrt{a_\infty}}{\sqrt{f_\infty}\sqrt{p+1}}\lim_{w\to+\infty}(g(w))^{\frac{k+1-p}{2}}=0
\end{aligned}$$

since $p>k+1$. This implies that

$$\lim_{t\to 0^+}J'(t)=\lim_{t\to 0^+}\frac{N-1}{2}\Lambda(\eta(t))=0.\tag{1.79}$$

Moreover

$$\begin{aligned}\Lambda'(w) &= \frac{\sqrt{2F(g(w))}F(g(w)) - f(g(w))g'(w)\int_0^w \sqrt{2F(g(s))}ds}{(F(g(w)))^2}\\ &= \frac{\sqrt{2}}{\sqrt{F(g(w))}} - \frac{f(g(w))g'(w)}{F(g(w))}\frac{\int_0^w \sqrt{2F(g(s))}ds}{F(g(w))}\\ &= \frac{\sqrt{2}}{\sqrt{F(g(w))}} - \frac{f(g(w))g'(w)}{F(g(w))}\Lambda(w),\end{aligned}$$

so that

$$\begin{aligned}J''(t) &= \frac{N-1}{2}\Lambda'(\eta(t))\eta'(t) = -\frac{N-1}{2}\Lambda'(\eta(t))\sqrt{2F(g(\eta(t)))}\\ &= -\frac{N-1}{\sqrt{2}}\left(\sqrt{2} - \frac{f(g(\eta(t)))g'(\eta(t))}{\sqrt{F(g(\eta(t)))}}\Lambda(\eta(t))\right)\\ &= (N-1)(-1 + B(\eta(t))\Lambda(\eta(t))).\end{aligned}$$

Then (1.78) implies that

$$\begin{aligned}\lim_{t\to 0^+} J''(t) &= (N-1)\Big(-1 + \lim_{t\to 0^+} B(\eta(t))\Lambda(\eta(t))\Big) = (N-1)\Big(-1 + \lim_{w\to+\infty} B(w)\Lambda(w)\Big)\\ &= (N-1)\left(-1 + \frac{2(p+1)}{(p+k+3)}\right) = \frac{(N-1)}{(p+k+3)}(p-k-1).\end{aligned} \tag{1.80}$$

From (1.79) and (1.80) we get

$$J(t) = \frac{(N-1)}{p+k+3}(p-k-1)t^2 + o(t^2) \tag{1.81}$$

for $t > 0$ sufficiently small. From formulas (1.77) and (1.81) we have

$$\begin{aligned}&\eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega))) =\\ &= \frac{\eta(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))}{(\mathrm{d}(x,\partial\Omega) - \mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega)))^{\frac{k+2}{k+1-p}}}\,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\Big(1 - \frac{\mathcal{H}(\sigma(x))J(\mathrm{d}(x,\partial\Omega))}{\mathrm{d}(x,\partial\Omega)}\Big)^{\frac{k+2}{k+1-p}}\\ &= \Gamma_0(1+o(1))\,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\Big(1 + \frac{k+2}{p-k-1}\mathcal{H}(\sigma(x))\frac{J(\mathrm{d}(x,\partial\Omega))}{\mathrm{d}(x,\partial\Omega)} + o\Big(\frac{J(\mathrm{d}(x,\partial\Omega))}{\mathrm{d}(x,\partial\Omega)}\Big)\Big)\\ &= \Gamma_0(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\Big(1 + \frac{k+2}{p-k-1}\mathcal{H}(\sigma(x))\frac{(N-1)(p-k-1)}{p+k+3}\frac{(\mathrm{d}(x,\partial\Omega))^2}{\mathrm{d}(x,\partial\Omega)}(1+o(1))\Big)(1+o(1))\end{aligned}$$

if $\mathrm{d}(x,\partial\Omega)$ is sufficiently small. Moreover

$$\begin{aligned}\eta(\mathrm{d}(x,\partial\Omega))o(\mathrm{d}(x,\partial\Omega)) &= \frac{\eta(\mathrm{d}(x,\partial\Omega))}{(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}}\,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,o(\mathrm{d}(x,\partial\Omega))\\ &= \Gamma_0(1+o(1))\,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}+1}\,o(1)\\ &= \Gamma_0\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,o(1)\end{aligned}$$

for $\mathrm{d}(x,\partial\Omega)$ small enough. Then (1.76) implies that

$$v(x) = \Gamma_0 \,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,(1+o(1)) + \Gamma_0 \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,(1+o(1))$$

for $\mathrm{d}(x,\partial\Omega)$ small enough. Then, in light of Lemma 3.1 and using (1.35), any blow-up solution $u \in C^2(\Omega)$ to (1.1) satisfies

$$\begin{aligned}
u(x) &= g(v(x)) \\
&= g\Big(\Gamma_0 \,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,(1+o(1)) + \Gamma_0 \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,(1+o(1))\Big) \\
&= \frac{g\Big(\Gamma_0 \,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,(1+o(1)) + \Gamma_0 \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,(1+o(1))\Big)}{\Big[\Gamma_0 \,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,(1+o(1)) + \Gamma_0 \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,(1+o(1))\Big]^{\frac{2}{k+2}}} \cdot \\
&\quad \cdot \Big[\Gamma_0 \,(\mathrm{d}(x,\partial\Omega))^{\frac{k+2}{k+1-p}}\,(1+o(1)) + \Gamma_0 \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{2k+3-p}{k+1-p}}\,(1+o(1))\Big]^{\frac{2}{k+2}} \\
&= g_\infty(1+o(1))\Gamma_0^{\frac{2}{k+2}}\,(\mathrm{d}(x,\partial\Omega))^{\frac{2}{k+1-p}}\left(1 + \frac{(k+2)(N-1)}{p+k+3}\mathcal{H}(\sigma(x))\mathrm{d}(x,\partial\Omega)(1+o(1))\right)^{\frac{2}{k+2}} \\
&= g_\infty\Gamma_0^{\frac{2}{k+2}}\,(\mathrm{d}(x,\partial\Omega))^{\frac{2}{k+1-p}}\,(1+o(1)) + \frac{2(N-1)}{p+k+3}g_\infty\Gamma_0^{\frac{2}{k+2}}\mathcal{H}(\sigma(x))\,(\mathrm{d}(x,\partial\Omega))^{\frac{k+3-p}{k+1-p}}\,(1+o(1))
\end{aligned}$$

as x approaches the boundary $\partial\Omega$. By performing some simple manipulations this yield the desired conclusions.

8.1 Proof of Theorem 1.6

The assertion of Theorem 1.6 follows by combining Propositions 8.2 and 8.3.

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