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Existence and concentration of positive solutions to generalized Chern-Simons-Schrödinger system with critical exponential growth ☆

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A R T I C L E I N F O

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A B S T R A C T

We are concerned with a class of generalized Chern-Simons-Schrödinger systems

$$\begin{cases} -\Delta u + \lambda V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(u), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0, \\ \partial_1 A_0 = A_2 |u|^2, \ \partial_2 A_0 = -A_1 |u|^2, \end{cases}$$

where $\lambda > 0$ denotes a sufficiently large parameter, $V : \mathbb{R}^2 \to \mathbb{R}$ admits a potential well $\Omega \triangleq \text{int} V^{-1}(0)$ and the nonlinearity f fulfills the critical exponential growth in the Trudinger-Moser sense at infinity. Under some suitable assumptions on V and f, based on variational method together with some new technical analysis, we are able to get the existence of positive solutions for some large $\lambda > 0$, and the asymptotic behavior of the obtained solutions is also investigated when $\lambda \to +\infty$.

1. Introduction

1.1. Overview

In this article, we focus on establishing the existence and concentrating behavior of positive solutions for a class of generalized Chern-Simons-Schrödinger (**CSS** in short) systems

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$$\begin{cases} -\Delta u + \lambda V(x)u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(u), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0, \\ \partial_1 A_0 = A_2|u|^2, \ \partial_2 A_0 = -A_1|u|^2, \end{cases} \tag{1.1}$$

where $\lambda > 0$ denotes a sufficiently large parameter, $V : \mathbb{R}^2 \to \mathbb{R}$ admits a potential well $\Omega \triangleq \mathrm{int}V^{-1}(0)$ and the nonlinearity f fulfills the critical exponential growth in the Trudinger-Moser sense at infinity. On the potential V, we shall firstly make the following assumptions

(V_1) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ with $V \geq 0$ on $\mathbb{R}^2$;
(V_2) there is $b > 0$ such that the set $\Sigma \triangleq \{x \in \mathbb{R}^2 : V(x) < b\}$ has positive finite Lebesgue measure;
(V_3) $\Omega = \mathrm{int}V^{-1}(0)$ is nonempty with smooth boundary with $\overline{\Omega} = V^{-1}(0)$, $V^{-1}(0) \triangleq \{x : V(x) = 0\}$.

In celebrated papers, Bartsch and his collaborators initially proposed the above hypotheses to study the nonlinear Schrödinger equations, see [4,5]. As we all know, the harmonic trapping potential

$$V(x) = \begin{cases} \omega_1|x_1|^2 + \omega_2|x_2|^2 - \omega, & \text{if } |(\sqrt{\omega_1}x_1, \sqrt{\omega_2}x_2)|^2 \geq \omega, \\ 0, & \text{if } |(\sqrt{\omega_1}x_1, \sqrt{\omega_2}x_2)|^2 \leq \omega, \end{cases}$$

with $\omega > 0$ satisfying (V_1)-(V_3), where $\omega_i > 0$ is called the anisotropy factor of the trap in quantum physics and trapping frequency of the ith-direction in mathematics, see e.g. [6,10,33]. In reality, the potential λV, instead of V, with the above mentioned assumptions (V_1)-(V_3) can be read as the steep potential well.

Over the past several decades, there were considerable attentions to the time-dependent CSS system in two spatial dimension

$$\begin{cases} iD_0\psi + (D_1D_1 + D_2D_2)\psi + g(x, |\psi|^2)\psi = 0, \\ \partial_0 A_1 - \partial_1 A_0 = -\mathrm{Im}(\overline{\psi} D_2\psi), \\ \partial_0 A_2 - \partial_2 A_0 = \mathrm{Im}(\overline{\psi} D_1\psi), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|\psi|^2, \end{cases} \tag{1.2}$$

where i is the imaginary unit, $D_j = \partial_j + iA_j$ is the covariant derivative for $j = 0, 1, 2$ with $\partial_0 = \frac{\partial}{\partial t}$, $\partial_1 = \frac{\partial}{\partial x_1}$, $\partial_2 = \frac{\partial}{\partial x_2}$ for $(t, x_1, x_2) \in \mathbb{R}^{1+2}$ and $A_j : \mathbb{R}^{1+2} \to \mathbb{R}$ acting as the gauge field and $\psi : \mathbb{R}^{1+2} \to \mathbb{C}$ denotes the complex scalar field. As a matter of fact, it is usually adopted to describe the non-relativistic dynamics behavior of massive number of particles in Chern-Simons gauge fields. This model plays an important role in the study of the high-temperature superconductor, Aharovnov-Bohm scattering, and quantum Hall effect, we refer the reader to [20–22]. Moreover, there exist some further physical motivations for considering CSS system (1.2), see e.g. [16,18,30,31].

Consider the standing wave ansatz $\psi(t, x) = e^{i\omega t}u(x)$ with a radially symmetric $u : \mathbb{R}^2 \to \mathbb{R}$, the CSS system (1.2) can reduce to a single equation. In fact, Byeon, Huh and Seok [8] disposed of the standing waves of type

$$\begin{aligned} &\psi(t, x) = u(|x|)e^{\mathrm{i}\lambda t}, \ A_0(t, x) = k(|x|), \\ &A_1(t, x) = \frac{x_2}{|x|^2}h(|x|), \ A_2(t, x) = -\frac{x_1}{|x|^2}h(|x|), \end{aligned} \tag{1.3}$$

where both k and h are radially symmetric and real value functions. We remark that (1.3) satisfies the Coulomb gauge condition with $\varsigma = ct + n\pi$, where n is an integer and c is a real constant. To look for solutions of CSS system (1.2) of the type (1.3), it is adequate to contemplate the following semilinear elliptic equation

$$-\Delta u + \omega u + \left(\int_{|x|}^{\infty} \frac{h(s)}{s} u^2(s) ds + \frac{h^2(|x|)}{|x|^2} \right) u = g(x, u) \text{ in } \mathbb{R}^2, \tag{1.4}$$

where $h(s) = \int_0^s \frac{r}{2} u^2(r) dr$. In [8], the authors contemplated the existence of ground state solutions of (1.4) with $g(x, u) = |u|^{p-2}u$ for $p > 4$ and $\omega > 0$ as well as existence of a solution for $p \in (2, 4)$ and the suitable $\omega > 0$. A more precise results for the case $p \in (2, 4)$ can be found in [36]. There are some other meaningful results on (1.4) and its variants, the reader can refer to [3,8,14,18,24,32,36,39–41,44,50] and the references therein.

Whereas, if the function $u : \mathbb{R}^2 \to \mathbb{R}$ in the standing wave ansatz $\psi(t, x) = e^{i\omega t} u(x)$ is not radially symmetric, the studies on (1.2) would be more complicated. In this situation, one usually tackles the case $A_j(t, x) = A_j(x)$ for all $(t, x_1, x_2) \in \mathbb{R}^{1+2}$ and $j = 0, 1, 2$. Observe that (1.2) can reduce to

$$\begin{cases} -\Delta u + \omega u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(x, u), \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \\ \partial_1 A_0 = A_2 |u|^2, \ \partial_2 A_0 = -A_1 |u|^2, \end{cases} \tag{1.5}$$

where $f(x, t) = g(x, |t|^2)t$ for every $(x, t) \in \mathbb{R}^2 \times \mathbb{R}$.

Suppose A_j satisfies the Coulomb gauge condition $\sum_{j=0}^{2} \partial_j A_j = 0$, then (1.5) becomes a special case of the original CSS equation (1.1), namely

$$\begin{cases} -\Delta u + \omega u + A_0 u + \sum_{j=1}^{2} A_j^2 u = f(x, u), \\ \partial_1 A_0 = A_2 |u|^2, \ \partial_2 A_0 = -A_1 |u|^2, \\ \partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2, \ \partial_1 A_1 + \partial_2 A_2 = 0. \end{cases} \tag{1.6}$$

By using some elementary calculations, one can obtain the explicit expression for A_j. On the one hand, combining $\partial_1 A_0 = A_2|u|^2$ and $\partial_2 A_0 = -A_1|u|^2$ in (1.6), it holds that

$$\Delta A_0 = \partial_1 \big(A_2 |u|^2\big) - \partial_2 \big(A_1 |u|^2\big),$$

and so

$$A_0[u](x) = \frac{x_1}{2\pi |x|^2} * \big(A_2 |u|^2\big) - \frac{x_2}{2\pi |x|^2} * \big(A_1 |u|^2\big). \tag{1.7}$$

On the other hand, we are derived from $\partial_1 A_2 - \partial_2 A_1 = -\frac{1}{2}|u|^2$ and $\partial_1 A_1 + \partial_2 A_2 = 0$ in (1.6) that

$$\Delta A_1 = \partial_2 \left(\frac{|u|^2}{2} \right) \text{ and } \Delta A_2 = -\partial_1 \left(\frac{|u|^2}{2} \right).$$

As a consequence, the components A_j for $j = 1, 2$ in (1.6) can be represented as

$$\begin{cases} A_1[u](x) = \frac{x_2}{2\pi|x|^2} * \left(\frac{|u|^2}{2}\right) = -\frac{1}{4\pi}\int\limits_{\mathbb{R}^2} \frac{(x_2-y_2)u^2(y)}{|x-y|^2}dy, \\ A_2[u](x) = -\frac{x_1}{2\pi|x|^2} * \left(\frac{|u|^2}{2}\right) = \frac{1}{4\pi}\int\limits_{\mathbb{R}^2} \frac{(x_1-y_1)u^2(y)}{|x-y|^2}dy. \end{cases} \tag{1.8}$$

In the sequel, we shall write A_j in place of $A_j[u]$ for $j \in \{0,1,2\}$ for simplicity as long as there is no misunderstanding. There are some further properties of A_j for $j \in \{0,1,2\}$ in Section 2 below.

In [19], Huh initially contemplated the system (1.6) with $f(x,t) = |t|^{p-2}t$ and $p > 6$, where the existence of infinitely many solutions was investigated. Afterwards, the existence, nonexistence and multiplicity of nontrivial solutions for (1.5) and its variants have been considerably studied by a lot of mathematicians, see [11,19,25,27,42,43,45,47] and the references therein for example even if these references are far to be exhaustive.

We have to note that the spatial dimension of Eq. (1.1) is very special, as for bounded domains $\Omega \subset \mathbb{R}^2$, the corresponding Sobolev embedding yields $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ for all $q \geq 1$, while $H_0^1(\Omega) \not\hookrightarrow L^\infty(\Omega)$. Loosely speaking, to get rid of the obstacle in this limiting case, the well-known Trudinger-Moser inequality [34,35,46] would be a candidate as the suitable substitute of the Sobolev inequality. Firstly, we shall introduce the case bounded domain Ω instead of the whole space $\mathbb{R}^2$. The authors in [34,35,46] established the following sharp maximal exponential integrability for functions in $H_0^1(\Omega)$:

$$\sup_{u\in H_0^1(\Omega):\|\nabla u\|_{L^2(\Omega)}\leq 1} \int\limits_\Omega e^{\alpha u^2}dx \leq C|\Omega| \text{ if } \alpha < 4\pi, \tag{1.9}$$

where the constant $C = C(\alpha) > 0$ and $|\Omega|$ stands for the Lebesgue measure of Ω. Subsequently, Lions [29] developed the concentration-compactness principle in the Trudinger-Moser inequality sense: Let (u_n) be a sequence of functions in $H_0^1(\Omega)$ with $\|\nabla u_n\|_{L^2(\Omega)} = 1$ such that $u_n \rightharpoonup u_0$ weakly in $H_0^1(\Omega)$, there holds

$$\sup_{n\in\mathbb{N}} \int\limits_\Omega e^{4\pi p u_n^2}dx < +\infty, \ \forall p < \frac{1}{1-\|\nabla u_0\|^2_{L^2(\Omega)}}. \tag{1.10}$$

Unfortunately, the domains $\Omega \subset \mathbb{R}^2$ satisfying $|\Omega| = \infty$ yield the supremum in (1.9) to be infinite, and hence the Trudinger-Moser inequality is not available for the unbounded domains. As to the whole space $\mathbb{R}^2$, the authors in [7,9] established the following version of the Trudinger-Moser inequality: For all $u \in H^1(\mathbb{R}^2)$ with $\|u\|_{L^2(\mathbb{R}^2)} \leq M < +\infty$, there is a positive constant $C = C(M,\alpha)$ such that

$$\sup_{u\in H^1(\mathbb{R}^2):\|\nabla u\|_{L^2(\mathbb{R}^2)}\leq 1} \int\limits_{\mathbb{R}^2} \left(e^{\alpha u^2} - 1\right)dx \leq C \text{ if } \alpha < 4\pi. \tag{1.11}$$

Souza and do Ó [13] extended the Lions's concentration-compactness principle to $\mathbb{R}^2$: Let (u_n) be in $W_0^{1,2}(\mathbb{R}^2)$ with $\|u_n\|_{W_0^{1,2}(\mathbb{R}^2)} = 1$ and suppose that $u_n \rightharpoonup u_0$ in $W_0^{1,2}(\mathbb{R}^2)$, there holds

$$\sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}^2} (e^{4\pi p u_n^2} - 1)dx < \infty, \ \forall 0 < p < p_{\alpha_0}(u) \triangleq \frac{1}{1-\|u_0\|^2_{W_0^{1,2}(\mathbb{R}^2)}} \tag{1.12}$$

Let us quote the results in [2,12] and their references therein concerning some other generalizations, extensions and applications of the Trudinger-Moser inequalities for bounded and unbounded domain.

Due to the Trudinger-Moser type inequality above, we shall say that a function f admits the critical exponential growth in the Trudinger-Moser sense at infinity. In fact, it is said that f possesses the *critical exponential growth* at infinity if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{t\to+\infty}\frac{|f(t)|}{e^{\alpha t^2}}=\begin{cases}0, & \forall\alpha>\alpha_0,\\ +\infty, & \forall\alpha<\alpha_0.\end{cases}\tag{1.13}$$

The above definition was introduced by Adimurthi and Yadava in [1], see also de Figueiredo, Miyagaki and Ruf [12] for example.

1.2. Main results

Inspired by all the works above, particularly by [37,39,45], we shall be concerned with the existence and concentration results for system (1.1) with steep potential well. Since we are interested in positive solutions, without loss of generality, we suppose that $f\in\mathcal{C}^0(\mathbb{R},\mathbb{R})$ vanishes in $(-\infty,0)$ and satisfies the following conditions

(f_1) $f\in\mathcal{C}^0(\mathbb{R},\mathbb{R}^+)$ and $f(z)=o(z^3)$ as $z\to0$, where $\mathbb{R}^+=[0,+\infty)$;
(f_2) there are $p\in(4,+\infty)$ and $\hat{\mu}>0$ such that $F(z)\ge\hat{\mu}z^p$ for all $z\in\mathbb{R}^+$, where $F(z)=\int_0^z f(s)ds$;
(f_3) the map $z\mapsto\frac{\theta f(z)z-2F(z)}{z^{\frac{6\theta-2}{\theta}}}$ is nondecreasing on $z\in(0,+\infty)$, where $\theta\in(1,+\infty)$ is a constant satisfying

$$\theta\in\begin{cases}\left(1,\frac{2}{6-p}\right), & \text{if } 4<p<6,\\ (1,+\infty), & \text{if } 6\le p<+\infty.\end{cases}$$

The first main result in this paper can be stated as follows.

Theorem 1.1. *Suppose* (V_1)-(V_3) *and* (1.13) *with* (f_1)-(f_3) *as well as the following conditions*

(V_4) *V is weakly differentiable a.e. in $\mathbb{R}^2$ and the map $t\mapsto t^{2(2\theta-1)}[2(\theta-1)V(tx)-(\nabla V(tx),tx)]$ is nondecreasing on $t\in(0,+\infty)$ as well as $(\theta-1)V(x)\ge|(\nabla V,x)|$ for all $x\in\mathbb{R}^2$, where $\theta>1$ comes from* (f_3) *and $(\cdot,\cdot)$ denotes the inner product in $\mathbb{R}^2$.*

If in addition the constant $\hat{\mu}$ appearing in (f_2) *is sufficiently large, then there exists a $\Lambda>0$ such that the system* (1.1) *admits at least one positive ground state solution for all $\lambda>\Lambda$.*

As a counterpart of [37, Theorem 1.3], we also dispose of the concentrating behavior of the positive ground state solution u_λ obtained in Theorem 1.1 when $\lambda\to+\infty$. More precisely, we are able to prove the following result.

Theorem 1.2. *Under the assumptions of Theorem 1.1 and let $(u_\lambda)\subset E_\lambda$ be the ground state solutions of system* (1.1) *established in Theorem 1.1, then, up to a subsequence if necessary, $u_\lambda\to u_0$ in $H^1(\mathbb{R}^2)$ as $\lambda\to+\infty$, where $u_0\in H^1(\mathbb{R}^2)\backslash\{0\}$ is a ground state solution of the following Dirichlet boundary problem*

$$\begin{cases}-\Delta u+A_0[u]u+\sum_{j=1}^2A_j^2[u]u=f(u) & \text{in }\Omega\\ \partial_1A_2[u]-\partial_2A_1[u]=-\frac12|u|^2 & \text{in }\Omega\\ \partial_1A_1[u]+\partial_2A_2[u]=0,\ \ A_1[u]\partial_1u+A_2[u]\partial_2u=0 & \text{in }\Omega,\\ \partial_1A_0[u]=A_2[u]|u|^2,\ \ \partial_2A_0[u]=-A_1[u]|u|^2 & \text{in }\Omega,\\ u=0 & \text{on }\partial\Omega,\end{cases}\tag{1.14}$$

and, moreover, $u_0=0$ in $\mathbb{R}^2\setminus\Omega$.

We would like to highlight here that the result in Theorem 1.1 can be applied for the nonlinearity

$$f(t)=\begin{cases}\hat{\mu}t^{p}, & t\in[0,t_0],\\ \hat{\mu}C_0te^{\alpha_0t^2}, & t\in[t_0,+\infty),\end{cases}\quad \text{and } f(t)=0 \text{ if } t\le 0,$$

for some suitable constants $t_0>0$ and $C_0>0$. As to the potential V, we provide an example as follows

$$V(x)=\begin{cases}0, & \text{if } |x|\le 1,\\ |x|^{\theta-1}, & \text{if } |x|>1.\end{cases}$$

Obviously, this example for V above is not sharp, but it reveals that the existence result in Theorem 1.1 seems reasonable.

Remark 1.3. The reader is invited to observe that the results explored in Theorems 1.1 and 1.2 can be regarded as some supplementaries and improvements to the counterparts in [37]. As a matter of fact, instead of the Nehari manifold constraint procedure adopted in the quoted paper, we heavily rely on the Nehari-Pohožaev manifold constraint method and so there are some new challenges to overcome.

Remark 1.4. As far as we are concerned, there exist some other interesting questions for some further explorations. On the one hand, we all know that the assumption (f_4) does not reveal the essential feature of the critical exponential growth (1.13) because it is a truly global one in the sense that f is p-superlinear growth at infinity and the parameter $\hat{\mu}$ must be large. If the constraint set $\mathcal{M}_\lambda$ defined in (3.1) below is a natural $\mathcal{C}^1$-manifold, then (f_4) would be replaced by $(f_5)-(f_6)$ which will be introduced later. On the other hand, the requirement $(\theta-1)V(x)\ge|(\nabla V,x)|$ for all $x\in\mathbb{R}^2$ in (V_4) is too limited, but we have no idea how to remove it so far.

It should be pointed out that the assumptions on f and V required in Theorem 1.1 are somehow restrictive. It is natural to ask that whether the existence result remains true when (f_3) and (V_4) are absent. Thus, our next main result would exhibit an affirmative answer. To achieve it, we impose the assumptions on the nonlinearity f below.

(f_4) there is a $\gamma>4$ such that $zf(z)-\gamma F(z)\ge 0$ for all $z\in\mathbb{R}^+$;
(f_5) there exist some constants $t_0>0$, $M_0>0$ and $\vartheta\in[0,1)$ such that

$$0<t^{\vartheta}F(t)\le M_0f(t),\ \forall t>t_0;$$

(f_6) $\lim\limits_{t\to+\infty}F(t)e^{-\alpha_0t^2}\triangleq\beta_0>0$, where $\alpha_0>0$ comes from (1.13).

Theorem 1.5. *Suppose* (V_1)-(V_3), (1.13), (f_1) *and* $(f_4)-(f_6)$ *as well as the following conditions*

(V_5) V *is weakly differentiable a.e. in* $\mathbb{R}^2$ *and satisfiers the inequality*

$$2V(x)+(\nabla V,x)\ge 0 \textit{ for all } x\in\mathbb{R}^2.$$

Then there exists a $\hat{\Lambda}>0$ *such that* (1.1) *has at least one positive solution for all* $\lambda>\hat{\Lambda}$.

There is no doubt that the assumptions $(f_4)-(f_6)$ and (V_5) are more general than those $(f_2)-(f_3)$ and (V_4), respectively.

Remark 1.6. It is worth mentioning here that we follow the approach adopted in [39] to conclude Theorem 1.5. Nevertheless, one cannot simply repeat the approaches explored in [39, Therorem 1.4] to get the result. Explaining it clearly, in contrast to this quoted result, there are three main contributions:

(1) Firstly, the work space in this paper is unnecessary to be radially symmetric. Indeed, we consider a generalized CSS system which urges us take some more careful calculations;

(2) Secondly, the critical exponential growth (1.13) in the nonlinearity is involved and so we have to make some delicate analysis to restore the compactness;

(3) Last but not the least, we contemplate the so-called steep potential well λV in (1.1). Consequently, we prefer to believe that the methods proposed in this paper would further prompt some related studies on the Chern-Simons-Schrödinger equations.

Finally, it is very similar to Theorem 1.2 that we can derive the asymptotical behavior of the positive solutions obtained by Theorem 1.5. Because of the detailed proof of Theorem 1.2, we shall just exhibit this result without proof.

Corollary 1.7. *Under the assumptions of Theorem 1.5 and let $(u_\lambda) \subset E_\lambda$ be the positive solutions of system (1.1) established in Theorem 1.5, then, up to a subsequence if necessary, $u_\lambda \to u_0$ in $H^1(\mathbb{R}^2)$ as $\lambda \to +\infty$, where $u_0 \in H^1(\mathbb{R}^2)\backslash\{0\}$ is a positive solution of (1.14) and $u_0 = 0$ in $\mathbb{R}^2\backslash\Omega$.*

According to the best knowledge of us, although from the perspectives of Theorems 1.1, 1.2 and 1.5 themselves, one could find similar results on Chern-Simons-Schrödinger equation in recent literatures, see e.g. [37,39,45], we would like to stress here that this paper deals with a wider class of nonlinearities involving critical exponential growth. What's more, due to the steep potential well, the techniques are not obvious to some extent. As one would observe later, some of particular barriers prevent us applying the variational method to prove the main results in a standard way.

The paper is organized as follows. In Section 2, we mainly introduce some preliminary results. In Sections 3 and 4, we show some crucial lemmas and exhibit the detailed proofs of Theorems 1.1, 1.2 and 1.5, respectively.

Notations. From now on until the end of this paper, otherwise mentioned explicitly, we will take advantage of the following notations:

- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant and $\mathbb{R}^+ \triangleq (0, +\infty)$.
- Let $(X, \|\cdot\|_X)$ be a Banach space with dual space $(X^{-1}, \|\cdot\|_{X^{-1}})$, and Ψ be functional on X.
- The (PS) sequence at a level $c \in \mathbb{R}$ ($(PS)_c$ sequence in short) corresponding to Φ means that $\Phi(x_n) \to c$ and $\Phi'(x_n) \to 0$ in X^{-1} as $n \to \infty$, where $\{x_n\} \subset X$.
- $|\cdot|_p$ stands for the usual norm of the Lebesgue space $L^p(\mathbb{R}^2)$ for all $p \in [1, +\infty]$.
- For any $\varrho > 0$ and every $x \in \mathbb{R}^2$, $B_\varrho(x) \triangleq \{y \in \mathbb{R}^2 : |y - x| < \varrho\}$.
- $|\Sigma|$ stands for the Lebesgue measure of a Lebesgue measurable set $\Sigma \subset \mathbb{R}^2$.
- $o_n(1)$ denotes the real sequences with $o_n(1) \to 0$ as $n \to +\infty$.
- "$\to$" and "$\rightharpoonup$" stand for the strong and weak convergence in the related function spaces, respectively;

2. Variational framework and preliminaries

In this section, we are going to search for the variational structure for the main theorems and then exhibit some preliminary results which enable us to treat our problems variationally.

2.1. Variational setting

First of all, we define the space

$$E \triangleq \left\{ u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} V(x)u^2 dx < +\infty \right\}.$$

By using (V_1), it would be simple to conclude that the space is a Hilbert space equipped with the inner product and norm

$$(u,v)_E = \int_{\mathbb{R}^2} [\nabla u \nabla v + V(x)uv]\, dx \text{ and } \|u\|_E = \left(\int_{\mathbb{R}^2} |\nabla u|^2 + V(x)u^2 dx \right)^{\frac{1}{2}}$$

for each $u, v \in E$. Particularly, one can deduce that the embedding $E \hookrightarrow H^1(\mathbb{R}^2)$ is continuous. Indeed, it follows from (V_2) that

$$\int_{\mathbb{R}^2\setminus\Sigma} u^2 dx \le \frac{1}{b} \int_{\mathbb{R}^2\setminus\Sigma} V(x)u^2 dx \le \frac{1}{b} \int_{\mathbb{R}^2} V(x)u^2 dx, \ \forall u \in H^1(\mathbb{R}^2). \tag{2.1}$$

On the other hand, we recall the celebrated Gagliardo-Nirenberg inequality:

$$|u|_r^r \le \kappa_{\mathrm{GN}} |\nabla u|_2^{r-2} |u|_2^2, \ \forall u \in H^1(\mathbb{R}^2), \ 2 < r < +\infty, \tag{2.2}$$

where $\kappa_{\mathrm{GN}} > 0$ is a constant. Combining (2.2) with $r = 4$ and the Young inequality,

$$\begin{aligned} \int_{\Sigma} |u|^2 dx &\le |\Sigma|^{\frac{1}{2}} \left(\int_{\Sigma} |u|^4 dx \right)^{\frac{1}{2}} \le |\Sigma|^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u|^4 dx \right)^{\frac{1}{2}} \\ &\le |\Sigma|^{\frac{1}{2}} \kappa_{\mathrm{GN}}^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} |u|^2 dx \right)^{\frac{1}{2}} \\ &\le \frac{1}{2} |\Sigma| \kappa_{\mathrm{GN}} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2} \int_{\mathbb{R}^2} |u|^2 dx, \ \forall u \in H^1(\mathbb{R}^2). \end{aligned} \tag{2.3}$$

As a consequence of (2.1) and (2.3), it holds that

$$\int_{\mathbb{R}^2} u^2 dx \le \max\left\{ \frac{2}{b}, |\Sigma|\kappa_{\mathrm{GN}} \right\} \|u\|_E^2, \ \forall u \in H^1(\mathbb{R}^2),$$

showing that the embedding $E \hookrightarrow H^1(\mathbb{R}^2)$ is continuous. So, there exists a constant $d_r > 0$ such that

$$|u|_r \le d_r \|u\|_E, \ \forall u \in E \text{ and } 2 \le r < +\infty. \tag{2.4}$$

For any $\lambda > 0$, define the Hilbert space $E_\lambda \triangleq (E, \|\cdot\|_{E_\lambda})$ with inner product and norm given by

$$(u,v)_{E_\lambda} = \int_{\mathbb{R}^2} [\nabla u \nabla v + \lambda V(x)uv]\, dx \text{ and } \|u\|_{E_\lambda} = \left(\int_{\mathbb{R}^2} \left[|\nabla u|^2 + \lambda V(x)|u|^2 \right] dx \right)^{\frac{1}{2}}$$

for all $u, v \in E$. Obviously, if $\lambda \geq 1$, one sees $\|u\|_E \leq \|u\|_{E_\lambda}$ for all $u \in E$. Using (V_2) again,

$$\begin{cases} \int\limits_{\Sigma} |u|^2 dx \leq |\Sigma|\kappa_{\mathrm{GN}} |\nabla u|_2^2 \leq |\Sigma|\kappa_{\mathrm{GN}} \|u\|_{E_\lambda}^2, \\ \int\limits_{\mathbb{R}^2 \setminus \Sigma} |u|^2 dx \leq \frac{1}{\lambda b} \int\limits_{\mathbb{R}^2 \setminus \Sigma} \lambda V(x)|u|^2 dx \leq \frac{1}{\lambda b} \int\limits_{\mathbb{R}^2} \lambda V(x)|u|^2 dx \leq \frac{1}{\lambda b}\|u\|_{E_\lambda}^2. \end{cases}$$

From which, for any $r \in (2, +\infty)$, there holds

$$\int\limits_{\mathbb{R}^3} |u|^r dx \leq \int\limits_{\mathbb{R}^2} |u|^2 dx \left(\int\limits_{\mathbb{R}^2} |\nabla u|^2 dx \right)^{\frac{r-2}{2}} \leq \max\left\{ |\Sigma|\kappa_{\mathrm{GN}}, \frac{1}{\lambda b} \right\} \|u\|_{E_\lambda}^r$$

Hence, for all $r \in (2, +\infty)$, we reach

$$|u|_r \leq \sqrt[r]{|\Sigma|\kappa_{\mathrm{GN}}} \|u\|_{E_\lambda} \text{ whenever } \lambda \geq b^{-1}|\Sigma|\kappa_{\mathrm{GN}} \triangleq \Lambda_0. \tag{2.5}$$

When the work space E_λ is built, we turn to investigate the variational structure of system (1.1). For this purpose, we have to focus on the Chern-Simons term in (1.1). To begin with, there exist some meaningful observations. According to the second equation and the last two equations in (1.1), for all $u \in E$, one has

$$\begin{aligned} \int\limits_{\mathbb{R}^2} A_0 |u|^2 dx &= 2 \int\limits_{\mathbb{R}^2} A_0(\partial_2 A_1 - \partial_1 A_2) dx \\ &= 2 \int\limits_{\mathbb{R}^2} (A_2 \partial_1 A_0 - A_1 \partial_2 A_0) dx = 2 \int\limits_{\mathbb{R}^2} (A_1^2 + A_2^2)|u|^2 dx. \end{aligned} \tag{2.6}$$

As a by-product of the well-known Hardy-Littlewood-Sobolev inequality [28, Theorem 4.3], we could conclude the following estimates to the gauge fields A_j for $j \in \{0, 1, 2\}$.

Lemma 2.1. *(see [19, Propositions 4.2-4.3]) Assume $1 < r < 2$ and $\frac{1}{r} - \frac{1}{\hat{r}} = \frac{1}{2}$, then*

$$|A_j|_{\hat{r}} \leq C_r |u|_{2r}^2 \text{ for } j = 1, 2, \ |A_0|_{\hat{r}} \leq C_r |u|_{2r}^2 |u|_4^2,$$

where $C_r > 0$ is a constant dependent of r.

Combining (2.5) and Lemma 2.1, for all $\lambda > \Lambda_0$, one easily sees that

$$|A_j u|_2 \leq |A_j|_{\hat{r}} |u|_{\frac{r}{r-1}} \leq C_r |u|_{2r}^2 |u|_{\frac{r}{r-1}} \leq \bar{C}_r \|u\|_{E_\lambda}^3, \text{ for } j = 1, 2, \tag{2.7}$$

because $2r > 2$ and $r/(r-1) > 2$, where $\bar{C}_r > 0$ is independent of $\lambda > \Lambda_0$. We also need the following Brézis-Lieb type lemma for the Chern-Simons term.

Lemma 2.2. *(see [17, Lemma 2.4]) If $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and $u_n \to u$ a.e. in $\mathbb{R}^2$ as $n \to \infty$, then one has $A_j[u_n] \to A_j[u]$ a.e. for $j = 1, 2$,*

$$\begin{cases} \lim\limits_{n \to \infty} \int\limits_{\mathbb{R}^2} A_0[u_n] u_n \psi dx = \int\limits_{\mathbb{R}^2} A_0[u] u \psi dx, \ \forall \psi \in H^1(\mathbb{R}^2), \\ \lim\limits_{n \to \infty} \int\limits_{\mathbb{R}^2} A_j^2[u_n] u_n \psi dx = \int\limits_{\mathbb{R}^2} A_j^2[u] u \psi dx, \ \forall \psi \in H^1(\mathbb{R}^2) \text{ with } j = 1, 2, \end{cases} \tag{2.8}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^2}\left[A_j^2[u_n]|u_n|^2 - A_j^2[u_n-u]|u_n-u|^2\right]dx = \int_{\mathbb{R}^2} A_j^2[u]|u|^2dx, \ \text{for } j=1,2. \tag{2.9}$$

Now, we are able to verify that the variational functional $I_\lambda : E_\lambda \to \mathbb{R}$ defined by

$$I_\lambda(u) = \frac{1}{2}\int_{\mathbb{R}^2}[|\nabla u|^2 + \lambda V(x)u^2]dx + \frac{1}{2}\int_{\mathbb{R}^2}\left(A_1^2[u]+A_2^2[u]\right)u^2dx - \int_{\mathbb{R}^2}F(u)dx$$

is well-defined and of class $\mathcal{C}^1(E_\lambda,\mathbb{R})$. Actually, due to (1.13) and (f_1), for all $\varepsilon>0$ and $\alpha>\alpha_0$, there is a constant $C_\varepsilon>0$ such that

$$|f(s)| \le \varepsilon|s| + C_\varepsilon|s|^{q-1}(e^{\alpha s^2}-1), \ \forall s\in\mathbb{R}, \tag{2.10}$$

where $q>2$ can be arbitrarily chosen later. By adopting (f_3), or (f_4), there holds

$$|F(s)| \le \varepsilon|s|^2 + C_\varepsilon|s|^{q}(e^{\alpha s^2}-1), \ \forall s\in\mathbb{R}. \tag{2.11}$$

Moreover, without mentioned any longer, let us exploit directly the following inequality (see e.g. [48, Lemma 2.1]):

$$(e^{\alpha s^2}-1)^m \le (e^{\alpha m s^2}-1), \ \forall s\in\mathbb{R}, \ \alpha>0 \text{ and } m>1.$$

With (2.10) and (2.11) in hands, together with (1.11), we could proceed as the calculations in [39,41] to deduce that the variational functional I_λ associated with (1.1) is well-defined and belongs to $\mathcal{C}^1(E_\lambda,\mathbb{R})$ such that

$$I'_\lambda(u)(v) = \int_{\mathbb{R}^2}\left[\nabla u\nabla v + \lambda V(x)uv + (A_1^2[u]+A_2^2[u]+A_0[u])uv\right]dx - \int_{\mathbb{R}^2}f(u)vdx$$

In particular, it follows from (2.6) that

$$I'_\lambda(u)(u) = \int_{\mathbb{R}^2}[|\nabla u|^2 + \lambda V(x)u^2]dx + 3\int_{\mathbb{R}^2}\left(A_1^2[u]+A_2^2[u]\right)u^2dx - \int_{\mathbb{R}^2}f(u)udx$$

2.2. Basic lemmas

Due to the above discussions, any (weak) solution of Eq. (1.1) corresponds to a critical point of I_λ. In order to search for critical points of I_λ, we introduce the following results.

Lemma 2.3. *If f satisfies (1.13) and (f_1). Let $(u_n)\subset H^1(\mathbb{R}^2)$ require $|\nabla u|_2^2 < \frac{4\pi}{\alpha_0}$ and $|u|_2^2 \le M_0$ for some $M_0\in(0,+\infty)$. Assume $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ and $u_n\to u$ a.e. in $\mathbb{R}^2$, then, up to subsequences,*

$$\begin{cases} \lim\limits_{n\to\infty}\int_{\mathbb{R}^2}\left[F(u_n)-F(u_n-u)\right]dx = \int_{\mathbb{R}^2}F(u)dx, \\ \lim\limits_{n\to\infty}\int_{\mathbb{R}^2}\left[f(u_n)u_n - f(u_n-u)(u_n-u)\right]dx = \int_{\mathbb{R}^2}f(u)udx. \end{cases} \tag{2.12}$$

Proof. Thanks to (1.11), the proof is standard and we omit the details. □

Lemma 2.4. *If f satisfies (1.13) and (f_1) as well as $(f_4)-(f_5)$. Suppose there is a sequence $(u_n) \subset E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ and $u_n \to u$ a.e. in $\mathbb{R}^2$. If in addition we assume that*

$$\sup_{n\in\mathbb{N}} \int\limits_{\mathbb{R}^2} f(u_n)u_n \mathrm{d}x \leq K_0 \tag{2.13}$$

for some $K_0 \in (0,+\infty)$ independent of $n \in \mathbb{N}$, then, going to a subsequence if necessary,

$$\lim_{n\to\infty} \int\limits_{\mathbb{K}} F(u_n)\mathrm{d}x = \int\limits_{\mathbb{K}} F(u)\mathrm{d}x \text{ for any compact set } \mathbb{K} \subset \mathbb{R}^2. \tag{2.14}$$

Moreover, passing to a subsequence if necessary, there holds

$$\lim_{n\to\infty} \int\limits_{\mathbb{R}^2} f(u_n)\psi \mathrm{d}x = \int\limits_{\mathbb{R}^2} f(u)\psi \mathrm{d}x \text{ for all } \psi \in C_0^\infty(\mathbb{R}^2). \tag{2.15}$$

Proof. We follow the essential ideas adopted in [12, Lemma 2.1] and so omit the details. □

It is similar to the proof of [42, Lemma A.1] that we can derive the following lemma.

Lemma 2.5. *(Pohožaev identity) Let $u \in E_\lambda$ be a critical point of the functional I_λ, then the identity $P_\lambda(u) \equiv 0$ holds true, where the functional $P_\lambda : E_\lambda \to \mathbb{R}$ is defined by*

$$P_\lambda(u) \triangleq \frac{1}{2} \int\limits_{\mathbb{R}^2} \lambda \left[2V(x) + (\nabla V, x)\right] u^2 dx + 2 \int\limits_{\mathbb{R}^2} \left(A_1^2[u] + A_2^2[u]\right) u^2 dx - 2 \int\limits_{\mathbb{R}^2} F(u) dx.$$

We conclude this section by the following two lemmas developed by P.-L. Lions [29].

Lemma 2.6. *Let $(\rho_n) \subset L^1(\mathbb{R}^2)$ be a bounded sequence and $\rho_n \geq 0$, then there exists a subsequence, still denoted by ρ_n, such that one of the following two possibilities occurs:*

(i) (Vanishing) $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} \rho_n dx = 0$ *for all $\varrho > 0$;*

(ii) (Non-Vanishing) *there are $\tau > 0$ and $\varrho < +\infty$ such that*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int\limits_{B_\varrho(y)} \rho_n dx = \tau.$$

Lemma 2.7. *Suppose that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^2)$ and $\{|\nabla u_n|\}$ is bounded in $L^2(\mathbb{R}^2)$ as well as*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int\limits_{B_\varrho(y)} |u_n|^2 dx = 0.$$

Then $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for $s \in (2,+\infty)$.

3. Some technical lemmas

3.1. The constraint manifold method

In this Subsection, we focus on the constraint manifold method for (1.1). First of all, to search for a ground state solution, let us consider the following minimization problem

$$m_\lambda \triangleq \inf_{u\in\mathcal{M}_\lambda} I_\lambda(u), \tag{3.1}$$

where $\mathcal{M}_\lambda = \{u \in E_\lambda\backslash(0) : G_\lambda(u) = 0\}$ with the functional $G_\lambda : E_\lambda \to \mathbb{R}$ defined by

$$\begin{aligned} G_\lambda(u) = \theta \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2}\int\limits_{\mathbb{R}^2} \lambda[2(\theta-1)V(x) - (\nabla V, x)]u^2 dx \\ + (3\theta-2)\int\limits_{\mathbb{R}^2} \left(A_1^2[u] + A_2^2[u]\right) u^2 dx - \int\limits_{\mathbb{R}^2} [\theta f(u)u - 2F(u)]dx \end{aligned}$$

Recalling the functional P_λ in Lemma 2.5, one sees that $G_\lambda(u) = \theta I'_\lambda(u)(u) - P_\lambda(u)$ for all $u \in E_\lambda$. In other words, if $u \in E_\lambda$ is a critical point of I_λ, then we are derived from Lemma 2.5 that $G_\lambda(u) = 0$. As a consequence, the set $\mathcal{M}_\lambda$ is a natural constraint and we then begin showing some properties for it and the minimization constant m_λ.

Before exhibiting them, we need the following elementary facts:

$$\xi_1(t, x) \triangleq \frac{1}{2}V(x) - \frac{t^{2(\theta-1)}}{2}V(t^{-1}x) - \frac{1-t^{6\theta-4}}{2(6\theta-4)}[2(\theta-1)V(x) - (\nabla V, x)] \tag{3.2}$$

for all $(t, x) \in (0, +\infty) \times \mathbb{R}^2$ and

$$\xi_2(t, z) \triangleq \frac{1-t^{6\theta-4}}{6\theta-4}[\theta f(z)z - 2F(z)] + t^{-2}F(t^\theta z) - F(z) \tag{3.3}$$

for all $(t, z) \in (0, +\infty) \times \mathbb{R}^+$.

Actually, it follows from (V_4) that

$$\begin{aligned} \frac{\partial}{\partial t}\xi_1(t, x) &= \frac{t^{6\theta-5}}{2}[2(\theta-1)V(x) - (\nabla V, x)] - \frac{t^{2\theta-3}}{2}[2(\theta-1)V(t^{-1}x) - (\nabla V(t^{-1}x), t^{-1}x)] \\ &= \frac{t^{6\theta-5}}{2}\left\{[2(\theta-1)V(x) - (\nabla V, x)] - \frac{[2(\theta-1)V(t^{-1}x) - (\nabla V(t^{-1}x), t^{-1}x)]}{t^{2(2\theta-1)}}\right\} \\ &\begin{cases} \le 0, & \text{if } t \in (0, 1], \\ \ge 0, & \text{if } t \in [1, +\infty). \end{cases} \end{aligned}$$

Hence, the function $t \mapsto \xi_1(t, x)$ is decreasing on $(0, 1)$ and increasing on $(1, +\infty)$ for all $x \in \mathbb{R}^2$ which indicate that $\xi_1(t, x) \ge \min_{t>0} \xi_1(t, x) = \xi_1(1, x) = 0$ for all $(t, x) \times (0, +\infty) \in \mathbb{R}^2$. Similarly, we are able to apply (f_3) to derive

$$\begin{aligned} \frac{\partial}{\partial t}\xi_2(t, z) &= t^{-3}[\theta f(t^\theta u)t^\theta u - 2F(t^\theta u)] - t^{6\theta-5}[\theta f(u)u - 2F(u)] \\ &= t^{6\theta-5}u^{\frac{6\theta-2}{\theta}}\left\{\frac{[\theta f(t^\theta u)t^\theta u - 2F(t^\theta u)]}{(t^\theta u)^{\frac{6\theta-2}{\theta}}} - \frac{[\theta f(u)u - 2F(u)]}{u^{\frac{6\theta-2}{\theta}}}\right\} \end{aligned}$$

$$\begin{cases} \leq 0, & \text{if } t \in (0,1], \\ \geq 0, & \text{if } t \in [1,+\infty). \end{cases}$$

It therefore reveals that $\xi_2(t,z) \geq \min\limits_{t>0} \xi_2(t,z) = \xi_2(1,z) = 0$ for all $(t,z) \times (0,+\infty) \in \mathbb{R}^+$.

Lemma 3.1. *Assume $(V_1)-(V_3)$ with (V_4) and (1.13) with $(f_1)-(f_3)$, then for any nonzero $u \in E_\lambda$, there is a unique $\bar{t} = \bar{t}(u) > 0$ such that $u_{\bar{t}} = \bar{t}^\theta u(\bar{t}\cdot) \in \mathcal{M}_\lambda$ for all $\lambda > \Lambda_0$, where $I_\lambda(u_{\bar{t}}) = \max\limits_{t>0} I_\lambda(u_t)$. In particular, there holds*

$$m_\lambda = d_\lambda \triangleq \inf_{u \in E_\lambda \backslash (0)} \max_{t>0} I_\lambda(u_t).$$

Proof. For any $u \in E_\lambda \backslash (0)$ and $t > 0$, we define $\tau(t) = I_\lambda(u_t)$, where

$$\begin{aligned} \tau(t) = &\frac{t^{2\theta}}{2} \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{t^{2(\theta-1)}}{2} \int\limits_{\mathbb{R}^2} \lambda V(t^{-1}x) u^2 dx + \frac{t^{6\theta-4}}{2} \int\limits_{\mathbb{R}^2} [A_1^2(u) + A_2^2(u)] u^2 dx \\ &- t^{-2} \int\limits_{\mathbb{R}^2} F(t^\theta u) dx \end{aligned}$$

It is simple to observe that

$$\tau'(t) = 0 \Longleftrightarrow t^{-1} G_\lambda(u_t) = 0 \Longleftrightarrow G_\lambda(u_t) = 0 \Longleftrightarrow u_t \in \mathcal{M}_\lambda.$$

Since $\theta > 1$ in (f_3) and $\lim\limits_{t \to 0^+} t^{-4\theta} F(t^\theta z) = 0$ for all $z \in \mathbb{R}$ by (f_1), we can derive $\lim\limits_{t \to 0^+} \tau(t) > 0$. Without loss of generality, we are assuming that $0 \in \Omega$ in (V_3) and thus $\lim\limits_{t \to +\infty} \int_{\mathbb{R}^3} \lambda V(t^{-1}x) u^2 dx = 0$. Adopting $p\theta > 6\theta - 2$ in (f_3) and (f_2), it holds that $\lim\limits_{t \to +\infty} \tau(t) = -\infty$. As a consequence, with the above two facts in hands, we take advantage of (f_1) and (f_2) to demonstrate that $\tau(t)$ possesses a critical point which corresponds to its maximum, that is, there exists a constant $\bar{t} > 0$ such that $\tau'(\bar{t}) = 0$. We next verify that $\bar{t}$ is unique. Arguing it indirectly, we would assume that there exist two constants $t_1, t_2 > 0$ with $t_1 \neq t_2$ such that $u_{t_i} \in \mathcal{M}_\lambda$ for $i \in \{1,2\}$. It concludes from some elementary computations that

$$\begin{aligned} &I_\lambda(u_{t_1}) - I_\lambda(u_{t_2}) - \frac{t_1^{6\theta-4} - t_2^{6\theta-4}}{(6\theta-4) t_1^{6\theta-4}} G_\lambda(u_{t_1}) \\ &= \xi_0\left(\frac{t_2}{t_1}\right) \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{t_1^{2(\theta-1)}}{2} \int\limits_{\mathbb{R}^2} \lambda \xi_1\left(\frac{t_2}{t_1}, t_1^{-1}x\right) u^2 dx + t_1^{-2} \int\limits_{\mathbb{R}^2} \xi_2\left(\frac{t_2}{t_1}, t_1 u\right) dx \end{aligned}$$

and

$$\begin{aligned} &I_\lambda(u_{t_2}) - I_\lambda(u_{t_1}) - \frac{t_2^{6\theta-4} - t_1^{6\theta-4}}{(6\theta-4) t_2^{6\theta-4}} G_\lambda(u_{t_2}) \\ &= \xi_0\left(\frac{t_1}{t_2}\right) \int\limits_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{t_2^{2(\theta-1)}}{2} \int\limits_{\mathbb{R}^2} \lambda \xi_1\left(\frac{t_1}{t_2}, t_2^{-1}x\right) u^2 dx + t_2^{-2} \int\limits_{\mathbb{R}^2} \xi_2\left(\frac{t_1}{t_2}, t_2 u\right) dx, \end{aligned}$$

where

$$\xi_0(t) \triangleq \frac{1-t^{2\theta}}{2} - \frac{\theta(1-t^{6\theta-4})}{6\theta-4} = \frac{2(\theta-1) - (3\theta-2)t^{2\theta} + \theta t^{6\theta-4}}{6\theta-4} \geq 0, \ \forall t \in (0,+\infty).$$

In view of (3.2) and (3.3), combining the above two formulas with $G_\lambda(u_{t_i}) = 0$ for $i \in \{1, 2\}$, we arrive at a contradiction if $t_1 \neq t_2$. Finally, the result $d_\lambda \leq m_\lambda$ is a direct consequence of the inequality

$$I_\lambda(u) - I_\lambda(u_t) - \frac{1 - t^{6\theta-4}}{6\theta - 4} G_\lambda(u) \geq 0,\ \forall u \in E_\lambda \text{ and } t > 0, \tag{3.4}$$

we immediately finish the proof of this lemma. □

According to Lemma 3.1, we know that $\mathcal{M}_\lambda$ is a nonempty set for some suitably large $\lambda > 0$. The following lemma ensures that the minimization constant m_λ would be well-defined. More precisely, we further show that m_λ is uniformly bounded from below and above by some positive constants which are independent of some suitably large $\lambda > 0$.

Lemma 3.2. *Assume $(V_1) - (V_3)$ with (V_4) and (1.13) with $(f_1) - (f_3)$, then there is a $\rho > 0$ independent of $\lambda > \Lambda_0$ such that*

$$\inf_{\lambda > \Lambda_0} m_\lambda \geq \rho. \tag{3.5}$$

If in addition the constant $\hat{\mu}$ in (f_2) satisfies

$$\hat{\mu} > \max\left\{ \frac{p(3\theta - 2)\mathbb{A}^{\frac{p\theta-2}{6\theta-4}}}{(p\theta - 2)\mathbb{B}} \left(\frac{p\theta - 6\theta + 2}{2m^*(p\theta - 2)} \right)^{\frac{p\theta - 6\theta + 2}{6\theta - 4}}, \frac{p\theta \mathbb{A}^{\frac{p\theta-2}{2\theta}}}{(p\theta - 2)\mathbb{B}} \left(\frac{p\theta - 2\theta - 2}{2m^*(p\theta - 2)} \right)^{\frac{p\theta - 2\theta - 2}{2\theta}} \right\}$$

with $\mathbb{A} \triangleq \int_{\mathbb{R}^2} \left\{ |\nabla \psi|^2 + [A_1^2(\psi) + A_2^2(\psi)]\psi^2 \right\} dx$ and $\mathbb{B} \triangleq \int_{\mathbb{R}^2} |\psi|^p dx$, where $\psi \in C_0^\infty(\mathbb{R}^2)$ is a cutoff function independent of λ, then

$$\sup_{\lambda > \Lambda_0} m_\lambda < m^* \triangleq \frac{4\pi \max\{4(\theta - 1), 2\theta - 1\}}{2\alpha_0(6\theta - 4)}. \tag{3.6}$$

Proof. On the one hand, let $t \to 0^+$ in (3.3), it holds that

$$\theta f(z)z - (6\theta - 2)F(z) \geq 0,\ \forall z \in \mathbb{R}^+.$$

Then, for all $u \in E_\lambda$, we depend on $(\theta - 1)V(x) \geq |(\nabla V, x)|$ for all $x \in \mathbb{R}^2$ in (V_4) to reach

$$\begin{aligned} I_\lambda(u) &= I_\lambda(u) - \frac{1}{6\theta - 4} G_\lambda(u) \\ &= \frac{\theta - 1}{3\theta - 2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{1}{2(6\theta - 4)} \int_{\mathbb{R}^2} \lambda[2(2\theta - 1)V(x) + (\nabla V, x)]u^2 dx \\ &\quad + \frac{1}{6\theta - 4} \int_{\mathbb{R}^2} [\theta f(u)u - (6\theta - 2)F(u)] dx \\ &\geq \frac{\theta - 1}{3\theta - 2} \int_{\mathbb{R}^2} |\nabla u|^2 dx + \frac{2\theta - 1}{2(6\theta - 4)} \int_{\mathbb{R}^2} \lambda V(x)u^2 dx \geq \frac{\max\{4(\theta - 1), 2\theta - 1\}}{2(6\theta - 4)} \|u\|_{E_\lambda}^2 \end{aligned} \tag{3.7}$$

implying that $m_\lambda \geq 0$ for all $\lambda > \Lambda_0$. If $m_\lambda = 0$, then there exists a sequence $(u_n) \subset \mathcal{M}_\lambda$ such that $\|u_n\|_{E_\lambda}^2 \to 0$. Denoting $v_n = u_n / \|u_n\|_{E_\lambda}$, then one simply has that $|\nabla v_n|_2^2 \leq 1$ and $|v_n|_2^2 \leq M_0 \in (0, +\infty)$ for some M_0 independent of $\lambda > \Lambda_0$ and $n \in \mathbb{N}$. Thus, using (1.11) and (2.10), we obtain

$$\int_{\mathbb{R}^2} f(u_n)u_n dx \le \varepsilon |u_n|_2^2 + C_\varepsilon \left(\int_{\mathbb{R}^2} |u_n|^{2q} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^2} (e^{2\alpha \|u_n\|_{E_\lambda}^2 v_n^2} - 1) dx \right)^{\frac{1}{2}}$$
$$\le \frac{\theta - 1}{2} \|u_n\|_{E_\lambda}^2 + C_\theta \|u_n\|_{E_\lambda}^q.$$

Since $(u_n) \subset \mathcal{M}_\lambda$ which is $G_\lambda(u_n) = 0$, we deduce that $\|u_n\|_{E_\lambda} \ge \left(\frac{\theta-1}{4C_\theta}\right)^{\frac{1}{q-2}}$ by $(\theta - 1)V(x) \ge |(\nabla V, x)|$ for all $x \in \mathbb{R}^2$ in (V_4), a contradiction. So, we have that $m_\lambda > 0$ for all $\lambda > \Lambda_0$. If $\inf_{\lambda > \Lambda_0} m_\lambda = 0$, then there is a sequence $(\lambda_n) \subset (\Lambda_0, +\infty)$ such that $m_{\lambda_n} \to 0$. Owing to (3.7), we can search for a sequence $(u_{\lambda_n}) \subset \mathcal{M}_{\lambda_n}$ such that $\|u_{\lambda_n}\|_{E_{\lambda_n}} \to 0$. Arguing as before, we obtain the same contradiction. Thus, we have $\inf_{\lambda > \Lambda_0} m_\lambda > 0$ finishing the verification of (3.5).

On the other hand, we begin verifying (3.6). Without loss of generality, we are assuming that $0 \in \Omega$. Because Ω is an open subset of $\mathbb{R}^3$, it holds that $B_{r_0}(0) \subset \Omega$ for some $r_0 > 0$. Given a constant $\hat{r}_0 > 0$ which will be determined later, we choose a cutoff function $\psi \in C_0^\infty(\mathbb{R}^2)$ in such a way that $\psi(x) \equiv 1$ if $|x| \le \hat{r}_0$ and $\psi(x) \equiv 0$ if $|x| \ge 2\hat{r}_0$. Due to Lemma 3.1 and (3.5), there is a $t_0 > 0$ which is independent of $\lambda > \Lambda_0$ such that

$$0 < \inf_{\lambda > \Lambda_0} m_\lambda \le \max_{\theta > 0} I_\lambda(\psi_t) = I_\lambda(\psi_{t_0}).$$

Letting $\hat{r}_0 = \frac{1}{2} t_0 r_0$, then

$$\int_{\mathbb{R}^3} V(t_0^{-1}x)\psi^2 dx = \int_{B_{t_0 r_0}(0)} V(t_0^{-1}x)\psi^2 dx + \int_{\mathbb{R}^2 \setminus B_{t_0 r_0}(0)} V(t_0^{-1}x)\psi^2 dx = 0$$

from where it follows that

$$I_\lambda(\psi_{t_0}) = \frac{t_0^{2\theta}}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{t_0^{6\theta-4}}{2} \int_{\mathbb{R}^2} [A_1^2(\psi) + A_2^2(\psi)]\psi^2 dx - t_0^{-2} \int_{\mathbb{R}^2} F(t_0^\theta \psi) dx.$$

Clearly, the proof of (3.6) would be done if $I_\lambda(\psi_{t_0}) < m^*$. Since $p\theta > 6\theta - 2$, then

$$\max_{t>1} \left\{ \frac{t^{2\theta}}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{t^{6\theta-4}}{2} \int_{\mathbb{R}^2} [A_1^2(\psi) + A_2^2(\psi)]\psi^2 dx - \frac{\hat{\mu} t^{p\theta-2}}{p} \int_{\mathbb{R}^2} |\psi|^p dx \right\}$$
$$\le \max_{t>0} \left\{ \frac{t^{6\theta-4}}{2} \int_{\mathbb{R}^2} \left\{ |\nabla \psi|^2 + [A_1^2(\psi) + A_2^2(\psi)]\psi^2 \right\} dx - \frac{\hat{\mu} t^{p\theta-2}}{p} \int_{\mathbb{R}^2} |\psi|^p dx \right\}$$
$$= \frac{p\theta - 6\theta + 2}{2(p\theta - 2)} \left(\frac{p(3\theta - 2)\mathbb{A}^{\frac{p\theta-2}{6\theta-4}}}{\hat{\mu}(p\theta - 2)\mathbb{B}} \right)^{\frac{6\theta-4}{p\theta-6\theta+2}} < m^*$$

and

$$\max_{t\in(0,1)} \left\{ \frac{t^{2\theta}}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{t^{6\theta-4}}{2} \int_{\mathbb{R}^2} [A_1^2(\psi) + A_2^2(\psi)]\psi^2 dx - \frac{\hat{\mu} t^{p\theta-2}}{p} \int_{\mathbb{R}^2} |\psi|^p dx \right\}$$
$$\le \max_{t>0} \left\{ \frac{t^{2\theta}}{2} \int_{\mathbb{R}^2} \left\{ |\nabla \psi|^2 + [A_1^2(\psi) + A_2^2(\psi)]\psi^2 \right\} dx - \frac{\hat{\mu} t^{p\theta-2}}{p} \int_{\mathbb{R}^2} |\psi|^p dx \right\}$$

$$= \frac{p\theta - 2\theta - 2}{2(p\theta - 2)} \left(\frac{p\theta \mathbb{A}^{\frac{p\theta-2}{2\theta}}}{\hat{\mu}(p\theta - 2)\mathbb{B}} \right)^{\frac{2\theta}{p\theta-2\theta-2}} < m^*.$$

As a consequence, it follows from (f_2) that

$$I_\lambda(\psi_{t_0}) \le \max_{t>0} \left\{ \frac{t^{2\theta}}{2} \int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{t^{6\theta-4}}{2} \int_{\mathbb{R}^2} [A_1^2(\psi) + A_2^2(\psi)]\psi^2 dx - \frac{\hat{\mu} t^{p\theta-2}}{p} \int_{\mathbb{R}^2} |\psi|^p dx \right\} < m^*$$

finishing the proof of (3.6). The proof is completed. □

Lemma 3.3. *Assume $(V_1) - (V_3)$ with (V_4) and (1.13) with $(f_1) - (f_3)$. Let $\lambda > \Lambda_0$ and $(u_n) \subset E_\lambda$ be a minimizing sequence of m_λ, then there exist $r \in (2, +\infty)$ and $\sigma_0 > 0$, independent of λ, such that $|u_n|_r \ge \sigma_0$, for all $n \ge 1$.*

Proof. First of all, we can make use of (3.6) and (3.7) to show that (u_n) is uniformly bounded in $n \in \mathbb{N}$ for all $\lambda > \Lambda_0$. Let us divide the proof into intermediate steps.

Step I: Let $\lambda > \Lambda_0$ and $(u_n) \subset E_\lambda$ be a minimizing sequence of m_λ, then there exist $r \in (2, +\infty)$ and $\sigma = \sigma(\lambda) > 0$ such that $|u_n|_r \ge \sigma$, for all $n \ge 1$.

Otherwise, we can apply Lemmas 2.6 and 2.7 to have that $u_n \to 0$ in $L^r(\mathbb{R}^2)$ for each $r \in (2, +\infty)$. According to the boundedness of (u_n) in E_λ, we see that (u_n) is uniformly bounded in $L^q(\mathbb{R}^2)$ for all $q \in (2, +\infty)$, too. As a consequence of (2.7), one simply arrives at

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left(A_1^2[u_n] + A_2^2[u_n] \right) u_n^2 dx = 0. \tag{3.8}$$

Since $J_\lambda(u_n) = m_\lambda + o_n(1)$ and $(u_n) \subset \mathcal{M}_\lambda$, combining (3.6) and (3.7), it has that $\limsup_{n\to\infty} \|u_n\|_{E_\lambda}^2 < \frac{4\pi}{\alpha_0}$. Thereby, we shall choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu' > 1$ sufficiently close to 1 in such a way that $\frac{1}{\nu} + \frac{1}{\nu'} = 1$ and

$$\alpha \nu' \|u_n\|_{E_\lambda}^2 < 4\pi(1 - \epsilon) \text{ for some suitable } \epsilon \in (0, 1).$$

Setting $\bar{u}_n = u_n / \|u_n\|_{E_\lambda}$, then $(u_n) \subset H^1(\mathbb{R}^2)$ and $|\nabla \bar{u}_n|_2^2 < 1$ as well as $|\bar{u}_n|_2^2 \le M_0$ by (2.5), where $M_0 \in (0, +\infty)$ is independent of $\lambda > \Lambda_0$ and $n \in \mathbb{N}$. It follows from (2.10) and the Holder's inequality that

$$\begin{aligned} \int_{\mathbb{R}^2} f(u_n) u_n dx &\le \varepsilon M_0 + C_\varepsilon \int_{\mathbb{R}^2} |u_n|^q (e^{\alpha u_n^2} - 1) dx \\ &\le \varepsilon M_0 + C_\varepsilon \left(\int_{\mathbb{R}^2} |u_n|^{q\nu} dx \right)^{\frac{1}{\nu}} \left(\int_{\mathbb{R}^2} (e^{4\pi(1-\epsilon)\bar{u}_n^2} - 1) dx \right)^{\frac{1}{\nu'}}. \end{aligned}$$

In view of (1.11), we obtain $\int_{\mathbb{R}^2} f(u_n) u_n dx \to 0$ by letting $n \to \infty$ and then tending $\varepsilon \to 0$. With this fact, we exploit (V_4) and $G_\lambda(u_n) = 0$ to reach $\|u_n\|_{E_\lambda} \to 0$. Due to (3.8), the above discussions permit us to conclude that $m_\lambda = \lim_{n\to\infty} I_\lambda(u_n) = 0$ which is impossible because of (3.5). This step is done.

Step II: Conclusion.

Let $r \in (2, +\infty)$ be as in Step I. Suppose by contradiction that the uniform control from below of $L^r(\mathbb{R}^2)$-norm is false. Then, for every $k \in \mathbb{N}$, $k \neq 0$, there exist $\lambda_k > \Lambda_0$ and a minimizing sequence $(u_{k,n})$ of m_{λ_k} such that

$$|u_{k,n}|_r < \frac{1}{k}, \text{ definitely.}$$

Then, by a diagonalization argument, for any $k \geq 1$, we can find an increasing sequence (n_k) in $\mathbb{N}$ and $u_{n_k} \in E_{\lambda_{n_k}}$ such that

$$u_{n_k} \in \mathcal{M}_{\lambda_k},\ J_{n_k}(u_{n_k}) = m_{\lambda_{n_k}} + o_k(1),\ |u_{n_k}|_r = o_k(1),$$

where $o_k(1)$ is a positive quantity which goes to zero as $k \to +\infty$. Then, we are able to arrive at a same contradiction in the Step I with (3.6), again. The proof is completed. □

Lemma 3.4. *Assume $(V_1) - (V_3)$ with (V_4) and (1.13) with $(f_1) - (f_3)$, then there is a $\Lambda > 0$ such that m_λ can be attained for all $\lambda > \Lambda$.*

Proof. Let $(u_n) \subset \mathcal{M}_\lambda$ be a sequence satisfying $I_\lambda(u_n) \to m_\lambda$ as $n \to \infty$. First of all, we are derived from (3.6) and (3.7) that (u_n) is uniformly bounded in E_λ for all $\lambda > \Lambda_0$. Passing to a subsequence if necessary, there is a function $u \in E_\lambda$ such that $u_n \rightharpoonup u$ in E_λ, $u_n \to u$ in $L^q_{\text{loc}}(\mathbb{R}^2)$ for all $2 < q < +\infty$ and $u_n \to u$ a.e. in $\mathbb{R}^2$.

Secondly, we shall find a suitable large $\Lambda > 0$ such that $u \neq 0$ for all $\lambda > \Lambda$. Owing to the above discussions, we know that $\|u_n\|^2_{E_\lambda} \leq C^*$ for a suitable $C^* > 0$, for any $n \geq 1$ and $\lambda > \Lambda_0$. Let $r > 2$ and $\sigma_0 > 0$ be given as in Lemma 3.3, recalling (V_3), there is a sufficiently large constant $\overline{R} > 1$ such that,

$$\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma} |u_n|^r dx \leq \frac{\sigma_0}{4}, \qquad \text{for all } \lambda > \Lambda_0 \text{ and for all } n \geq 1. \tag{3.9}$$

Since $V(x) \geq b$ on Σ^c by (V_3), we have

$$\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} |u_n|^2 dx \leq \frac{1}{\lambda b} \int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} \lambda V(x)|u_n|^2 dx \leq \frac{C^*}{\lambda b}$$

It easily infers that

$$\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} |u_n|^r dx \leq \left(\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} |u_n|^2 dx \right)^{\frac{1}{2}} \left(\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} |u_n|^{2(r-1)} dx \right)^{\frac{1}{2}},$$

and so one can find a $\Lambda > \Lambda_0$ such that

$$\int\limits_{B^c_{\overline{R}}(0)\cap\Sigma^c} |u_n|^r dx \leq \frac{\sigma_0}{4}, \qquad \text{for all } \lambda > \Lambda \text{ and for all } n \geq 1. \tag{3.10}$$

Finally, we fix $\lambda > \Lambda_0$, if $u_n \rightharpoonup u \equiv 0$, we can deduce that

$$\int\limits_{B_{\overline{R}}(0)} |u_n|^r dx \leq \frac{\sigma_0}{4}, \quad \text{for all } n \text{ sufficiently large.} \tag{3.11}$$

Clearly, (3.9), (3.10) and (3.11) are in contradictions with Lemma 3.3.

Finally, we conclude that $u_n \to u$ along a subsequence as $n \to \infty$ for every $\lambda > \Lambda$. Taking (3.6) and (3.7) into account again, it holds that $\sup_{n\in\mathbb{N}} \|u_n\|^2_{E_\lambda} < \frac{4\pi}{\alpha_0}$. Define $w_n \triangleq u_n - u$, then we are able to apply (2.8)-(2.9) and (2.12) to derive

$$\lim_{n\to\infty} I_\lambda(w_n) = \lim_{n\to\infty} \left[I_\lambda(u_n) - I_\lambda(u)\right] = m_\lambda - I_\lambda(u) \tag{3.12}$$

and

$$\lim_{n\to\infty} G_\lambda(w_n) = \lim_{n\to\infty} \left[G_\lambda(u_n) - G_\lambda(u)\right] = -G_\lambda(u). \tag{3.13}$$

We claim that $G_\lambda(u) \le 0$. Otherwise, it has that $\lim_{n\to\infty} G_\lambda(w_n) < 0$ by (3.13). Without loss of generality, we are assuming that $G_\lambda(w_n) < 0$ for all $n \in \mathbb{N}$. From which, one knows that $w_n \ne 0$ and so Lemma 3.1 permits us to determine a $t_n > 0$ such that $G_\lambda((w_n)_{t_n}) = 0$. Combining (3.4) and (3.12)-(3.13),

$$\begin{aligned} m_\lambda - I_\lambda(u) + \frac{1}{6\theta-4}G_\lambda(u) &= \lim_{n\to\infty} \left[I_\lambda(w_n) - \frac{1}{6\theta-4}G_\lambda(w_n)\right] \\ &\ge \lim_{n\to\infty} \left[I_\lambda\big((w_n)_{t_n}\big) - \frac{t_n^{6\theta-4}}{6\theta-4}G_\lambda(w_n)\right] > \lim_{n\to\infty} I_\lambda((w_n)_{t_n}) \ge m_\lambda, \end{aligned}$$

which gives that

$$I_\lambda(u) - \frac{1}{6\theta-4}G_\lambda(u) < 0.$$

It is similar to (3.7) that we would get a contradiction. Hence, we have arrived at $G_\lambda(u) \le 0$. Adopting Lemma 3.1 again, there exists a $t > 0$ such that $u_t \in \mathcal{M}_\lambda$. Owing to (3.4) and the Fatou's lemma,

$$\begin{aligned} m_\lambda = \lim_{n\to\infty} I_\lambda(u_n) &= \lim_{n\to\infty} \left[I_\lambda(u_n) - \frac{1}{6\theta-4}G_\lambda(u_n)\right] \ge I_\lambda(u) - \frac{1}{6\theta-4}G_\lambda(u) \\ &\ge I_\lambda(u_\theta) - \frac{t^{6\theta-4}}{6\theta-4}G_\lambda(u) \ge I_\lambda(u_t) \ge m_\lambda, \end{aligned}$$

which yields that $u_n \to u$ in E_λ. Consequently, $I_\lambda(u) = m_\lambda$ and $G_\lambda(u) = 0$. The proof is completed. □

3.2. The minimax argument

In this Subsection, we shall dispose of the minimax argument to find a positive solution for (1.1) with a wider class of V and f. Without (V_4) and (f_3), one could not take advantage of the minimization constraint manifold method explored in Section 3. Whereas, because of (f_4), it seems impossible to prove that the (PS) sequence is uniformly bounded. As a consequence, we shall depend on an indirect approach developed by Jeanjean [23].

Proposition 3.5. *(See [23, Theorem 1.1 and Lemma 2.3]) Let $(X, \|\cdot\|)$ be a Banach space and $T \subset R^+$ be an interval, consider a family of C^1 functionals on X of the form*

$$\Phi_\mu(u) = A(u) - \mu B(u), \quad \forall \mu \in T,$$

with $B(u) \ge 0$ and either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\| \to +\infty$. Assume that there exists two points $v_1, v_2 \in X$ such that

$$c_\mu = \inf_{\gamma\in\Gamma} \sup_{\theta\in[0,1]} \Phi_\mu(\gamma(\theta)) > \max\{\Phi_\mu(v_1), \Phi_\mu(v_1)\}, \quad \forall \mu \in T,$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for almost every $\mu \in T$, there is a sequence $(u_n(\mu)) \subset X$ such that

(*a*) *$(u_n(\mu))$ is bounded in X;*
(*b*) *$\Phi_\mu(u_n(\mu)) \to c_\mu$ and $\Phi'_\mu(u_n(\mu)) \to 0$;*
(*c*) *the map $\mu \to c_\mu$ is non-increasing and left continuous.*

Letting $T = [\delta, 1]$, where $\delta \in (0, 1)$ is a positive constant. To apply Proposition 3.5, we will introduce a family of $\mathcal{C}^1$-functionals on $X = E_\lambda$ with the form

$$I_{\lambda,\mu}(u) = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)u^2]dx + \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2[u] + A_2^2[u]\right) u^2 dx - \mu \int_{\mathbb{R}^3} F(u)dx. \tag{3.14}$$

Define $I_{\lambda,\mu}(u) = A(u) - \mu B(u)$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^2} [|\nabla u|^2 + \lambda V(x)u^2]dx + \frac{1}{2}\int_{\mathbb{R}^2} \left(A_1^2[u] + A_2^2[u]\right) u^2 dx \to +\infty \text{ as } \|u\|_{E_\lambda} \to +\infty,$$

and

$$B(u) = \int_{\mathbb{R}^2} F(u)dx \geq 0.$$

Clearly, $I_{\lambda,\mu}$ is of class $\mathcal{C}^1$-functionals with

$$I'_{\lambda,\mu}(u)(v) = \int_{\mathbb{R}^2} \left[\nabla u \nabla v + \lambda V(x)uv + (A_1^2[u] + A_2^2[u] + A_0[u])uv\right] dx - \mu \int_{\mathbb{R}^3} f(u)vdx$$

for all $u, v \in E_\lambda$.

For simplicity, from now on until the end of this section, we shall always suppose the assumptions in Theorem 1.5 when there is no misunderstanding.

Lemma 3.6. *The functional $I_{\lambda,\mu}$ possesses a mountain-pass geometry, that is,*

(*a*) *there exists $v \in E_\lambda \setminus \{0\}$ independent of μ such that $I_{\lambda,\mu}(v) \leq 0$ for all $\mu \in [\delta, 1]$;*
(*b*) *$c_{\lambda,\mu} \triangleq \inf_{\eta\in\Gamma} \sup_{\theta\in[0,1]} I_{\lambda,\mu}(\gamma(\eta)) > \max\{I_{\lambda,\mu}(0), I_{\lambda,\mu}(v)\}$ for all $\mu \in [\delta, 1]$, where*

$$\Gamma = \{\eta \in \mathcal{C}([0,1], E_\lambda) : \eta(0) = 0, \eta(1) = v\}.$$

Proof. The proof is very similar to the calculations on showing the existence of critical points in the proof of Lemma 3.1, so we omit the details. □

Next, we try to look for a uniform upper bound for $c_{\lambda,\mu}$ with respect to λ. Because $\Omega = \mathrm{int}V^{-1}(0)$ is open, without loss of generality, we can suppose that $0 \in \Omega$ and there exists a constant $\rho_0 > 0$ such that $B_{\rho_0}(0) \subset \Omega$. Motivated by [7,15,26,49], we introduce the Moser sequence defined by

$$\bar{w}_n(x) \triangleq \frac{1}{\sqrt{2\pi}} \begin{cases} \sqrt{\log n}, & \text{if } 0 \le |x| \le \frac{\rho_0}{n}, \\ \frac{\log(\frac{\rho_0}{|x|})}{\sqrt{\log n}}, & \text{if } \frac{\rho_0}{n} < |x| \le \rho_0, \\ 0, & \text{if } |x| > \rho_0. \end{cases}$$

It is easy to see that $(\bar{w}_n) \subset E_\lambda$. Moreover, thanks to [37, Lemma 3.5], we derive the following lemma.

Lemma 3.7. *It holds that* $\|\bar{w}_n\|_{E_\lambda} \to 1$ *and* $N(\bar{w}_n) \triangleq \int_{\mathbb{R}^2} \left(A_1^2[\bar{w}_n] + A_2^2[\bar{w}_n]\right) \bar{w}_n^2 dx \to 0$ *as* $n \to \infty$.

Lemma 3.8. *Let* $\lambda > \Lambda_0$, *then there exists* $\tilde{C} < \frac{2\pi}{\alpha_0}$, *independent of* λ *such that* $c_{\lambda,\mu} < \tilde{C}$, $\forall \mu \in [\delta, 1]$.

Proof. The essential idea comes from [37, Lemma 3.6], we exhibit the details for the sake of the reader's convenience. Setting $w_n = \bar{w}_n/\|\bar{w}_n\|_{E_\lambda}$, then $\|w_n\|_{E_\lambda} \equiv 1$ and $N(w_n) = N(\bar{w}_n)/\|\bar{w}_n\|^6_{E_\lambda} \to 0$ as $n \to \infty$ by Lemma 3.7. Due to the definition of $c_{\lambda,\mu}$, to end with the proof, it is enough to show that there exists some $n_0 \in \mathbb{N}$ such that

$$\tilde{C} \triangleq \max_{t \ge 0} \left\{ \frac{t^2}{2} + \frac{t^6}{2} N(w_{n_0}) - \mu \int_{\mathbb{R}^2} F(tw_{n_0}) dx \right\} < \frac{2\pi}{\alpha_0}.$$

Indeed, it simply observes that

$$c_\lambda \le \max_{t \ge 0} J_\lambda(tw_{n_0}) \le \max_{t \ge 0} \left\{ \frac{t^2}{2} + \frac{t^6}{2} N(w_{n_0}) - \mu \int_{\mathbb{R}^2} F(tw_{n_0}) dx \right\}.$$

On the contrary, suppose that for all $n \in \mathbb{N}$, there is a constant $t_n > 0$ such that

$$\frac{t_n^2}{2} + \frac{t_n^6}{2} N(w_n) - \mu \int_{\mathbb{R}^2} F(t_n w_n) dx \ge \frac{2\pi}{\alpha_0} \tag{3.15}$$

and such that

$$t_n^2 + 3t_n^6 N(w_n) = \mu \int_{\mathbb{R}^2} f(t_n w_n) t_n w_n dx. \tag{3.16}$$

From (f_5) and (f_6), for all $\epsilon \in (0, \beta_0)$, there exists a constant $R_\epsilon = R(\epsilon) > 0$ such that

$$f(z)z \ge M_0^{-1}(\beta_0 - \epsilon) t^{\vartheta+1} e^{\alpha_0 |t|^2}, \ \forall z \ge R_\epsilon.$$

Thanks to (3.15), $\{t_n\}$ is bounded below by some positive constant. For some sufficiently large $n \in \mathbb{N}$, one knows that $t_n w_n \ge R_\epsilon$ on $B_{\rho_0/n}(0)$. Then, on one hand, by (3.16) and (g_1), we can obtain that

$$t_n^2 + 3t_n^6 N(w_n) \ge C M_0^{-1} \mu(\beta_0 - \epsilon) \int_{B_{\rho_0/n}(0)} (t_n w_n)^{\vartheta+1} e^{\alpha_0 |t_n w_n|^2} dx$$

$$\geq CM_0^{-1}\mu(\beta_0-\epsilon)t_n^{\vartheta+1}\left(\frac{\log n}{2\pi\|\bar{w}_n\|_{E_\lambda}^2}\right)^{\frac{\vartheta+1}{2}}\exp\left(\frac{\alpha_0 t_n^2\log n}{2\pi\|\bar{w}_n\|_{E_\lambda}^2}\right)n^{-2} \tag{3.17}$$

which, together with the fact that $N(w_n)\leq C\|w_n\|_{E_\lambda}^6\leq C<+\infty$, reveals that (t_n) is uniformly bounded in $n\in\mathbb{N}$. If not, we may assume that $t_n\to+\infty$ and then

$$C\log t_n\geq t_n^2\left(\frac{\alpha_0}{2\pi\|\bar{w}_n\|_{E_\lambda}^2}-\frac{2}{t_n^2}\right)\log n$$

which, together with Lemma 3.7, yields a contradiction. So, up to a subsequence if necessary, there exists a constant $t_0\in(0,+\infty)$ such that $t_n\to t_0$.

On the other hand, by (3.15), we have $t_0^2\geq 4\pi/\alpha_0$. Taking $\epsilon=\beta_0/2$ in (3.17), one has

$$(1-\vartheta)\log t_0+o(1)\geq C+C\log(\log n)+C\left(\frac{\alpha_0}{2\pi}t_0^2-2\right)\log n+o(1)\geq C+C\log(\log n)+o(1),$$

yields a contradiction if $n\in\mathbb{N}$ is sufficiently large. The proof is completed. □

Combining the arguments explored in Lemmas 3.6 and 3.8, there is a constant $\hat{\rho}>0$ such that

$$\hat{\rho}\leq\inf_{\lambda>\Lambda_0}c_{\lambda,\mu}\leq\sup_{\lambda>\Lambda_0}c_{\lambda,\mu}<\frac{2\pi}{\alpha_0},\ \ \forall\mu\in[\delta,1]. \tag{3.18}$$

Lemma 3.9. *Let (u_n) be a bounded (PS) sequence of the functional $I_{\lambda,\mu}$ at the level $c>0$, then for each $\hat{M}\in\left(c,\frac{2\pi}{\alpha_0}\right)$, there is a $\hat{\Lambda}=\Lambda(\hat{M})>0$ such that (u_n) contains a strongly convergent subsequence in E_λ for all $\lambda>\hat{\Lambda}$.*

Proof. Since (u_n) is bounded in E_λ, then there exists a $u\in E_\lambda$ such that $u_n\rightharpoonup u$ in E_λ, $u_n\to u$ in $L^s_{\text{loc}}(\mathbb{R}^2)$ with $s\in[1,+\infty)$ and $u_n\to u$ *a.e.* in $\mathbb{R}^2$. To conclude the proof clearly, we shall split it into several steps:

Step 1: Define $v_n\triangleq u_n-u$, then there exists a $\hat{\Lambda}=\Lambda(\hat{M})>0$ such that $v_n\to 0$ in $L^q(\mathbb{R}^2)$ for all $q\in(2,+\infty)$ along a subsequence as $n\to\infty$ when $\lambda>\hat{\Lambda}$.

Actually, since (v_n) is uniformly bounded in $n\in\mathbb{N}$ for all $\lambda>\Lambda_0$, then we have one of the following two possibilities for some $r>0$:

$$\begin{cases}\text{(i)}\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^2}\displaystyle\int_{B_r(y)}|v_n|^2dx>0,\\ \text{(ii)}\ \lim\limits_{n\to\infty}\sup\limits_{y\in\mathbb{R}^2}\displaystyle\int_{B_r(y)}|v_n|^2dx=0.\end{cases}$$

As a consequence, the conclusion would be clear if we could demonstrate that the case (i) cannot occur for sufficiently large $\lambda>0$. Now, we suppose, by contradiction, that (i) was true. Proceeding as the very similar way in Lemma 3.4, there is a constant $\hat{\delta}>0$ independent of $\lambda>\Lambda_0$ such that

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^2}\int_{B_r(y)}|v_n|^2dx\geq\hat{\delta}$$

for some $r>0$. Since (u_n) is uniformly bounded in E_λ, without loss of generality, we can assume that $\lim\limits_{n\to\infty}\|u_n\|_{E_\lambda}^2\leq\Theta$ for some $\Theta\in(0,+\infty)$. Clearly, there holds $\lim\limits_{n\to\infty}\|v_n\|_{E_\lambda}^2\leq 4\Theta$. Recalling $v_n\to 0$ in

$L^q_{\rm loc}(\mathbb{R}^2)$ with $q \in (2,+\infty)$ and $|\mathcal{A}_R| \to 0$ as $R \to +\infty$ by (V_2), where $\mathcal{A}_R \triangleq \{x \in \mathbb{R}^2\backslash B_R(0) : V(x) < b\}$, we can determine a sufficiently large but fixed $R > 0$ to satisfy

$$\limsup_{n\to\infty} \int_{B_R(0)} |v_n|^2 dx < \frac{\hat{\delta}}{4} \tag{3.19}$$

and

$$|\mathcal{A}_R| < \left(\frac{\hat{\delta}}{16\Theta}\right)^{\frac{q}{q-2}} (|\Sigma|\kappa_{\rm GN})^{-\frac{2}{q-2}}. \tag{3.20}$$

Combining (2.5) and (3.20), one sees that

$$\limsup_{n\to\infty} \int_{\mathcal{A}_R} |v_n|^2 dx \le \limsup_{n\to\infty} \left(\int_{\mathcal{A}_R} |v_n|^q dx\right)^{\frac{2}{q}} |\mathcal{A}_R|^{\frac{q-2}{q}} \le 4\Theta(|\Sigma|\kappa_{\rm GN})^{\frac{2}{q}} |\mathcal{A}_R|^{\frac{q-2}{q}} < \frac{\hat{\delta}}{4}. \tag{3.21}$$

Let us choose $\hat{\Lambda} = \max\left\{1, \Lambda_0, \frac{16\Theta}{\hat{\delta} b}\right\}$, then for all $\lambda > \hat{\Lambda}$, we reach

$$\limsup_{n\to\infty} \int_{\mathcal{B}_R} |v_n|^2 dx \le \limsup_{n\to\infty} \frac{1}{\lambda b} \int_{\mathcal{B}_R} \lambda V(x)|v_n|^2 dx \le \frac{4\Theta}{\lambda b} < \frac{\hat{\delta}}{4}, \tag{3.22}$$

where $\mathcal{B}_R \triangleq \{x \in \mathbb{R}^2\backslash B_R(0) : V(x) \ge b\}$. We gather (3.19), (3.20) and (3.22) to derive

$$\begin{aligned} \hat{\delta} &\le \lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{B_r(y)} |v_n|^2 dx \le \limsup_{n\to\infty} \int_{\mathbb{R}^2} |v_n|^2 dx \\ &= \limsup_{n\to\infty} \left(\int_{\mathbb{R}^2\backslash B_R(0)} |v_n|^2 dx + \int_{B_R(0)} |v_n|^2 dx \right) \le \frac{3\hat{\delta}}{4} \end{aligned}$$

which is impossible. The proof of this step is done.

Step 2: $u \ne 0$, $I'_{\lambda,\mu}(u) = 0$ and $I_{\lambda,\mu}(u) \ge 0$.

We suppose, by contradiction, that $u \equiv 0$ and thus the Step 1 gives us that $u_n \to 0$ in $L^q(\mathbb{R}^2)$ for all $q \in (2,+\infty)$ when $\Lambda > \hat{\Lambda}$. Since (u_n) is a bounded (PS) sequence of the functional $I_{\lambda,\mu}$, we know that (2.13) holds true from some $K_0 \in (0,+\infty)$, then (2.14) and (2.15) reveal that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} F(u_n) dx = 0 \text{ and } \lim_{n\to\infty} \int_{\mathbb{R}^2} f(u_n) u_n dx = 0. \tag{3.23}$$

Moreover, we are derived from (2.7) that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left(A_1^2[u_n] + A_2^2[u_n]\right) u_n^2 dx = 0. \tag{3.24}$$

As a consequence, according to $I_{\lambda,\mu}(u_n) = c + o_n(1)$ it holds that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^2}[|\nabla u_n|^2+\lambda V(x)u_n^2]dx=2c<\frac{4\pi}{\alpha_0}.$$

Thereby, we shall choose $\alpha>\alpha_0$ sufficiently close to α_0 and $\nu'>1$ sufficiently close to 1 in such a way that $\frac{1}{\nu}+\frac{1}{\nu'}=1$ and

$$\alpha\nu'|\nabla u_n|_2^2<4\pi(1-\epsilon) \text{ for some suitable } \epsilon\in(0,1).$$

Setting $\bar{u}_n=\sqrt{\frac{\alpha\nu'}{4\pi(1-\epsilon)}}u_n$, then $(u_n)\subset H^1(\mathbb{R}^2)$ and $|\nabla\bar{u}_n|_2^2<1$ as well as $|\bar{u}_n|_2^2\le M_0$ by (2.5), where $M_0\in(0,+\infty)$ is independent of $\lambda>\hat{\Lambda}$ and $n\in\mathbb{N}$. It follows from (2.10) and the Holder's inequality that

$$\begin{aligned}\int_{\mathbb{R}^2}f(u_n)u_ndx&\le\varepsilon M_0+C_\varepsilon\int_{\mathbb{R}^2}|u_n|^q(e^{\alpha u_n^2}-1)dx\\&\le\varepsilon M_0+C_\varepsilon\left(\int_{\mathbb{R}^2}|u_n|^{q\nu}dx\right)^{\frac{1}{\nu}}\left(\int_{\mathbb{R}^2}(e^{4\pi(1-\epsilon)\bar{u}_n^2}-1)dx\right)^{\frac{1}{\nu'}}.\end{aligned}$$

Recalling (1.11), we derive $\int_{\mathbb{R}^2}f(u_n)u_ndx\to0$ by letting $n\to\infty$ and then tending $\varepsilon\to0$. Combining this fact and (3.23)-(3.24), we exploit $I'_{\lambda,\mu}(u_m)\to0$ to obtain $\|u_n\|_{E_\lambda}\to0$. Consequently, one could observe that $0<c=\lim_{n\to\infty}I_{\lambda,\mu}(u_m)=0$ which is absurd. So, $u\neq0$ holds true. As a direct consequence of (2.8) and (2.15) together with $I'_{\lambda,\mu}(u_n)\to0$, one sees $I'_{\lambda,\mu}(u)=0$. It is very similar to Lemma 2.5 that $P_{\lambda,\mu}(u)=0$, where $P_{\lambda,\mu}:E_\lambda\to\mathbb{R}$ is given by

$$P_{\lambda,\mu}(v)\triangleq\frac12\int_{\mathbb{R}^2}\lambda\left[2V(x)+(\nabla V,x)\right]v^2dx+2\int_{\mathbb{R}^2}\left(A_1^2[v]+A_2^2[v]\right)v^2dx-2\mu\int_{\mathbb{R}^2}F(v)dx.$$

Since $I'_{\lambda,\mu}(u)(u)=0$ and $P_{\lambda,\mu}(u)=0$, we apply (f_4) and (V_5) to obtain

$$\begin{aligned}I_{\lambda,\mu}(u)&=I_{\lambda,\mu}(u)-\frac12I'_{\lambda,\mu_n}(u)(u)+\frac{\gamma-2}{4}P_{\lambda,\mu}(u)\\&=\frac{\gamma-2}{8}\int_{\mathbb{R}^2}\lambda\left[2V(x)+(\nabla V,x)\right]u^2dx+\frac{\gamma-4}{2}\int_{\mathbb{R}^2}\left(A_1^2[u]+A_2^2[u]\right)u^2dx\\&\quad+\frac{\mu}{2}\int_{\mathbb{R}^2}\left[f(u)u-\gamma F(u)\right]dx\\&\ge\frac{\gamma-2}{8}\int_{\mathbb{R}^2}\lambda\left[2V(x)+(\nabla V,x)\right]u^2dx+\frac{\gamma-4}{2}\int_{\mathbb{R}^2}\left(A_1^2[u]+A_2^2[u]\right)u^2dx\end{aligned}$$

implying that $I_{\lambda,\mu}(u)\ge0$. The proof of this Step is done.

Step 3: Passing to a subsequence if necessary, $u_n\to u$ in E_λ as $n\to\infty$.

By the Fatou's lemma, there holds

$$0<\|u\|_{E_\lambda}\le\liminf_{n\to\infty}\|u_n\|_{E_\lambda}. \tag{3.25}$$

Up to a subsequence if necessary, we define

$$v_n\triangleq\frac{u_n}{\|u_n\|_{E_\lambda}}\text{ and }v=\frac{u}{\lim_{n\to\infty}\|u_n\|_{E_\lambda}}.$$

Obviously, $0 < \|v\|_{E_\lambda} \le 1$ by (3.25). If $\|v\|_{E_\lambda} = 1$, we deduce that $\|u_n\|_{E_\lambda} \to \|u\|_{E_\lambda}$ which together with $u_n \rightharpoonup u$ in E_λ yields that $u_n \to u$ in E_λ. Hence, the proof is finished. Suppose that $0 < \|v\|_{E_\lambda} < 1$. In this situation, combining Steps 1-2, (2.7), (2.14) and the Fatou's lemma, we obtain

$$\begin{aligned}\frac{4\pi}{\alpha_0} &> 2c \ge 2[c - I_\lambda(u)] = \limsup_{n\to\infty}(\|u_n\|^2_{E_\lambda} - \|u\|^2_{E_\lambda}) = \limsup_{n\to\infty}\|u_n\|^2_{E_\lambda}\left(1 - \left\|\frac{u}{\|u_n\|_{E_\lambda}}\right\|^2_{E_\lambda}\right) \\ &\ge (1 - \|v\|^2_{E_\lambda})\limsup_{n\to\infty}\|u_n\|^2_{E_\lambda}\end{aligned}$$

which gives that

$$\limsup_{n\to\infty}\|u_n\|^2_{E_\lambda} < \frac{4\pi}{\alpha_0(1 - \|v\|^2_{E_\lambda})}.$$

Then, we would choose $\alpha > \alpha_0$ sufficiently close to α_0 and $\nu' > 1$ sufficiently close to 1 in such a way that $\frac{1}{v} + \frac{1}{v'} = 1$ and

$$\alpha\nu'\|u_n\|^2_{E_\lambda} < \frac{4\pi(1-\epsilon)}{1 - \|v\|^2_{E_\lambda}} \triangleq 4\pi p_\epsilon, \text{ for some suitable } \epsilon \in (0,1),$$

where $0 < p_\epsilon = (1-\epsilon)/(1 - \|v\|^2_{E_\lambda}) < P_{\alpha_0}(v)$. So, by (1.12) and $|u_n|^2 = \|u_n\|^2_{E_\lambda}|v_n|^2$, we have that

$$\sup_{n\in\mathbb{N}}\int\limits_{\mathbb{R}^2}(e^{\alpha\nu'|u_n|^2} - 1)dx \le \sup_{n\in\mathbb{N}}\int\limits_{\mathbb{R}^2}(e^{4\pi p_\epsilon|v_n|^2} - 1)dx < +\infty. \tag{3.26}$$

To finish the proof, we claim that

$$\int\limits_{\mathbb{R}^2} f(u_n)(u_n - u)dx \to 0. \tag{3.27}$$

Indeed, since $I'_{\lambda,\mu}(u_n)(u - u_n) \to 0$, we apply the convexity of the functional $J(u) \triangleq \frac{\|u\|^2_{E_\lambda}}{2}$ to get

$$\begin{aligned}\frac{1}{2}\|u\|^2_{E_\lambda} &= J(u) \ge J(u_n) + J'(u_n)(u - u_n) \\ &= \frac{1}{2}\|u_n\|^2_{E_\lambda} + \int\limits_{\mathbb{R}^2}[\nabla u_n\nabla(u - u_n) + \lambda V(x)u_n(u - u_n)]dx \\ &= \frac{1}{2}\|u_n\|^2_{E_\lambda} + I'_{\lambda,\mu}(u_n)(u - u_n) - \mu\int\limits_{\mathbb{R}^2} f(u_n)(u_n - u)dx \\ &\quad - \int\limits_{\mathbb{R}^2}\left(A_1^2[u_n] + A_2^2[u_n] + A_0[u_n]\right)u_n(u - u_n)dx\end{aligned}$$

which gives that $\limsup_{n\to\infty}\|u_n\|^2_{E_\lambda} \le \|u\|^2_{E_\lambda}$, where we have applied the Step 1 to (2.7). Hence, it reaches $u_n \to u$ in E_λ by Fatou's lemma. The remainder is to verify the validity of (3.27). Actually, one could gather (2.10) and (3.26) jointly with the Step 1 to have that

$$\int_{\mathbb{R}^2} |f(u_n)(u_n - u)|dx \le \varepsilon|u_n|_2|u_n - u|_2 + C_\varepsilon \int_{\mathbb{R}^2} |u_n - u||u_n|^{q-1}(e^{\alpha u_n^2} - 1)dx$$

$$\le \varepsilon|u_n|_2|u_n - u|_2 + C_\varepsilon|u_n|_{q\nu}^{q-1} \left(\int_{\mathbb{R}^2} |u_n - u|^{q\nu}dx\right)^{\frac{1}{q\nu}} \left(\int_{\mathbb{R}^2} (e^{\alpha\nu' u_n^2} - 1)dx\right)^{\frac{1}{\nu'}} \to 0$$

by letting $n \to \infty$ and then tending $\varepsilon \to 0$. The proof is completed. □

4. Proof of main theorems

4.1. Proof of Theorem 1.1

The proof would be done if u obtained in Lemma 3.4 satisfies $I'_\lambda(u) = 0$ in E_λ^{-1}. Motivated by [38], we argue it indirectly. If $I'_\lambda(u) \ne 0$, there exists a $\varphi \in C_0^\infty(\mathbb{R}^2)$ such that $I'_\lambda(u)(\varphi) < -1$. Let $\varepsilon > 0$ be small enough and satisfy

$$I'_\lambda(u_t + \tau\varphi)(\varphi) \le -\frac{1}{2}, \text{ for } |t-1| + |\tau| \le \varepsilon. \tag{4.1}$$

Let $\chi \in C_0^\infty(\mathbb{R}, [0,1])$ be a cut-off function satisfying $\chi(t) \equiv 1$ for every $|t-1| \le \frac{\varepsilon}{2}$ and $\chi(t) \equiv 0$ for all $|t-1| \ge \varepsilon$. For any $t > 0$, we define

$$\eta(t) \triangleq \begin{cases} u_t, & \text{if } |t-1| \ge \varepsilon, \\ u_t + \varepsilon\chi(t)\varphi, & \text{if } |t-1| < \varepsilon. \end{cases}$$

Obviously, $\eta \in \mathcal{C}(E_\lambda)$ and one can fix $\varepsilon > 0$ sufficiently small such that $\|\eta(t)\|_{E_\lambda} > 0$ for $|t-1| < \varepsilon$. By (4.1), it is easy to show that

$$\max_{t>0} I_\lambda(\eta(t)) < m_\lambda.$$

Proceeding as the proof of Lemma 3.1, we have $G_\lambda(\eta(1-\varepsilon)) > 0$ and $G_\lambda(\eta(1+\varepsilon)) < 0$. Since $G_\lambda(\eta(t))$ is continuous, there exists $t_0 \in (1-\varepsilon, 1+\varepsilon)$ such that $G_\lambda(\eta(t_0)) = 0$ which is $\eta(t_0) \in \mathcal{M}_\lambda$. Therefore, $m_\lambda \le I_\lambda(\eta(t_0)) \le \max_{t>0} I_\lambda(\eta(t)) < m_\lambda$, a contradiction. As to the positivity of u, it is standard and we omit it here. The proof is completed.

Next, we will deal with the concentrating behavior of ground state solutions obtained in Theorem 1.1. For any $u \in H_0^1(\Omega)$, we denote by $\tilde{u} \in H^1(\mathbb{R}^2)$ its trivial extension, namely

$$\tilde{u} \triangleq \begin{cases} u & \text{in } \Omega, \\ 0 & \text{in } \Omega^c = \{x : x \in \mathbb{R}^3 \backslash \Omega\}. \end{cases}$$

For $i \in \{0, 1, 2\}$, we can define so $A_i[\tilde{u}]$ as in (1.7) and (1.8). Observe, for example that

$$\int_{\mathbb{R}^2} \left(A_1^2[\tilde{u}] + A_2^2[\tilde{u}]\right) |\tilde{u}|^2 dx = \int_{\Omega} \left(A_1^2[\tilde{u}] + A_2^2[\tilde{u}]\right) |u|^2 dx.$$

We now define $I_0|_\Omega : H_0^1(\Omega) \to \mathbb{R}$ as

$$I_0|_\Omega(u) = \frac{1}{2}\int\limits_\Omega \left[|\nabla u|^2 + (A_1^2[\tilde{u}] + A_2^2[\tilde{u}])|u|^2\right] dx - \int\limits_\Omega F(u)dx$$

and consider the minimization problem

$$m_0|_\Omega \triangleq \inf_{u\in\mathcal{M}_0|_\Omega} I_0|_\Omega(u)$$

where

$$\mathcal{M}_0|_\Omega = \{u \in H_0^1(\Omega)\backslash\{0\} : G_0|_\Omega(u) = 0\}$$

denotes the corresponding manifold and $G_0|_\Omega : H_0^1(\Omega) \to \mathbb{R}$ is given by

$$G_0|_\Omega(u) = \theta\int\limits_\Omega |\nabla u|^2 dx + (3\theta - 2)\int\limits_{\mathbb{R}^2} \left(A_1^2[\tilde{u}] + A_2^2[\tilde{u}]\right) u^2 dx - \int\limits_\Omega [\theta f(u)u - 2F(u)]dx$$

We note that, up to the above trivial extension, there holds that $\mathcal{M}_0|_\Omega \subset \mathcal{M}_\lambda$ for all $\lambda > 0$.

For each $\lambda > \Lambda_0$, we denote by $u_\lambda \in E_\lambda$ a ground state solution of system (1.1), that is, $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = m_\lambda$. Then, we prove Theorem 1.2 as follows.

4.2. Proof of Theorem 1.2

Let $\lambda_n \to +\infty$ as $n \to +\infty$ and $(u_{\lambda_n}) \subset E_{\lambda_n}$ be a sequence of ground state solutions of system (1.1), that is, $I'_{\lambda_n}(u_{\lambda_n}) = 0$ and $I_{\lambda_n}(u_{\lambda_n}) = m_{\lambda_n}$. Up to a subsequence if necessary, by (3.5) and $\mathcal{M}_0|_\Omega \subset \mathcal{M}_\lambda$, for all $\lambda > 0$,

$$0 < \rho \le \lim_{n\to\infty} I_{\lambda_n}(u_{\lambda_n}) \triangleq \tilde{m}_\Omega \le m_0|_\Omega < +\infty. \tag{4.2}$$

Clearly, (u_{λ_n}) is bounded in $H^1(\mathbb{R}^2)$. Thereby, up to a subsequence if necessary, there is a $u_0 \in H^s1(\mathbb{R}^2)$ such that $u_{\lambda_n} \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$ and $u_{\lambda_n} \to u_0$ a.e. in $\mathbb{R}^2$. By means of (2.8) and (2.15), we conclude that $I_0|'_\Omega(u_0) = 0$. We claim that $u_0 \equiv 0$ in Ω^c. Otherwise, there is a compact subset $\Theta_{u_0} \subset \Omega^c$ with $\mathrm{dist}(\Theta_{u_0}, \partial\Omega^c) > 0$ such that $u_0 \neq 0$ on Θ_{u_0} and by Fatou's lemma

$$\liminf_{n\to\infty} \int\limits_{\mathbb{R}^2} u_n^2 dx \ge \int\limits_{\Theta_{u_0}} {u_0}^2 dx > 0. \tag{4.3}$$

Moreover, there exists $\varepsilon_0 > 0$ such that $V(x) \ge \varepsilon_0$ for any $x \in \Theta_{u_0}$ by the assumptions (V_1) and (V_2). According to (4.2)-(4.3), we depend on a similar calculation in (3.7) to reach

$$c_\Omega \ge \frac{2\theta - 1}{2(6\theta - 4)} \liminf_{n\to\infty} \int\limits_{\mathbb{R}^2} \lambda_n V(x)u^2 dx \ge \frac{(2\theta - 1)\varepsilon_0}{2(6\theta - 4)} \left(\int\limits_{\Theta_u} {u_0}^2 dx\right) \liminf_{n\to\infty} \lambda_n = +\infty,$$

a contradiction. Therefore, $u_0 \in H_0^1(\Omega)$ by the fact that $\partial\Omega$ is smooth and $I_0|'_\Omega(u_0) = 0$. Similar to the proof of Lemma 3.4, one knows that $u_0 \neq 0$. Proceeding as the proof of Lemma 2.5, it holds that $G_0|_\Omega(u_0) = G_{\lambda_n}(\tilde{u}_0) = 0$. In view of (4.2), by $u_0 \in H_0^1(\Omega)$, we use the Fatou's lemma to obtain

$$m_0|_\Omega \geq \tilde{m}_\Omega = \liminf_{n\to\infty}\left[I_{\lambda_n}(u_{\lambda_n}) - \frac{1}{6\theta-4}G_{\lambda_n}(u_{\lambda_n})\right]$$
$$\geq I_0|_\Omega(u_0) - \frac{1}{6\theta-4}G_0|_\Omega(u_0) = I_0|_\Omega(u_0) \geq m_0|_\Omega$$

yielding that $u_{\lambda_n} \to u_0$ in $H^1(\mathbb{R}^2)$ and $I_0|_\Omega(u_0) = m_0|_\Omega$. The proof is finished.

4.3. Proof of Theorem 1.5

Let us recall Proposition 3.5, Lemma 3.6 and Lemma 3.9, there exist two sequences $(\mu_n) \subset [\delta, 1]$ and $(u_n) \subset E_\lambda \backslash \{0\}$ such that

$$I'_{\lambda,\mu_n}(u_n) = 0,\ I_{\lambda,\mu_n}(u_n) = c_{\lambda,\mu_n} \text{ and } \mu_n \to 1^-. \tag{4.4}$$

Since $I'_{\lambda,\mu_n}(u_n) = 0$, we are derived from a similar argument in Lemma 2.5 that $P_{\lambda,\mu_n}(u_n) \equiv 0$, where

$$P_{\lambda,\mu_n}(v) \triangleq \frac{1}{2}\int_{\mathbb{R}^2} \lambda\left[2V(x) + (\nabla V, x)\right] v^2 dx + 2\int_{\mathbb{R}^2} \left(A_1^2[v] + A_2^2[v]\right) v^2 dx - 2\mu_n \int_{\mathbb{R}^2} F(v)dx,\ \forall v \in E_\lambda.$$

We claim that (u_n) is uniformly bounded in E_λ for every $\lambda > \Lambda_0$. Actually, some elementary calculations provide us that

$$\begin{aligned} c_{\lambda,\mu_n} &= I_{\lambda,\mu_n}(u_n) - \frac{1}{2}I'_{\lambda,\mu_n}(u_n)(u_n) + \frac{\gamma-2}{4}P_{\lambda,\mu_n}(u_n) \\ &= \frac{\gamma-2}{8}\int_{\mathbb{R}^2} \lambda\left[2V(x) + (\nabla V, x)\right] u_n^2 dx + \frac{\gamma-4}{2}\int_{\mathbb{R}^2} \left(A_1^2[u_n] + A_2^2[u_n]\right) u_n^2 dx \\ &\quad + \frac{\mu_n}{2}\int_{\mathbb{R}^2} \left[f(u_n)u_n - \gamma F(u_n)\right] dx \end{aligned}$$

jointly with (3.18) and (f_4) implies that the two quantities

$$\left(\int_{\mathbb{R}^2} \lambda\left[2V(x) + (\nabla V, x)\right] u_n^2 dx\right) \text{ and } \left(\int_{\mathbb{R}^2} \left(A_1^2[u_n] + A_2^2[u_n]\right) u_n^2 dx\right)$$

are uniformly bounded in $n \in \mathbb{N}$. From which, we are derived from $P_{\lambda,\mu_n}(u_n) = 0$ that $\left(\mu_n \int_{\mathbb{R}^2} F(u_n)dx\right)$ is uniformly bounded in $n \in \mathbb{N}$, too. With these discussions, combining $I_{\lambda,\mu_n}(u_n) = c_{\lambda,\mu_n}$ and (3.18), we derive that (u_n) is uniformly bounded in E_λ.

Then, we claim that (u_n) is a $(PS)_{c_{\lambda,1}}$ sequence of the functional $I_\lambda = I_{\lambda,1}$. Actually, taking into account $\mu_n \to 1^-$ and Proposition 3.5-(c),

$$\lim_{n\to\infty} I_{\lambda,1}(u_n) = \left(\lim_{n\to\infty} I_{\lambda,\mu_n}(u_n) + (\mu_n - 1)\int_{\mathbb{R}^2} F(u_n)dx\right) = \lim_{n\to\infty} c_{\lambda,\mu_n} = c_{\lambda,1},$$

where we have used the fact that $(F(u_n))$ is uniformly bounded in $L^1(\mathbb{R}^2)$. Similarly, we deduce from $I'_{\lambda,\mu_n}(u_n)(\psi) = 0$ that $(f(u_n)\psi)$ is uniformly bounded in $L^1(\mathbb{R}^2)$ and so

$$\lim_{n\to\infty}\frac{|I'_{\lambda,1}(u_n)(\psi)|}{\|\psi\|_{E_\lambda}}=\lim_{n\to\infty}\frac{\left|I'_{\lambda,\mu_n}(u_n)(\psi)+(\mu_n-1)\int_{\mathbb{R}^2}f(u_n)\psi dx\right|}{\|\psi\|_{E_\lambda}}$$
$$\leq\lim_{n\to\infty}\frac{|\mu_n-1|\left|\int_{\mathbb{R}^2}f(u_n)\psi dx\right|}{\|\psi\|_{E_\lambda}}=0,\ \forall\psi\in E_\lambda.$$

As a consequence, one has that (u_n) is a $(PS)_{c_{\lambda,1}}$ sequence of the functional $I_\lambda=I_{\lambda,1}$.

Finally, combining the above two steps and (3.18), we can apply Lemma 3.9 to finish the proof.

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