CONCENTRATING NORMALIZED SOLUTIONS FOR 2D NONLOCAL SCHRÖDINGER EQUATIONS: CRITICAL EXPONENTIAL CASE

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Abstract. We deal with the existence of solutions to a class of nonlocal Schrödinger problems with different types of potentials

$$\begin{cases} -\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u|^2 dx = a^2, \end{cases}$$

where $a \neq 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier, $\kappa > 0$ is a parameter, $W \in \mathcal{C}(\mathbb{R}^2)$ is the nonnegative external potential, $\mu \in (0, 2)$ and F denotes the primitive function of $f \in \mathcal{C}(\mathbb{R})$ which fulfills the critical exponential growth in the Trudinger-Moser sense at infinity. We prove that the problems admit at least a positive solution which concentrating behavior is also analyzed.

Contents

1. Introduction

In this paper, we aim to get existence of positive solutions to the following nonlocal Schrödinger equation with different types of potentials

$$-\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \quad \text{in } \mathbb{R}^2, \tag{1.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{1.2}$$

where $a \neq 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier, $\kappa > 0$ is a parameter, $W \in \mathcal{C}(\mathbb{R}^2)$ is the nonnegative external potential, $\mu \in (0, 2)$ and F denotes the primitive function of $f \in \mathcal{C}(\mathbb{R})$ which fulfills the critical exponential growth in the Trudinger-Moser sense at infinity.

Inspired by the well-known Trudinger-Moser type inequality, we recall that a function f has the critical exponential growth at infinity if there exists a constant $\alpha_0 > 0$ such that

$$\lim_{|s|\to+\infty} \frac{|f(s)|}{e^{\alpha s^2}} = \begin{cases} 0, & \forall \alpha > \alpha_0, \\ +\infty, & \forall \alpha < \alpha_0. \end{cases} \tag{1.3}$$

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This definition was introduced by Adimurthi and Yadava [2], see also de Figueiredo, Miyagaki and Ruf [30] for example.

Hereafter, we shall suppose that the nonlinearity f satisfies (1.3) and the assumptions below

(f_1) $f \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ and $f(s) \equiv 0$ for all $s \in (-\infty, 0]$;

(f_2) There is a $q \in \left(2, \frac{6-\mu}{2}\right)$ such that $f(s)/s^{q-1}$ is an increasing function of s on $(0, +\infty)$.

(f_3) There is a $c_0 > 0$ such that $f(s) \geq c_0 s^{q-1}$ for all $s \in [0, +\infty)$.

We would like to highlight here that a great many functions f satisfy the above assumptions, with $\alpha_0 = 4\pi$ and $c_0 = 1$, for example,

$$f(s) = \begin{cases} 0, & s \leq 0, \\ s^{q-1} e^{4\pi s^2}, & s > 0, \end{cases}$$

where $q \in \left(2, \frac{6-\mu}{2}\right)$. The similar assumptions for a nonlinearity f satisfying (1.3) and $(f_1) - (f_3)$ with the above example can be found in [12, 58].

Over the past few decades, a lot of attentions have been paid to the standing wave solutions to the time-dependent nonlinear Choquard equation of the type

$$i\frac{\partial \psi}{\partial t} = \Delta\psi - W(x)\psi + [|x|^{-\mu} * F(\psi)]f(\psi) \text{ in } \mathbb{R}^+ \times \mathbb{R}^N, \tag{1.4}$$

where $\psi : \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ acts as the time-dependent wave function, $W : \mathbb{R}^N \to \mathbb{R}$ stands for the real external potential and nonlinear term $f(\psi)$ describes the interaction effect among particles. Inserting the standing wave ansatz $\psi(x,t) = \exp(-i\omega t)u(x)$ with $\omega \in \mathbb{R}$ and $x \in \mathbb{R}^N$ into (1.4), then $u : \mathbb{R}^N \to \mathbb{R}$ satisfies the Choquard equation

$$-\Delta u + \bar{W}(x)u = [|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^N, \tag{1.5}$$

where and in the sequel $\bar{W}(x) = W(x) + \omega$ for all $x \in \mathbb{R}^N$.

As we all know, there exist two directions in the studies of standing waves of the Choquard equation (1.4). On the one hand, one can choose the frequency $\omega \in \mathbb{R}$ to be fixed and investigate the existence of nontrivial solutions for Eq. (1.5) obtained as the critical points of the variational functional $I : H^1(\mathbb{R}^N) \to \mathbb{R}$ given by

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} \left[|\nabla u|^2 + (W(x)+\omega)u^2\right] dx - \frac{1}{2}\int_{\mathbb{R}^N} [|x|^{-\mu} * F(u)]F(u)dx.$$

Actually, Eq. (1.5) is closely related to the Choquard equation arising from the studies of Bose-Einstein condensation and can be exploited to describe the finite-range many-body interactions between particles since $|x|^{-\mu}$ can be reviewed as the classic Riesz potential. Letting $N \geq 3$ and $f(s) = |s|^{p-2}s$ for all $s \in \mathbb{R}$, Eq. (1.5) is simply of the form

$$-\Delta u + u = (|x|^{-\mu} * |u|^p)|u|^{p-2}u, \ x \in \mathbb{R}^N. \tag{1.6}$$

To describe a polaron at rest in the quantum field theory, Pekar [51] introduced the Choquard-Pekar equation which is $N = 3$, $\mu = 1$ and $p = 2$ in Eq. (1.6). Choquard adopted this equation to characterization an electron trapped in its own hole as an approximation to the Hartree-Fock theory for the one component plasma [41]. Subsequently, Lieb [40] and Lions [43] contemplated the existence and uniqueness of positive solutions to (1.6) by variational methods. The authors in [45, 47] concluded the regularity, positivity and radial symmetry of the ground state solutions and investigated the decay properties at infinity. It should be pointed out that (1.6) was also proposed by Moroz *et al.* in [46] as a model for self-gravitating particles in the context as it can be regarded as the classic Schrödinger-Newton equation, see e.g. [52, 62]. Actually, Eq. (1.6) and its variants have received more and more attentions by many mathematicians because of the appearance of the convolution type nonlinearities in these years. Let us refer the reader to [1, 4, 7, 8, 10, 11, 14, 38, 47, 48, 55] and the references therein, particularly by [49], for some very abundant and meaningful review of the Choquard equations.

On the other hand, one can contemplate the case $\omega \in \mathbb{R}$ to be unknown. In such a situation, $\omega \in \mathbb{R}$ is supposed to act as a Lagrange multiplier and the L^2-norm of obtained solutions would be prescribed since there is a conservation of mass which is said that the wave function $\psi(x,t)$ with its corresponding Cauchy initial function $\psi(0,x)$ which preserves L^2-mass in the following sense

$$\int_{\mathbb{R}^N} |\psi(t,x)|^2 dx = \int_{\mathbb{R}^N} |\psi(0,x)|^2 dx, \ \forall t \in (0,\infty).$$

From the physical point of view, this spirit of research holds particular significance as it accounts for the conservation of mass. Moreover, it provides valuable insights into the dynamic properties of the standing waves of (1.5), for instance stability or instability in [24, 28]. In this article, we shall focus primarily on this direction.

In [34], Jenajean made full use of a minimax approach and compactness argument to conclude the existence of solutions for the following Schrödinger problem

$$\begin{cases} -\Delta u + \omega u = g(u) \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2. \end{cases} \tag{1.7}$$

Subsequently, there exist some further complements and generalizations in [36]. In [59], letting $g(t) = \tau|t|^{q-2}t + |t|^{p-2}t$ with $2 < q \le 2 + \frac{4}{N} \le p < 2^*$, Soave obtained the existence of solutions for problem (1.7), where $2^* = \frac{2N}{N-2}$ if $N \ge 3$ and $2^* = \infty$ if $N = 2$. For this type of combined nonlinearities, Soave also [60] proved the existence of ground state and excited solutions when $p = 2^*$. For more interesting results for problem (1.7), we will refer the reader to [6, 21, 35, 37, 63] and the references therein.

In spirit of [34], when $\bar{W}(x) \equiv 0$ for all $x \in \mathbb{R}^N$ in Eq. (1.5), the authors in [39] deduced the existence of nontrivial solutions solutions to the following nonlocal problem of Choquard type

$$\begin{cases} -\Delta u + \omega u = [|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^N, \\ \displaystyle\int_{\mathbb{R}^N} |u|^2 dx = a^2, \end{cases} \tag{1.8}$$

provided $a > 0$ is sufficiently small, where f possesses the Sobolev subcritical growth at infinity. Afterwards, Bartsch *et al.* [17] investigated the existence of solutions for problem (1.8) which is simpler and more transparent than that of [39]. As to the case that the nonlinearity f admits the critical growth, Ye, Shen and Yang [65] dealt with the existence of normalized ground state solutions for the Hartree problem with a perturbation. There are some other interesting results with respect to problem (1.8), see [14, 17, 29, 39] for example.

Alternatively, the reader may observe that the spatial dimension of problem (1.1) is two, the case therefore is very special because $2^* = \infty$ in this situation. Explaining it more specifically, the fact $H^1(\mathbb{R}^2) \not\hookrightarrow L^\infty(\mathbb{R}^2)$ shall make the problems special and quite delicate. Thus, it is not so direct to dispose of the nonlinearity involving a critical exponential growth trivially. Letting $W(x) \equiv 0$ for all $x \in \mathbb{R}^2$ in Eq. (1.1), Deng and Yu [29] supposed that the nonlinearity satisfies (1.3) and the following assumptions

- (F_1) $f : \mathbb{R} \to \mathbb{R}$ is continuous;
- (F_2) $f(t) = o(|t|^\tau)$ as $|t| \to 0$ for some $\tau > 3$;
- (F_3) there exists a positive constant $\theta > \frac{6-\mu}{2}$ such that $0 < \theta F(t) \le f(t)t$ for $t \ne 0$;
- (F_4) there exist constants $\sigma > \frac{6-\mu}{2}$ and $\xi > 0$ such that $F(t) \ge \xi\, |t|^\sigma$ for all $t \in \mathbb{R}$.

Then, for some sufficiently small mass $a > 0$ and and $\sigma > 0$ large enough, the authors exploited the arguments in [34] to investigate the existence of normalized solution. Moreover, the ground state solution was also considered when f is in addition supposed to have some monotone type assumptions. Afterwards, Alves and Shen [14] handled the existence of nontrivial solutions to the problem below

$$\begin{cases} -\Delta u + \omega u = [|x|^{-\mu} * (|x|^{-\beta} F(u))]|x|^{-\beta} f(u) \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u|^2 dx = a^2, \end{cases} \tag{1.9}$$

where $\beta > 0$, $0 < \mu < 2$ with $0 < 2\beta + \mu < 2$ and f admits the supercritical exponential growth (see [9, 11]). Whereas, it is worth pointing out here that either the assumption (F_3) or

(F_4') $\liminf\limits_{s\to+\infty} \frac{F(s)}{e^{\alpha_0 s^2}} > 0$, where $\alpha_0 > 0$ comes from (1.3),

plays a pivotal role in [14]. Actually, either the assumption (F_4) or (F_4') is mainly exploited to restore the compactness caused by the critical exponential growth and the whole space $\mathbb{R}^2$. As a conseuqnece, these assumptions seem indispensable to some extent in the mentioned works.

Motivated by all of the quoted papers above, particularly by [12, 13, 57, 58], we are going to consider the existence of normalized solutions to a class of nonlocal Schrödinger equations with different potentials and critical exponential growth. Speaking it clearly, on the one hand, let's suppose that $W(x) := V(\varepsilon x)$ for all $\varepsilon > 0$ and $x \in \mathbb{R}^2$ in Eq. (1.1) with the assumption below

(V) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ and $0 < V_0 \triangleq \inf\limits_{x\in\mathbb{R}^2} V(x) < V_\infty \triangleq \liminf\limits_{|x|\to\infty} V(x) < +\infty$, where $V(0) = V_0$.

Now, we can state the first main result in this paper as follows.

Theorem 1.1. *Suppose (V), (1.3), $(f_1) - (f_3)$ and $\mu \in (0, 2)$, then there exist some constants $\kappa^* > 0$, $a^* > 0$ and $\varepsilon^* > 0$ such that, for every $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$, the problem*

$$\begin{cases} -\Delta u + V(\varepsilon x) u = \sigma u + \kappa[|x|^{-\mu} * F(u)] f(u) \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{cases} \tag{1.10}$$

has a couple of weak solution $(\bar{u}, \bar{\sigma}) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ such that $\bar{u}(x) > 0$ for all $x \in \mathbb{R}^2$ and $\bar{\sigma} < 0$. Moreover, if z_ε denotes the global maximum of $\bar{u}$, then, up to a subsequence if necessary,

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

On the other hand, we shall suppose that $W(x) = \lambda V(x)$ for all $\lambda > 0$ and $x \in \mathbb{R}^2$ and the function $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies the following conditions

- (V_1) $V \in \mathcal{C}(\mathbb{R}^2, \mathbb{R})$ with $V(x) \geq 0$ on $\mathbb{R}^2$;
- (V_2) $\Omega \triangleq \mathrm{int} V^{-1}(0)$ is nonempty and bounded with smooth boundary, and $\overline{\Omega} = V^{-1}(0)$;
- (V_3) there exists a $b > 0$ such that the set $\Xi \triangleq \{x \in \mathbb{R}^2 : V(x) < b\}$ is nonempty and admits finite measure.

The main result in this direction is the following.

Theorem 1.2. *Suppose $(V_1) - (V_3)$, (1.3), $(f_1) - (f_3)$ and $\mu \in (0, 2)$, then there are $\kappa_* > 0$, $a_* > 0$ and $\lambda_* > 1$ such that, for all $\kappa \in (0, \kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$, the problem*

$$\begin{cases} -\Delta u + \lambda V(x) u = \sigma u + \kappa[|x|^{-\mu} * F(u)] f(u) \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{cases} \tag{1.11}$$

has a couple of weak solution $(\underline{u}, \underline{\sigma}) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ such that $\underline{u}(x) > 0$ for all $x \in \mathbb{R}^2$ and $\underline{\sigma} < 0$. If we denote $(\underline{u}_\lambda, \underline{\sigma}_\lambda)$ by the couple of weak solutions established above for all $\lambda > \lambda_*$, then for all *fixed* $a > a_*$*, passing to a subsequence if necessary,* $\underline{u}_\lambda \to \underline{u}_0$ *in* $H^1(\mathbb{R}^2)$ *and* $\underline{\sigma}_\lambda \to \underline{\sigma}_0$ *in* $\mathbb{R}$ *as* $\lambda \to +\infty$*, where* $\underline{\sigma}_0 < 0$ *and* $(\underline{u}_0, \underline{\sigma}_0)$ *is a couple of weak solution to the problem below*

$$\begin{cases} -\Delta u = \sigma u + \kappa \left(\int_\Omega \frac{F(u(y))}{|x-y|^\mu} dy \right) f(u), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega, \\ \int_\Omega |u|^2 dx = a^2. \end{cases} \tag{1.12}$$

Remark 1. We mention here that the above two types of potentials V appearing in Theorems 1.1 and 1.2 have been contemplated by many mathematicians over the past several decades, see e.g. [16, 23, 31–33, 54] and [18, 20, 22, 27, 44], respectively. As a matter of fact, the former one is known as the Rabinowitz's potential, while the latter one is called by the steep potential well.

Remark 2. Concerning the existence of normalized solutions to some classes of local equations with Rabinowitz's potential, we prefer to refer the reader to [3, 5, 13, 15, 57]. Moreover, the reader may find the latest paper [58] focuses on the normalized solutions to Schrödinger-Newton system with steep potential well. Whereas, it seems the first time to consider the existence of normalized solutions to Choquard equations with the above two types of potentials in a unified way.

Remark 3. It should be pointed out that we could not conclude the proofs of Theorems 1.1 and 1.2 simply by repeating the approaches adopted in the previous papers mentioned in Remark 2. On the one hand, we successfully generalize the local case in [3, 5, 13, 15, 57] to the nonlocal one and so there are some additional difficulties. On the other hand, thanks to the special structure of the work space in [58], the key compact imbedding holds true in advance and it mainly deals with the boundedness of minimizing sequence, while we easily get the boundedness and there are some subtle efforts to recover the compactness in the proof of Theorem 1.2. As a consequence, we tend to believe that this article may prompt some further studies on normalized solutions to a class of nonlocal Schrödinger equations.

To conclude this section, we simply sketch the main ideas to arrive at the proofs of Theorems 1.1 and 1.2. Owing to the arguments adopted in [12, 13, 57, 58], for each fixed constant $R > 0$, we introduce the following continuous function $f_R : \mathbb{R} \to \mathbb{R}$ defined by

$$f_R(s) = \begin{cases} 0, & \text{if } s \le 0, \\ f(s), & \text{if } 0 \le s \le R, \\ \frac{f(R)}{R^{q-1}} s^{q-1}, & \text{if } R \le s < +\infty, \end{cases} \tag{1.13}$$

where the constant $q \in \left(2, \frac{6-\mu}{2}\right)$ comes from (f_2). From now on until the end of the paper, we define $F_R(s) = \int_0^s f_R(t)dt$ for each $s \in \mathbb{R}$ to be the primitive function of f_R. It follows from a direct computation with (f_2) that

$$qF_R(s) \le f_R(s)s, \ \forall s \ge 0. \tag{1.14}$$

Moreover, we can exploit the monotone assumption in (f_2) again to see that

$$f_R(s) \le \frac{f(R)}{R^{q-1}} s^{q-1}, \ \forall s \ge 0. \tag{1.15}$$

With such a nonlinearity f_R defied in (1.13), we turn to contemplate the following auxiliary problem

$$\begin{cases} -\Delta u + W(x)u = \sigma u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2, \\ \int_{\mathbb{R}^2} |u|^2 dx = a^2. \end{cases} \tag{1.16}$$

By (1.15), we can see that Problem (1.16) above involves L^2-subcritical growth since $q < \frac{6-\mu}{2}$. So, the solvability of Problem (1.16) becomes available. At this stage, we invite the reader to observe that if the couple (u_R, σ_R) is a solution of Problem (1.16), then it is indeed the solution to the original Problems (1.1)-(1.2) as long as $|u_R|_\infty \le R$ due to the definition of f_R in (1.13). Have this in mind, we shall derive the proofs of Theorems 1.1 and 1.2 combining the solvability of Problem (1.16) and the L^∞-estimate.

This article is organized as follows.

In Section 1.2, we will introduce some preliminary results handling the convolution parts.

Sections 2 and 3 are devoted to the existence results for the auxiliary Problem (1.16) with two different types of potentials.

Finally, the detailed proofs of Theorems 1.1 and 1.2 shall be exhibited in Section 4.

1.1. **Notations.** From now on in this paper, otherwise mentioned, we use the following notations:

- $B_r(x) \subset \mathbb{R}^2$ is an open ball centered at $x \in \mathbb{R}^2$ with radius $r > 0$ and $B_r = B_r(0)$.
- $C, C_1, C_2, \cdots$ denote any positive constant, whose value is not relevant.
- For all $x \in \mathbb{R}^2$, we define
$$u^+(x) \triangleq \max\{u(x), 0\} \ge 0 \text{ and } u^-(x) \triangleq \min\{u(x), 0\} \le 0.$$
- $|\cdot|_p$ denotes the usual norm of the Lebesgue space $L^p(\mathbb{R}^2)$, for every $p \in [1, +\infty]$. $\|\cdot\|_{H^i}$ denotes the usual norm of the Hilbert space for $i \in \{1, 2\}$.
- $o_n(1)$ denotes a real sequence with $o_n(1) \to 0$ as $n \to +\infty$.
- " $\to$ " and " $\rightharpoonup$ " stand for the strong and weak convergence in the related function spaces, respectively.
- We recall the celebrated Gagliardo-Nirenberg inequality, given an $l \in [2, +\infty)$,
$$|u|_l^l \le \mathbb{C}|u|_2^{(1-\gamma_l)l}|\nabla u|_2^{\gamma_l l} \text{ in } H^1(\mathbb{R}^2),\ \gamma_l = 2\left(\frac{1}{2} - \frac{1}{l}\right), \tag{1.17}$$
where the constant $\mathbb{C} > 0$ is just dependent of l.

1.2. **Two basic facts.** In this section, we are going to exhibit some preliminary results adopted to prove the main results. From now on until the end of the present article, we shall always suppose that $0 < \mu < 2$ just for simplicity. Let us first introduce the well-known Hardy-Littlewood-Sobolev inequality.

Lemma 1.3. *(Hardy-Littlewood-Sobolev inequality [42, Theorem 4.3]). Suppose that $s, r > 1$ and $0 < \mu < N$ with $\frac{1}{s} + \frac{\mu}{N} + \frac{1}{r} = 2$, $\varphi \in L^s(\mathbb{R}^N)$ and $\psi \in L^r(\mathbb{R}^N)$. Then, there exists a sharp constant $C = C(s, N, \mu, r) > 0$, independent of φ and ψ, such that*
$$\int_{\mathbb{R}^N} [|x|^{-\mu} * \varphi(x)]\psi(x)dx \le C\|\varphi\|_s\|\psi\|_r. \tag{1.18}$$

Since it mainly concerns in the whole space $\mathbb{R}^2$ in this paper, we will assume that $N = 2$ in (1.18) throughout this paper.

Let us conclude this section by introducing the celebrated Brézis-Lieb lemma for the nonlocal term of Choquard type.

Lemma 1.4. *([47, Lemma 2.4]). Let $p \in [\frac{4-\mu}{2}, +\infty)$ and $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $L^{\frac{4p}{4-\mu}}(\mathbb{R}^2)$. If $u_n \to u$ almost everywhere on $\mathbb{R}^2$ as $n \to \infty$, then*
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left[\left(|x|^{-\mu} * |u_n|^p\right)|u_n|^p - \left(|x|^{-\mu} * |u_n - u|^p\right)|u_n - u|^p\right] dx = \int_{\mathbb{R}^2} \left(|x|^{-\mu} * |u|^p\right)|u|^p dx. \tag{1.19}$$
Moreover, for all $\varphi \in C_0^\infty(\mathbb{R}^2)$, it holds that
$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left(|x|^{-\mu} * |u_n|^p\right)|u_n|^{p-2}u_n\varphi dx = \int_{\mathbb{R}^2} \left(|x|^{-\mu} * |u|^p\right)|u|^{p-2}\varphi dx. \tag{1.20}$$

2. Truncated problem: Rabinowitz's type potential

In this section, we are going to dispose of the existence of positive solutions for the following nonlocal Schrödinger equation

$$-\Delta u + V(\varepsilon x)u = \sigma u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2, \tag{2.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{2.2}$$

where the potential $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies (V), $\varepsilon, \kappa > 0$ are parameters, $a > 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier and the nonlinearity f_R is defined in (1.13).

In general, to solve Problems (2.1)-(2.2), we look for critical points of the following variational functional

$$J_{\varepsilon,R}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V(\varepsilon x)|u|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx \tag{2.3}$$

restricted to the sphere $S(a)$ defined by

$$S(a) = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} |u|^2 dx = a^2\right\}. \tag{2.4}$$

Taking advantage of (V) and (1.15) together with (1.18), it is simple to verify that the functional $J_{\varepsilon,R}$ is of class $\mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$ and it derivative is given by

$$J'_{\varepsilon,R}(u)v = \int_{\mathbb{R}^2} [\nabla u \nabla v + V(\varepsilon x)uv]\, dx - \kappa\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)v dx,\ \forall u, v \in H^1(\mathbb{R}^2).$$

We note that since V is a positive and bounded function by (V), then the work space $H^1(\mathbb{R}^2)$ with its usual norm $\|\cdot\|_{H^1}$ will be adopted for simplicity in the present section.

The existence result concerning the Problems (2.1)-(2.2) is the following:

Theorem 2.1. *Suppose (V), (1.3), $(f_1) - (f_3)$ and $\mu \in (0, 2)$, then there exists an $R^* > 0$ such that for all $R > R^*$, there exist some $a^* = a^*(R) > 0$ and $\varepsilon^* = \varepsilon^*(R) > 0$ such that, for each fixed $\kappa \in (0, 1)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$, the minimization problem*

$$\Upsilon_{\varepsilon,R}(a) \triangleq \min_{u\in S(a)} J_{\varepsilon,R}(u) \tag{2.5}$$

can be attained by some function in $H^1(\mathbb{R}^2)$. Hence, there is $(u_R, \sigma_R) \in H^1(\mathbb{R}^2) \times \mathbb{R}$ such that it is a couple solution of Problems (2.1)-(2.2), where $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ and $\sigma_R < 0$.

The proof of the above theorem will be divided into several lemmas. Before exhibiting them, we will always suppose that the potential V and the nonlinearity f_R satisfy (V) and (1.3) with $(f_1) - (f_3)$ in this section, respectively.

Lemma 2.2. *For all fixed $R > 0$, the variational functional $J_{\varepsilon,R}$ is coercive and bounded from below on $S(a)$ for each $\kappa \in (0, 1)$, $a > 0$ and $\varepsilon > 0$, where $J_{\varepsilon,R}$ and $S(a)$ are appearing in (2.3) and (2.4), respectively.*

Proof. By (1.14)-(1.15) and (1.18), for all $u \in S(a)$, we use (1.17) with $l = \frac{4q}{4-\mu} > 2$ to reach

$$J_{\varepsilon,R}(u) \geq \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 dx - \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\left(\int_{\mathbb{R}^2} |\nabla u|^2 dx\right)^{q-\frac{4-\mu}{2}}.$$

As $q \in \left(2, \frac{6-\mu}{2}\right)$, clearly $q - \frac{4-\mu}{2} < 1$, then the statement concludes. □

As a direct consequence of Lemma 2.2, for every fixed $R > 0$, $\kappa \in (0,1)$, $a > 0$ and $\varepsilon > 0$, the real number $\Upsilon_{\varepsilon,R}(a)$ in (2.5) is well-defined and it shall be exploited to look for nontrivial solutions for Problems (2.1)-(2.2). Alternatively, we need to conclude that $\Upsilon_{\varepsilon,R}(a)$ is uniformly bounded above with respect to $\kappa \in (0,1)$ and $\varepsilon > 0$ and so there is the result below.

Lemma 2.3. *There exists an $R^* > 0$ such that for all fixed $R > R^*$, there is an $a^* = a^*(R) > 0$ satisfying for all $a > a^*$, there exists a constant $\Theta_R = \Theta(R) < 0$, independent of ε, such that $\Upsilon_{\varepsilon,R}(a) \le \Theta_R$ for all $\kappa \in (0,1)$ and $\varepsilon > 0$.*

Proof. According to the definition of f_R in (1.13), there holds

$$\frac{f_R(s)}{s^{q-1}} = \begin{cases} \dfrac{f(s)}{s^{q-1}}, & 0 \le s \le R, \\ \dfrac{f(R)}{R^{q-1}}, & R \le s < +\infty. \end{cases} \tag{2.6}$$

Since f satisfies (1.3), we apply (f_2) to see that $\lim\limits_{R\to+\infty} \dfrac{f(R)}{R^{q-1}} = +\infty$ which indicates that there exists an $R^* > 0$ such that, for all $R > R^*$, it holds that $\dfrac{f(R)}{R^{q-1}} \ge c_0$. As a consequence, owing to (2.6) and (f_3), we arrive at

$$f_R(s) \ge c_0 s^{q-1}, \ \forall s \ge 0 \text{ and } R > R^*. \tag{2.7}$$

We now fix a positive function $\psi \in C_0^\infty(\mathbb{R}^2) \cap S(1)$, combining (2.7) and (V), there holds

$$J_{\varepsilon,R}(t\psi) \le \frac{t^2}{2}\int_{\mathbb{R}^2} |\nabla \psi|^2 dx + \frac{|V|_\infty}{2} t^2 - \frac{\kappa c_0^2 t^{2q}}{2q^2}\int_{\mathbb{R}^2} (|x|^{-\mu} * |\psi|^q)|\psi|^q dx \to -\infty$$

as $t \to +\infty$, where we have exploited the fact that $V(\varepsilon x) \le |V|_\infty$ for all $\varepsilon > 0$ and $x \in \mathbb{R}^2$ by (V). Choosing a sufficiently large $t^* = t^*(R) > 0$ and letting $a^* = t^*|\psi|_2$, it permits us to look for a constant $\Theta_R = \Theta(R) < 0$, dependent of R, such that

$$J_{\varepsilon,R}(u) \le \Theta_R, \ \forall R > R^*, \ \kappa \in (0,1), \ a > a^* \text{ and } \varepsilon > 0,$$

provided $u \in S(a)$, as asserted. The proof is completed. □

Similar to [3, 5, 13, 15, 57], we have the following result in the nonlocal case of Choquard type.

Lemma 2.4. *Let $a_2 > a_1 > a^*$. Then, $\dfrac{\Upsilon_{\varepsilon,R}(a_2)}{a_2^2} < \dfrac{\Upsilon_{\varepsilon,R}(a_1)}{a_1^2}$ for all fixed $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$.*

Proof. Let $\xi > 1$ such that $a_2 = \xi a_1$ and $(u_n) \subset S(a_1)$ be a minimizing sequence with respect to the number $\Upsilon_{\varepsilon,R}(a_1)$, that is,

$$J_{\varepsilon,R}(u_n) \to \Upsilon_{\varepsilon,R}(a_1) \text{ as } n \to +\infty.$$

Setting $v_n = \xi u_n$, obviously $v_n \in S(a_2)$. According to (f_2), the function $t \mapsto \frac{F_R(t)}{t^q}$ is increasing on $(0,+\infty)$, we obtain the inequality

$$F_R(ts) \ge t^q F_R(s), \ \forall s > 0 \text{ and } t \ge 1,$$

and so, by using $\Upsilon_{\varepsilon,R}(a_2) \le J_{\varepsilon,R}(v_n) = J_{\varepsilon,R}(\xi u_n)$, we have that

$$\begin{aligned}\Upsilon_{\varepsilon,R}(a_2) &\le \xi^2 J_{\varepsilon,R}(u_n) + \frac{\kappa}{2}\int_{\mathbb{R}^2} \left\{\xi^2[|x|^{-\mu} * F_R(u_n)]F_R(u_n) - [|x|^{-\mu} * F_R(\xi u_n)]F_R(\xi u_n)\right\} dx \\ &\le \xi^2 J_{\varepsilon,R}(u_n) + \frac{\kappa(\xi^2 - \xi^{2q})}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)]F_R(u_n) dx.\end{aligned}$$

To continue the proof, we claim that

Claim 2.5. *There exist a positive constant $C > 0$, independent of $n \in \mathbb{N}$, and a positive integer $n_0 \in \mathbb{N}$ such that* $\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx \geq C$ *for all* $n \geq n_0$.

Otherwise, there exists a subsequence of $(u_n) \subset S(a_1)$, still denoted by itself, such that

$$\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx \to 0 \text{ as } n \to +\infty.$$

Now, we apply Lemma 2.3 to obtain

$$\Theta_R \geq \Upsilon_{\varepsilon,R}(a_1) + o_n(1) = J_{\varepsilon,R}(u_n) \geq -\frac{\kappa}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx,\ n \in \mathbb{N},$$

which is absurd and Claim 2.5 is proved. Thanks to Claim 2.5 and the fact that $\xi^2 - \xi^{2q} < 0$, we therefore reach

$$\Upsilon_{\varepsilon,R}(a_2) \leq \xi^2 J_{\varepsilon,R}(u_n) + \kappa(\xi^2 - \xi^{2q})C,$$

for $n \in \mathbb{N}$ large. Letting $n \to +\infty$, it follows that

$$\Upsilon_{\varepsilon,R}(a_2) \leq \xi^2\Upsilon_{\varepsilon,R}(a_1) + \kappa(\xi^2 - \xi^{2q})C < \xi^2\Upsilon_{\varepsilon,R}(a_1),$$

that is,

$$\frac{\Upsilon_{\varepsilon,R}(a_2)}{a_2^2} < \frac{\Upsilon_{\varepsilon,R}(a_1)}{a_1^2},$$

proving the lemma. □

Lemma 2.6. *Let $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ be fixed, assume $(u_n) \subset H^1(\mathbb{R}^2)$ is a minimizing sequence associated with $\Upsilon_{\varepsilon,R}(a)$ for $a > a^*$. Then, there exist bounded sequence $(\sigma_n) \subset \mathbb{R}$ and $\sigma_R < 0$ such that for some subsequence, still denoted by itself, one has $\lim_{n\to+\infty} \sigma_n = \sigma_R$ and*

$$\|\Upsilon'_{\varepsilon,R}(u_n) - \sigma_n\Psi'(u_n)\|_{(H^1(\mathbb{R}^2))^{-1}} \to 0 \text{ as } n \to +\infty,$$

where $\Psi : H^1(\mathbb{R}^2) \to \mathbb{R}$ is given by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2dx.$$

Proof. Setting the functional $\Psi : H^1(\mathbb{R}^2) \to \mathbb{R}$ given by

$$\Psi(u) = \frac{1}{2}\int_{\mathbb{R}^2}|u|^2dx,$$

we see that $S(a) = \Psi^{-1}(\{a^2/2\})$. Then, by Willem [64, Proposition 5.12], there exists $(\sigma_n) \subset \mathbb{R}$ such that

$$\|\Upsilon'_{\varepsilon,R}(u_n) - \sigma_n\Psi'(u_n)\|_{(H^1(\mathbb{R}^2))^{-1}} \to 0 \text{ as } n \to +\infty, \tag{2.8}$$

Since (u_n) is bounded in $H^1(\mathbb{R}^2)$, it concludes that (σ_n) is also a bounded sequence, then we can assume that $\sigma_n \to \sigma_R$ as $n \to +\infty$ along a subsequence. This together with (2.8) leads to

$$\Upsilon'_{\varepsilon,R}(u_n) - \sigma_R\Psi'(u_n) = o_n(1) \text{ in } (H^1(\mathbb{R}^2))^{-1}.$$

Now, we are going to prove that $\sigma_R < 0$. First of all, let us recall that

$$\int_{\mathbb{R}^2}\left[|\nabla u_n|^2 + V(\varepsilon_n x)|u_n|^2\right]dx - \kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)]f_R(u_n)u_ndx = \sigma_R a^2 + o_n(1).$$

Since $J_{\varepsilon,R}(u_n) = \Upsilon_{\varepsilon,R}(a) + o_n(1)$, one gets

$$2\Upsilon_{\varepsilon,R}(a) + \kappa\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(u_n)][F_R(u_n) - f_R(u_n)u_n]dx = \sigma_R a^2 + o_n(1).$$

By (f_2), it holds that

$$2\Upsilon_{\varepsilon,R}(a) + \left(\frac{1}{q} - 1\right) \kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] f_R(u_n) u_n dx \geq \sigma_R a^2 + o_n(1).$$

As $f(s)s \geq 0$ for all $s \in \mathbb{R}$, $q > 2$, we get

$$2\Upsilon_{\varepsilon,R}(a) \geq \sigma_R a^2.$$

Now, according to $\Upsilon_{\varepsilon,R}(a) \leq \Theta_R < 0$ for every $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon > 0$ by Lemma 2.3, it follows that $\sigma_R < 0$. The proof is completed. □

Our next result is a compactness theorem on $S(a)$ and then it is possible to find a minimizer for $\Upsilon_{\varepsilon,R}(a)$.

Theorem 2.7. *Let $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ be fixed as above. Suppose that $(u_n) \subset S(a)$ is a minimizing sequence of $\Upsilon_{\varepsilon,R}(a)$ for each fixed $a > a^*$, then $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ as $n \to \infty$. If $u \neq 0$, then $u_n \to u$ in $H^1(\mathbb{R}^2)$ along a subsequence as $n \to \infty$.*

Proof. Since $J_{\varepsilon,R}$ is coercive on $S(a)$, the sequence (u_n) is bounded, and so, $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^2)$ for some subsequence. If $u \neq 0$ and $|u|_2 = \hat{a} \neq a$, we must have $\hat{a} \in (0,a)$. By the Brézis-Lieb Lemma (see e.g. [64]),

$$|u_n|_2^2 = |u_n - u|_2^2 + |u|_2^2 + o_n(1).$$

Furthermore, arguing as (1.19) to see that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \{[|x|^{-\mu} * F_R(u_n)] F_R(u_n) - [|x|^{-\mu} * F_R(u_n - u)] F_R(u_n - u)\} dx = \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)] F_R(u) dx.$$

Setting $v_n = u_n - u$, $d_n = |v_n|_2$ and supposing that $|v_n|_2 \to d$, we reach $a^2 = \hat{a}^2 + d^2$. From $d_n \in (0,a)$ for n large enough,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) = J_{\varepsilon,R}(u_n) = J_{\varepsilon,R}(v_n) + J_{\varepsilon,R}(u) + o_n(1) \geq \Upsilon_{\varepsilon,R}(d_n) + \Upsilon_{\varepsilon,R}(\hat{a}) + o_n(1).$$

thereby, by Lemma 2.4,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) \geq \frac{d_n^2}{a^2} \Upsilon_{\varepsilon,R}(a) + \Upsilon_{\varepsilon,R}(\hat{a}) + o_n(1).$$

Letting $n \to +\infty$, one finds

$$\Upsilon_{\varepsilon,R}(a) \geq \frac{d^2}{a^2} \Upsilon_{\varepsilon,R}(a) + \Upsilon_{\varepsilon,R}(\hat{a}). \tag{2.9}$$

Since $\hat{a} \in (0,a)$, employing Lemma 2.4 in (2.9) again, we arrive at the following inequality

$$\Upsilon_{\varepsilon,R}(a) > \frac{d^2}{a^2} \Upsilon_{\varepsilon,R}(a) + \frac{\hat{a}^2}{a^2} \Upsilon_{\varepsilon,R}(a) = \left(\frac{d^2}{a^2} + \frac{\hat{a}^2}{a^2}\right) \Upsilon_{\varepsilon,R}(a) = \Upsilon_{\varepsilon,R}(a),$$

which is absurd. This asserts that $|u|_2 = a$, or equivalently, $u \in S(a)$.

As $|u_n|_2 = |u|_2 = a$, $u_n \rightharpoonup u$ in $L^2(\mathbb{R}^2)$ and $L^2(\mathbb{R}^2)$ is reflexive, it is well-known that

$$u_n \to u \text{ in } L^2(\mathbb{R}^2). \tag{2.10}$$

This combined with interpolation theorem in the Lebesgue space and (1.14)-(1.15) gives

$$\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] F_R(u_n) dx \to \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)] F_R(u_n) dx. \tag{2.11}$$

These limits together with $\Upsilon_{\varepsilon,R}(a) = \lim_{n\to+\infty} J_{\varepsilon,R}(u_n)$ provide

$$\Upsilon_{\varepsilon,R}(a) \geq J_{\varepsilon,R}(u).$$

As $u \in S(a)$, we infer that $J_{\varepsilon,R}(u) = \Upsilon_{\varepsilon,R}(a)$, then

$$\lim_{n\to+\infty} J_{\varepsilon,R}(u_n) = J_{\varepsilon,R}(u),$$

that combines with (2.10) and (2.11) to give

$$\|u_n\|^2_{H^1} \to \|u\|^2_{H^1},$$

The last limit permits to conclude that $u_n \to u$ in $H^1(\mathbb{R}^2)$. The proof is completed. □

As we can observe that it is crucial to verify that the weak limit $u \neq 0$ before exploiting the compact result established in Theorem 2.7. To arrive at it, we need to introduce the following variational functionals $J_{0,R}$ and $J_{\infty,R}$ defined by

$$\begin{cases} J_{0,R}(u) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_0|u|^2\right] dx - \dfrac{\kappa}{2}\displaystyle\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx, \\ J_{\infty,R}(u) = \dfrac{1}{2}\displaystyle\int_{\mathbb{R}^2} \left[|\nabla u|^2 + V_\infty|u|^2\right] dx - \dfrac{\kappa}{2}\displaystyle\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx, \end{cases} \tag{2.12}$$

restricted to the sphere $S(a)$ defined in (2.4). One easily sees that $J_{0,R}, J_{\infty,R} \in \mathcal{C}^1(H^1(\mathbb{R}^2), \mathbb{R})$ and

$$\begin{cases} J'_{0,R}(u)v = \displaystyle\int_{\mathbb{R}^2} (\nabla u\nabla v + V_0 uv)\, dx - \kappa\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)vdx, \\ J'_{\infty,R}(u)v = \displaystyle\int_{\mathbb{R}^2} (\nabla u\nabla v + V_\infty uv)\, dx - \kappa\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)vdx, \end{cases} \quad \forall u, v \in S(a).$$

We also need to consider the minimization problems below

$$\begin{cases} \Upsilon_{0,R}(a) = \displaystyle\min_{u\in S(a)} J_{0,R}(u) \\ \Upsilon_{\infty,R}(a) = \displaystyle\min_{u\in S(a)} J_{0,\infty}(u). \end{cases} \tag{2.13}$$

Owing to the definitions of $\Upsilon_{0,R}(a)$ and $\Upsilon_{\infty,R}(a)$, by (V), it is clear to check that

$$\Upsilon_{0,R}(a) < \Upsilon_{\infty,R}(a),\ \forall R > R^*,\ \kappa \in (0,1) \text{ and } a > a^*. \tag{2.14}$$

Lemma 2.8. *If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed, then it holds $\lim_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) = \Upsilon_{0,R}(a)$ for all $a > a^*$. Particularly, there is a small $\varepsilon^* = \varepsilon^*(R) > 0$ such that $\Upsilon_{\varepsilon,R}(a) < \Upsilon_{\infty,R}(a)$ for all $\varepsilon \in (0, \varepsilon^*)$.*

Proof. To begin with the proof, we claim that

Claim 2.9. *If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed, there is a $U_0 \in H^1(\mathbb{R}^2)$ such that*

$$U_0 \in S(a) \text{ and } J_{0,R}(U_0) = \Upsilon_{0,R}(a),\ \forall a > a^*.$$

Indeed, we suppose that $(U_n) \subset S(a)$ is a minimizing sequence of $\Upsilon_{0,R}(a)$. Similar to Lemma 2.2, (U_n) is bounded and there is a $\bar{U}_0$ such that $U_n \rightharpoonup \bar{U}_0$ along a subsequence. It follows from the Vanishing lemma, c.f. [64, Lemma 1,21], that there are $\rho > 0$ and $(z_n) \subset \mathbb{R}^2$ such that

$$\liminf_{n\to\infty} \int_{B_\rho(y_n)} |U_n|^2 dx > 0.$$

Otherwise, $U_n \to 0$ in $L^p(\mathbb{R}^2)$ for every $p \in (2, +\infty)$ which together with (1.18) and (1.14)-(1.15) implies that $\lim_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(U_n)]F_R(U_n)dx = 0$. So, $\Upsilon_{0,R}(a) = \lim_{n\to\infty} \int_{\mathbb{R}^2} |\nabla U_n|^2 dx \geq 0$ but it cannot occur using a similar arguments in Lemma 2.3. Now, we define $\bar{U}_n \triangleq U_n(\cdot + z_n)$ and it is still a minimizing sequence of $\Upsilon_{0,R}(a)$. Hence, $\bar{U}_n \rightharpoonup U_0 \neq 0$ in $H^1(\mathbb{R}^2)$ along a subsequence, and then the Claim is proved by Theorem 2.7.

Since $U_0 \in S(a)$ in Claim 2.9, we arrive at

$$\Upsilon_{\varepsilon,R}(a) \le J_{\varepsilon,R}(U_0) = \frac{1}{2}\int_{\mathbb{R}^N}(|\nabla U_0|^2 + V(\varepsilon x)|U_0|^2)dx - \frac{\kappa}{2}\int_{\mathbb{R}^2}[|x|^{-\mu} * F_R(U_0)]F_R(U_0)dx.$$

Taking the limit as $\varepsilon \to 0^+$ and recalling $V(0) = \inf_{z\in\mathbb{R}^2} V(z) = V_0$, then we make full use of the Lebesgue's Dominated Convergence theorem as well as Claim 2.9 to get

$$\limsup_{\epsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) \le J_{0,R}(U_0) = \Upsilon_{0,R}(a). \tag{2.15}$$

On the other hand, by (V), one finds

$$J_{0,R}(u) \le J_{\varepsilon,R}(u),\ \forall u \in H^1(\mathbb{R}^2),$$

implying that

$$\Upsilon_{0,R}(a) \le \Upsilon_{\varepsilon,R}(a),\ \forall \varepsilon > 0.$$

Thereby,

$$\Upsilon_{0,R}(a) \le \liminf_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a). \tag{2.16}$$

From (2.15) and (2.16), it holds that

$$\lim_{\varepsilon\to 0^+} \Upsilon_{\varepsilon,R}(a) = \Upsilon_{0,R}(a).$$

The limit above combined with (2.14) yields that there is a $\varepsilon^* > 0$ such that $\Upsilon_{0,R}(a) < \Upsilon_{\infty,R}(a)$ for all $\varepsilon \in (0, \varepsilon^*)$. The proof is completed. □

Lemma 2.10. *If $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon > 0$ are fixed. Assume $(u_n) \subset S(a)$ is a minimizing sequence with respect to $\Upsilon_{\varepsilon,R}(a)$ for all $a > a^*$, then there is a function $u \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u$ along a subsequence in $H^1(\mathbb{R}^2)$. Moreover, we have that $u \neq 0$ provided that $\varepsilon \in (0, \varepsilon^*)$.*

Proof. The first part is a direct consequence of Lemma 2.2 and hence we omit it here. Suppose by the contradiction that $u_n \rightharpoonup 0$ in $H^1(\mathbb{R}^2)$. Then,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) = J_{\varepsilon,R}(u_n) = J_{\infty,R}(u_n) + \frac{1}{2}\int_{\mathbb{R}^2}[V(\varepsilon x) - V_\infty]|u_n|^2dx.$$

From (V), given $\eta > 0$, there is a sufficiently large $\rho > 0$ such that

$$V(z) \ge V_\infty - \eta \text{ for } |z| \ge \rho.$$

Thereby, in view of $(u_n) \subset S(a)$,

$$\Upsilon_{\varepsilon,R}(a) + o_n(1) \ge \Upsilon_{\infty,R}(a) - \eta a^2 + \int_{\mathcal{O}_\rho}[V(\varepsilon x) - V_\infty]|u_n|^2dx,$$

where $\mathcal{O}_\rho \triangleq \{z \in \mathbb{R}^2 : |z| < \varepsilon^{-1}\rho\}$. Letting $n \to \infty$ and then tending $\eta \to 0^+$, we derive

$$\Upsilon_{\varepsilon,R}(a) \ge \Upsilon_{\infty,R}(a),$$

which contradicts with Lemma 2.8. The proof of the lemma is completed. □

At this stage, we can show the detailed proof of Theorem 2.1.

Proof of Theorem 2.1. Using Lemma 2.2, we can choose a minimizing sequence $(u_n) \subset S(a)$ associated with $\Upsilon_{\varepsilon,R}(a)$ and there is a $u_R \in H^1(\mathbb{R}^2)$ such that $u_n \rightharpoonup u_R$ in $H^1(\mathbb{R}^2)$. In light of the suitable $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon \in (0, \varepsilon^*)$, we can rely on Theorem 2.7 and Lemma 2.10 to see that $u_n \to u_R$ in $H^1(\mathbb{R}^2)$ and so u_R is a minimizer of $\Upsilon_{\varepsilon,R}(a)$ for every $R > R^*$, $\kappa \in (0,1)$ and $\varepsilon \in (0, \varepsilon^*)$ whenever $a > a^*$. Thanks to the Lagrange multiplier theorem, there is a $\sigma_R \in \mathbb{R}$ such that (u_R, σ_R) is a couple of weak solutions to Eq. (2.1), where $\sigma_R < 0$ follows directly by Lemma

2.6. We clearly know that $u_R \geq 0$ by (f_1), then some very similar arguments adopted in [48] reveal that $u_R(x) > 0$ for all $x \in \mathbb{R}^2$. The proof is completed. □

Let us finish this section by exhibiting the following theorem.

Theorem 2.11. *Let u_R be given as in Theorem 2.1, if z_ε denotes the global maximum of u_R, then, up to a subsequence if necessary,*

$$\lim_{\varepsilon\to 0^+} V(\varepsilon z_\varepsilon) = V_0.$$

Proof. Let $\varepsilon_n \to 0^+$, we relabel u_R as u_n to be a solution of the problem below

$$\left\{ \begin{array}{l} -\Delta u + V(\varepsilon_n x)u = \sigma u + \kappa[|x|^{-\mu} * F(u)]f(u) \text{ in } \mathbb{R}^2, \\ \displaystyle\int_{\mathbb{R}^2} |u(x)|^2 dx = a^2, \end{array} \right. \tag{2.17}$$

for some $\sigma = \sigma_n \leq \frac{2}{a^2}\Upsilon_{\varepsilon_n,R}(a)$ by Lemma 2.6. From Lemma 2.8, we know that

$$\lim_{n\to\infty} J_{\varepsilon_n,R}(u_n) = \lim_{n\to\infty} \Upsilon_{\varepsilon_n,R}(a) = \Upsilon_{0,R}(a). \tag{2.18}$$

Now we claim that

Claim 2.12. *There exists a sequence $(\bar{y}_n) \subset \mathbb{R}^2$ such that $v_n = u_n(\cdot + \bar{y}_n)$ contains a strongly convergent subsequence in $H^1(\mathbb{R}^2)$. Moreover, up to a subsequence if necessary, $y_n = \varepsilon_n \bar{y}_n \to y$ as $n \to \infty$, where $V(y) = V_0 = \inf\limits_{z\in\mathbb{R}^2} V(x)$.*

Indeed, there are some $\rho > 0$, $\beta > 0$ and $(\bar{y}_n) \subset \mathbb{R}^2$ such that

$$\int_{B_\rho(\bar{y}_n)} |u_n|^2 dx \geq \beta, \ \forall n \in \mathbb{N}. \tag{2.19}$$

Otherwise, one has that $u_n \to 0$ in $L^p(\mathbb{R}^2)$ for all $2 < p < +\infty$ which together with (1.14)-(1.15) and (1.18) implies that $\lim\limits_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx = 0$. As a consequence, by means of (2.18), we derive $\Upsilon_{0,R}(a) = \lim\limits_{n\to\infty} \int_{\mathbb{R}^2} |\nabla u_n|^2 dx \geq 0$ violating Lemma 2.3. Thereby, (2.19) holds true and we could fix $v_n = u_n(\cdot + \bar{y}_n)$. There is a $v \neq 0$ such that $v_n \rightharpoonup v$ in $H^1(\mathbb{R}^2)$ along a subsequence. Since $(v_n) \subset S(a)$ and $J_{\varepsilon_n,R}(u_n) \geq J_{0,R}(u_n) = J_{0,R}(v_n) \geq \Upsilon_{0,R}(a)$, then one can invoke from (2.18) that (v_n) is a minimizing sequence of $\Upsilon_{0,R}(a)$. It is very similar to Theorem 2.7 that $v_n \to v$ in $H^1(\mathbb{R}^2)$ along a subsequence. Next, we shall verify that (y_n) is bounded in $n \in \mathbb{N}$. Suppose, by contradiction, that $|y_n| \to +\infty$ and so

$$\begin{aligned} \Upsilon_{0,R}(a) &= \lim_{n\to\infty} \left\{ \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla v_n|^2 + V(\varepsilon_n x + y_n)|v_n|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v_n)]F_R(v_n)dx \right\} \\ &= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla v|^2 + V_\infty |v|^2\right) dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v)]F_R(v)dx \\ &\geq \Upsilon_{\infty,R}(a) \end{aligned}$$

which is absurd by (2.14), where we have used (2.18) and $v_n \to v$ in $H^1(\mathbb{R}^2)$. Thus, passing to a subsequence if necessary, we can assume that $y_n \to y$ in $\mathbb{R}^2$. A similar argument shows that

$$\Upsilon_{0,R}(a) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla v|^2 + V(y)|v|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v)]F_R(v)dx \geq \Upsilon_{V(y),R}(a).$$

If $V(y) > V_0$, as the byproduct of Theorem 2.7, we could conclude that $\Upsilon_{V(y),R}(a) > \Upsilon_{0,R}(a)$. So, we must have that $V(y) = V_0$ proving the Claim.

Recalling (2.17), $(v_n) \subset S(a)$ is a sequence of solutions to the equation

$$-\Delta u + V(\varepsilon_n x + y_n)u = \sigma_n u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2$$

with

$$\lim_{n\to\infty} \sigma_n \le \frac{2}{a^2}\Upsilon_{\varepsilon_n,R}(a) \le \Theta_R < 0.$$

Owing to Claim 2.12, the same arguments explored in [4, Lemma 4.3] become available in this scenario to verify that

$$\lim_{n\to\infty} v_n(x) = 0 \text{ uniformly in } n \in \mathbb{N}.$$

From which, given a $\tau > 0$, there are some $\rho_0 > 0$ and $n_0 \in \mathbb{N}$ such that

$$v_n(x) \le \tau,\ \forall |x| \ge \rho_0 \text{ and } n \ge n_0.$$

Clearly, it holds that $|v_n|_\infty \not\to 0$. In fact, we can derive from (2.19) that $|v_n|_\infty^2 \ge \beta \text{meas}(B_\rho(0))$. At this stage, let us fix $\tau > 0$ such that $|v_n|_\infty \ge 2\tau$ and let $\hat{y}_n \in \mathbb{R}^2$ satisfy $v_n(\hat{y}_n) = |v_n|_\infty$ for all $n \in \mathbb{N}$. Therefore, according to the above discussions, there holds $|\hat{y}_n| \le \rho_0$ for all $n \in \mathbb{N}$. Furthermore, if we denote z_n by $u_n(z_n) = |u_n|_\infty$ for all $n \in \mathbb{N}$, then $z_n = \hat{y}_n + \bar{y}_n$ and

$$\lim_{n\to\infty} V(\varepsilon_n z_n) = \lim_{n\to\infty} V(\varepsilon_n \hat{y}_n + \varepsilon_n \bar{y}_n) = \lim_{n\to\infty} V(\varepsilon_n \hat{y}_n + y_n) = V(y) = V_0$$

finishing the proof. □

3. Truncated problem: steep potential well

In this section, we shall conclude the existence of positive solutions for the following nonlocal Schrödinger equation

$$-\Delta u + \lambda V(x)u = \sigma u + \kappa[|x|^{-\mu} * F_R(u)]f_R(u) \text{ in } \mathbb{R}^2, \tag{3.1}$$

under the constraint

$$\int_{\mathbb{R}^2} |u|^2 dx = a^2, \tag{3.2}$$

where the potential $V : \mathbb{R}^2 \to \mathbb{R}$ satisfies the assumptions $(V_1) - (V_3)$, $\lambda, \kappa > 0$ are parameters, $a > 0$, $\sigma \in \mathbb{R}$ is known as the Lagrange multiplier and the nonlinearity f_R is defined in (1.13).

Before solving Problems (3.1)-(3.2), we have to determine a suitable work space. Proceeding as [53, 56, 58], given a fixed $\lambda > 0$, by (V_1), we define the space

$$E_\lambda \triangleq \left\{ u \in L^2_{\text{loc}}(\mathbb{R}^2) : |\nabla u| \in L^2(\mathbb{R}^2) \text{ and } \int_{\mathbb{R}^2} \lambda V(x)|u|^2 dx < +\infty \right\}$$

which is indeed a Hilbert space equipped with the inner product and norm

$$(u, v)_{E_\lambda} = \int_{\mathbb{R}^2} \left[\nabla u \nabla v + \lambda V(x)uv\right] dx \text{ and } \|u\|_{E_\lambda} = \sqrt{(u, u)_{E_\lambda}},\ \forall u, v \in E_\lambda.$$

From here onwards, we shall denote E and $\|\cdot\|_E$ by E_λ and $\|\cdot\|_{E_\lambda}$ for $\lambda = 1$, respectively. It is simple to observe that $\|\cdot\|_E \le \|\cdot\|_{E_\lambda}$ for every $\lambda \ge 1$. Therefore, owing to [56, Lemma 2.4], E_λ could be continuously imbedded into $H^1(\mathbb{R}^2)$ for all $\lambda \ge 1$.

Define the variational functional $\mathcal{J}_{\lambda,R} : E_\lambda \to \mathbb{R}$ by

$$\mathcal{J}_{\lambda,R}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|\nabla u|^2 + \lambda V(x)|u|^2\right] dx - \frac{\kappa}{2}\int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]F_R(u)dx. \tag{3.3}$$

Obviously, combining (1.15) and (1.18), one can easily show that $\mathcal{J}_{\lambda,R}$ belongs to $\mathcal{C}^1(E_\lambda, \mathbb{R})$ and it derivative is given by

$$\mathcal{J}'_{\lambda,R}(u)v = \int_{\mathbb{R}^2} [\nabla u \nabla v + \lambda V(x)uv]\, dx - \kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u)]f_R(u)v dx,\ \forall u, v \in E_\lambda.$$

In order to solve Problems (3.1)-(3.2), we consider the following minimization problem

$$\bar{\Upsilon}_{\lambda,R}(a)=\min_{u\in S(a)}\mathcal{J}_{\lambda,R}(u), \tag{3.4}$$

where, with $\lambda\geq 1$, the sphere in defined by

$$S(a)=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}|u|^2dx=a^2\right\}. \tag{3.5}$$

The existence result for Problems (3.1)-(3.2) in this section can be stated as follows.

Theorem 3.1. *Suppose $(V_1)-(V_3)$, (1.3), $(f_1)-(f_3)$ and $\mu\in(0,2)$, then there is an $R_*>0$ such that for every $R>R_*$, there exist some $a_*=a_*(R)>0$ and $\lambda_*=\lambda_*(R)>1$ such that, for all $\kappa\in(0,1)$, $a>a_*$ and $\lambda>\lambda_*$, the minimization problem (3.4) could be achieved by some function in E_λ. Moreover, there is $(u_R,\sigma_R)\in H^1(\mathbb{R}^2)\times\mathbb{R}$ such that it is a couple solution of Problems (3.1)-(3.2), where $u_R(x)>0$ for all $x\in\mathbb{R}^2$ and $\sigma_R<0$.*

Arguing as what we have done is Section 2, we introduce several lemmas to prove Theorem 3.1. For simplicity, when there is no misunderstanding, we will also suppose that the potential V and the nonlinearity f_R satisfy $(V_1)-(V_3)$ and (1.3) with $(f_1)-(f_3)$ in this section, respectively.

Using the same calculations in the proof of Lemma 2.2, we have the result below.

Lemma 3.2. *For all fixed $R>0$, the variational functional $\mathcal{J}_{\lambda,R}$ is coercive and bounded from below on $S(a)$ for each $\kappa\in(0,1)$, $a>0$ and $\lambda\geq 1$, where $\mathcal{J}_{\lambda,R}$ and $S(a)$ are appearing in (3.3) and (3.5), respectively.*

Proof. The proof is totally same as that of Lemma 2.2 and so we omit it here. □

Employing some necessary modifications in the proof of Lemma 2.3, we are able to conclude the lemma below.

Lemma 3.3. *There exists an $R_*>0$ such that for all fixed $R>R_*$, there is an $a_*=a_*(R)>0$ satisfying for all $a>a_*$, there exists a constant $\bar{\Theta}_R=\bar{\Theta}(R)<0$, independent of λ, such that $\bar{\Upsilon}_{\lambda,R}(a)\leq\bar{\Theta}_R$ for all $\kappa\in(0,1)$ and $\lambda\geq 1$.*

Proof. The main idea originates from [58, Lemma 3.3], we show the details for the convenience of the reader. Without loss of generality, we are assuming that $0\in\text{int}V^{-1}(0)$. Therefore, there exists a sufficiently small $r>0$ such that $B_r(0)\subset\text{int}V^{-1}(0)$. Choose $\psi\in C_0^\infty(B_r(0))$ to be a function satisfying $\int_{B_r(0)}|\psi|^2dx=1$ and so $\psi\in S(1)$. Thanks to the definition of Ω, it holds that

$$\int_{\mathbb{R}^2}V(x)|\psi|^2dx=\int_{\Omega}V(x)|\psi|^2dx+\int_{\Omega^c}V(x)|\psi|^2dx=0. \tag{3.6}$$

Proceeding as the proof of Lemma 2.3, we could determine a sufficiently large $t_*=t_*(R)>0$ and then $t_*=t_*|\psi|_2$ to find a constant $\bar{\Theta}_R<0$, dependent of R, such that

$$\mathcal{J}_{\lambda,R}(u)\leq\bar{\Theta}_R,\ \forall R>R_*,\ \kappa\in(0,1),\ a>a_*\text{ and }\lambda\geq 1,$$

provided $u\in S(a)$. The proof is completed. □

Owing to the essential feature of steep potential well, there is no need to certify the similar result in Lemma 2.4. In other words, we shall conclude the counterpart of Theorem 2.7 directly.

Theorem 3.4. *Let $R>R_*$, $\kappa\in(0,1)$ and $\lambda\geq 1$ be fixed. Suppose $(u_n)\subset S(a)$ is a minimizing sequence of $\bar{\Upsilon}_{\lambda,R}(a)$ for all $a>a_*$, then $u_n\rightharpoonup u$ in E_λ as $n\to\infty$. If in addition $u\neq 0$, there is a sufficiently large $\lambda_*'=\lambda_*'(R)>1$ such that $u_n\to u$ in E_λ along a subsequence as $n\to\infty$ for all $\lambda>\lambda_*'$.*

Proof. The first part is same as its counterpart in Theorem 2.7 and we omit it here. To derive the remaining part, we define $v_n \triangleq u_n - u \rightharpoonup 0$ in E_λ. Let us recall from (V_3) that the nonempty set $\Xi \triangleq \{x \in \mathbb{R}^2 : V(x) < b\}$ has finite measure, then

$$\begin{aligned}\int_{\mathbb{R}^2} |v_n|^2 dx &= \int_{\mathbb{R}^2\setminus\Xi} |v_n|^2 dx + \int_{\Xi} |v_n|^2 dx + = \int_{\mathbb{R}^2\setminus\Xi} |v_n|^2 dx + o_n(1)\\ &\leq \frac{1}{\lambda b}\int_{\mathbb{R}^2\setminus\Xi} \lambda V(x)|v_n|^2 dx + o_n(1) \leq \frac{1}{\lambda b}\|v_n\|^2_{E_\lambda} + o_n(1)\end{aligned}$$

which together with (1.17) with $l = \frac{4q}{4-\mu} > 2$ gives that

$$\int_{\mathbb{R}^2} |v_n|^l dx \leq \mathbb{C}(\lambda b)^{-\frac{(1-\gamma_l)l}{2}} \|v_n\|^l_{E_\lambda} + o_n(1).$$

From the inequality above, combining (1.14)-(1.15) and (1.18), we see that

$$\kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(v_n)]F_R(v_n)dx \leq \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R)}{q^{4-\mu}R^{(4-\mu)(q-1)}}(\lambda b)^{-\frac{4-\mu}{2}}\|v_n\|^{2q}_{E_\lambda} + o_n(1). \tag{3.7}$$

On the other hand, obviously $|v_n|_2 \in (0,a)$, then Lemma 3.3 indicates that $\bar{\Upsilon}_{\lambda,R}(|v_n|_2) \leq 0$. Moreover, $\|v_n\|_{E_\lambda}$ is bounded, namely there exists a $\zeta = \zeta(R) > 0$ such that $\|v_n\|_{E_\lambda} \leq \zeta$ for all $n \in \mathbb{N}$. Combining these facts jointly with (3.7), it holds that

$$0 \geq \left[\frac{1}{2} - \frac{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R)}{2q^{4-\mu}R^{(4-\mu)(q-1)}}(\lambda b)^{-\frac{4-\mu}{2}}\zeta^{2(q-1)}\right]\|v_n\|^2_{E_\lambda} + o_n(1). \tag{3.8}$$

Consequently, we shall determine a sufficiently large $\lambda'_* = \lambda'_*(R) > 1$ to satisfy $\|v_n\|^2_{E_\lambda} = o_n(1)$ whenever $\lambda > \lambda'_*$. The proof is completed. □

To apply Theorem 3.4 successfully, we need the following lemma to show that the weak limit $u \neq 0$.

Lemma 3.5. *Under the assumptions of Theorem 3.4 above, then there exists a $\lambda_* = \lambda_*(R) > \lambda'_*$ such that $u \neq 0$ for all $\lambda > \lambda_*$.*

Proof. We collect the methods used in [53, 56, 58] to reach the proof. Firstly, we claim that

Claim 3.6. *For some $q_0 \in (2, +\infty)$, there exists a constant $\beta_0 > 0$, independent of $\lambda \geq 1$, such that*

$$\lim_{n\to\infty} \sup_{z\in\mathbb{R}^2} \int_{B_\rho(z)} |u_n|^{q_0} dx = \beta_0.$$

To demonstrate this Claim, we can suppose that there exists a constant $\beta_\lambda = \beta(\lambda) > 0$ such that $\lim\limits_{n\to\infty} \sup\limits_{y\in\mathbb{R}^2} \int_{B_\varrho(y)} |u_n|^{q_0} dx = \beta_\lambda$. Otherwise, $u_n \to 0$ in $L^s(\mathbb{R}^2)$ for every $s \in (2, +\infty)$ jointly with (1.14)-(1.15) and (1.18) yields that $\lim\limits_{n\to\infty} \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)]F_R(u_n)dx = 0$. Hence, we can conclude that $\bar{\Upsilon}_{\lambda,R}(a) = \lim\limits_{n\to\infty} \mathcal{J}_{\lambda,R}(u_n) \geq 0$ and it is impossible because of Lemma 3.3. With such a β_λ, we are able to finish the verification of this Claim. Suppose, by contradiction, that the uniform control from below of $L^{q_0}(\mathbb{R}^2)$-norm is false. Consequently, for any $k \in \mathbb{N}$, $k \neq 0$, there are $\lambda_k > 1$ and a minimizing sequence $(u_{k,n})$ of $\bar{\Upsilon}_{\lambda_k,R}$ such that

$$|u_{k,n}|_{q_0} < \frac{1}{k}, \text{ definitely.}$$

Then, by a diagonalization argument, for any $k \geq 1$, it permits us to find an increasing sequence $(n_k) \subset \mathbb{N}$ and $(u_{n_k}) \subset E_{\lambda_{n_k}}$ such that

$$(u_{n_k}) \subset S(a),\ \mathcal{J}_{\lambda_{n_k},R}(u_{n_k}) = \bar{\Upsilon}_{\lambda_{n_k},R}(a) + o_k(1) \text{ and } |u_{n_k}|_{q_0} = o_k(1).$$

where $o_k(1) \to 0$ as $k \to +\infty$. In this situation, we could repeat the calculations above to reach a contradiction $\bar{\Upsilon}_{\lambda_{n_k},R}(a) \geq 0$, again. So, the Claim is proved.

Thanks to Claim 3.6, there exist a sequence $(z_n) \subset \mathbb{R}^2$ and a subsequence (u_n), still denoted by itself, such that

$$\int_{B_\rho(z_n)} |u_n|^2 dx = \frac{1}{2}\beta_0. \tag{3.9}$$

Claim 3.7. *The sequence (z_n) above is uniformly bounded in $n \in \mathbb{N}$.*

Otherwise, we suppose by contradiction to choose a subsequence if necessary that $|z_n| \to \infty$. Define

$$\Xi_n^1 \triangleq \{x \in B_\rho(z_n) : V(x) < b\} \text{ and } \Xi_n^2 \triangleq \{x \in B_\rho(z_n) : V(x) \geq b\}.$$

Since the set $\Xi \triangleq \{x \in \mathbb{R}^2 : V(x) < b\}$ is nonempty and has finite measure, one concludes that

$$\text{meas}(\Xi_n^1) \leq \text{meas}(\{x \in \mathbb{R}^2 : |x| \geq |y_n| - 2, V(x) < b\}) \to 0 \text{ as } n \to \infty.$$

For $\lambda \geq 1$, one sees $|u_n|_r$ with $r > 2$ is uniformly bounded in $n \in \mathbb{N}$ by Lemma 3.2 and then

$$\int_{\Xi_n^1} |u_n|^2 dx \leq [\text{meas}(\Xi_n^1)]^{\frac{r-2}{r}} |u_n|_r^2 = o_n(1)$$

which together with (3.9) reveals that

$$\int_{\Xi_n^2} |u_n|^2 dx = \int_{B_\rho(z_n)} |u_n|^2 dx - \int_{\Xi_n^1} |u_n|^2 dx = \frac{1}{2}\beta_0 + o_n(1).$$

Thanks to $V(x) \geq 0$ for all $x \in \mathbb{R}^2$ by (V_1), using the definition of Ξ_n^2, we obtain

$$\int_{\mathbb{R}^2} V(x)|u_n|^2 dx \geq \int_{\Xi_n^2} V(x)|u_n|^2 dx \geq b \int_{\Xi_n^2} |u_n|^2 dx = \frac{1}{2} b\beta_0 + o_n(1). \tag{3.10}$$

It follows from (1.14)-(1.15) and (1.18) that

$$\sup_{n\in\mathbb{N}} \left\{ \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] F_R(u_n) dx \right\} \leq C,\ \forall \lambda \geq 1, \tag{3.11}$$

where $C > 0$ is independent of $n \in \mathbb{N}$ and $\lambda \geq 1$. So, we deduce by (3.10) and (3.11) that

$$\bar{\Upsilon}_{\lambda,R}(a) \geq \frac{1}{2} \int_{\mathbb{R}^2} \lambda V(x)|u_n|^2 dx - C + o_n(1) \geq \frac{\lambda b\beta_0}{4} - C + o_n(1) \tag{3.12}$$

where the positive constants b, β_0 and C are independent of $\lambda \geq 1$. Adopting Lemma 3.3 again, there is a sufficiently large $\lambda_* = \lambda_*(R) > \lambda'_*(R)$ such that (3.12) is impossible provided $\lambda > \lambda_*$. Hence, the Claim is proved.

Owing to Claim 3.7, passing to a subsequence if necessary, we suppose that $z_n \to z_0$ in $\mathbb{R}^2$. Since $u_n \to u$ in $L^2_{\text{loc}}(\mathbb{R}^2)$, then we can arrive at the proof of this lemma. □

Proof of Theorem 3.1. There is a minimizing sequence $(u_n) \subset S(a)$ associated with $\bar{\Upsilon}_{\lambda,R}(a)$ by Lemma 3.2 and thus $u_n \rightharpoonup u_R$ in E_λ for some $\lambda \geq 1$. As a consequence of Theorem 3.4 and Lemma 3.5, for all $R > R_*$, $\kappa \in (0,1)$, $a > a_*$ and $\lambda > \lambda_*$, we see that $u_n \to u_R$ in E_λ and so u_R is a minimizer of $\bar{\Upsilon}_{\lambda,R}(a)$. By exploiting the Lagrange multiplier theorem again, a similar argument in Lemma 2.6 makes sure a $\sigma_R < 0$ that (u_R, σ_R) is a couple of weak solutions to Eq. (3.1). Finally, the reader can derive $u_R > 0$ as in Theorem 2.1. The proof is completed. □

As we can observe from the proof of Theorem 3.1, the couple (u_R, σ_R) exists for all $R > R_*$, $\kappa \in (0,1)$, $a > a_*$ and $\lambda > \lambda_*$. In other words, if $R > R_*$, $\kappa \in (0,1)$ and $a \in (0, a_*)$ are fixed, the couple (u_R, σ_R) would also rely on $\lambda > \lambda_*$. It is therefore that we shall relabel it by $(u_\lambda, \sigma_\lambda)$ when $R > R_*$, $\kappa \in (0,1)$ and $a \in (0, a_*)$ are fixed.

Letting $\lambda \to +\infty$, we have the following result.

Theorem 3.8. *Let $(u_\lambda, \sigma_\lambda) \in E_\lambda \times \mathbb{R}$ denote by the couple of weak solutions established above for all $\lambda > \lambda_*$, passing to a subsequence if necessary, $u_\lambda \to u_0$ in $H^1(\mathbb{R}^2)$ and $\sigma_\lambda \to \sigma_0$ in $\mathbb{R}$ as $\lambda \to +\infty$, where $\sigma_0 < 0$ and (u_0, σ_0) is a couple of weak solution to the problem below*

$$\begin{cases} -\Delta u = \sigma u + \kappa \left(\displaystyle\int_\Omega \frac{F_R(u(y))}{|x-y|^\mu} dy \right) f_R(u), \ x \in \Omega, \\ u(x) = 0, \ x \in \partial\Omega, \\ \displaystyle\int_\Omega |u|^2 dx = a^2. \end{cases} \tag{3.13}$$

Proof. Let $\lambda_n \to +\infty$, we contemplate the subsequence of $(u_\lambda, \sigma_\lambda) \in E_\lambda \times \mathbb{R}$, namely $(u_{\lambda_n}, \sigma_{\lambda_n})$ satisfies $(u_{\lambda_n}) \subset S(a)$ and $\mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) = \bar{\Upsilon}_{\lambda_n,R}$. By Lemma 3.3, the sequence (u_{λ_n}) is uniformly bounded in $n \in \mathbb{N}$. Similar to the proof of Lemma 2.6, there holds

$$\sigma_{\lambda_n} = \frac{1}{a^2} \left\{ \int_{\mathbb{R}^2} \left[|\nabla u_n|^2 + \lambda_n V(x) |u_n|^2 \right] dx - \kappa \int_{\mathbb{R}^2} [|x|^{-\mu} * F_R(u_n)] f_R(u_n) u_n dx \right\} + o_n(1)$$

showing that (σ_{λ_n}) is uniformly bounded in $n \in \mathbb{N}$. Up to a subsequence if necessary, $u_{\lambda_n} \rightharpoonup u_0$ in $H^1(\mathbb{R}^2)$ and $\sigma_{\lambda_n} \to \sigma_0$ in $\mathbb{R}$ as $n \to +\infty$. In view of Lemmas 2.6 and 3.3 again, there holds

$$\sigma_0 = \lim_{n\to\infty} \sigma_{\lambda_n} \le \lim_{n\to\infty} \frac{2}{a^2} \bar{\Upsilon}_{\lambda_n,R}(a) \le \bar{\Theta}_R < 0.$$

To continue the proof, we claim that

Claim 3.9. *$u_0 \equiv 0$ in $\Omega^c \triangleq \mathbb{R}^2 \backslash \Omega$ and so $u_0 \in S_\Omega(a) \triangleq \left\{ u \in H_0^1(\Omega) : \int_\Omega |u|^2 dx = a^2 \right\}$.*

Otherwise, there exists a compact subset $\hat{\Omega}_{u_0} \subset \Omega^c$ with $\text{dist}(\hat{\Omega}_{u_0}, \partial\Omega^c) > 0$ such that $u_0 \not\equiv 0$ on $\hat{\Omega}_{u_0}$ and by Fatou's lemma

$$a^2 = \liminf_{n\to\infty} \int_{\mathbb{R}^2} u_{\lambda_n}^2 dx \ge \int_{\hat{\Theta}_{u_0}} {u_0}^2 dx > 0. \tag{3.14}$$

Moreover, there exists $\zeta_0 > 0$ such that $V(x) \ge \zeta_0$ for every $x \in \hat{\Omega}_{u_0}$ by the assumptions (V_1) and (V_2). Combining Lemma 3.3, (1.14) and (3.14), we derive

$$\begin{aligned} 0 &\ge \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R} = \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \\ &= \liminf_{n\to\infty} \left\{ \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) - \frac{1}{q} \left[\mathcal{J}'_{\lambda_n,R}(u_{\lambda_n}) u_{\lambda_n} - \sigma_{\lambda_n} a^2 \right] \right\} \\ &\ge \frac{q-2}{2q} \zeta_0 \left(\int_{\hat{\Theta}_{u_0}} u_0^2 dx \right) \liminf_{n\to\infty} \lambda_n + \frac{\sigma_0}{q} a^2 \\ &= +\infty \end{aligned}$$

which is impossible. Consequently, $u_0 \in H_0^1(\Omega)$ by the fact that $\partial\Omega$ is smooth. By taking some similar calculations explored in (3.8) to show $u_{\lambda_n} \to u_0$ in $H^1(\mathbb{R}^2)$ and so $u_0 \in S_\Omega(a)$.

Claim 3.10. *$\mathcal{J}_{\Omega,R}(u_0) = \bar{\Upsilon}_{\Omega,R}(a)$, where $\bar{\Upsilon}_{\Omega,R} \triangleq \inf_{u \in S_\Omega(s)} \mathcal{J}_{\Omega,R}(u)$ and the variational functional $\mathcal{J}_{\Omega,R} : H_0^1(\Omega) \to \mathbb{R}$ is defined by*

$$\mathcal{J}_{\Omega,R}(u) = \frac{1}{2} \int_\Omega |\nabla u|^2 dx - \frac{\kappa}{2} \int_\Omega \int_\Omega \frac{F_R(u(x)) F_R(u(y))}{|x-y|^\mu} dx dy, \ \forall u \in H_0^1(\Omega).$$

Actually, it is simple to see that $S_\Omega(a) \subset S(a)$ and so $\bar{\Upsilon}_{\Omega,R}(a) \geq \bar{\Upsilon}_{\lambda_n,R}(a)$. As a consequence, there holds $\bar{\Upsilon}_{\Omega,R}(a) \geq \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R}(a)$. On the other hand, we gather these facts together with the Fatou's lemma to obtain

$$\bar{\Upsilon}_{\Omega,R}(a) \geq \liminf_{n\to\infty} \bar{\Upsilon}_{\lambda_n,R}(a) = \liminf_{n\to\infty} \mathcal{J}_{\lambda_n,R}(u_{\lambda_n}) \geq \mathcal{J}_{\Omega,R}(u_0) \geq \bar{\Upsilon}_{\Omega,R}(a)$$

proving the Claim.

Finally, we shall prove that $\mathcal{J}'_\Omega(u_0) - \sigma_0 u_0 = 0$ in $(H_0^1(\Omega))^{-1}$. To see it, for every $\psi \in C_0^\infty(\Omega)$, Combining (1.20) and $\sigma_{\lambda_n} \to \sigma_0$, it holds that

$$\lim_{n\to\infty} \left\{ \mathcal{J}'_{\lambda_n,R}(u_{\lambda_n})\psi - \sigma_{\lambda_n} \int_{\mathbb{R}^2} u_{\lambda_n} \psi dx \right\} = 0, \ \forall \psi \in C_0^\infty(\Omega),$$

we can arrive at the desired result. The proof is completed. □

4. Proofs of main results

In this section, we are concerned with the existence and concentrating behavior of positive solutions to the nonlocal Schrödinger equation (1.1) under the mass-constraint (1.2).

Firstly, we shall provide some growth conditions with the nonlinearity f and f_R which play foremost roles in this section. It can infer from $(f_1) - (f_2)$ that

$$\lim_{s\to 0^+} \frac{f_R(s)}{s} = 0 \text{ and } \lim_{s\to 0^+} \frac{f(s)}{s} = 0. \tag{4.1}$$

Actually, taking advantage of (f_1) and (f_2) with $q > 2$ again to obtain

$$0 \leq \lim_{s\to 0^+} \frac{f_R(s)}{s} = \lim_{s\to 0^+} \frac{f(s)}{s} = \lim_{s\to 0^+} \frac{f(s)}{s^{q-1}} s^{q-2} \leq f(1) \lim_{s\to 0^+} s^{q-2} = 0.$$

Combining (1.3) and (4.1), given a fixed $\varepsilon > 0$, for every $\bar{p} > 2$ and $\nu > 1$, we are able to search for two constants such that $\tilde{b}_1 = \tilde{b}_1(\bar{p}, \alpha, \varepsilon) > 0$ and $\tilde{b}_2 = \tilde{b}_2(\bar{p}, \alpha, \varepsilon) > 0$ satisfying

$$|f(s)| \leq \varepsilon|s| + \tilde{b}_1 |s|^{\bar{p}-1}(e^{4\pi\nu s^2} - 1), \ \forall s \in \mathbb{R}, \tag{4.2}$$

and

$$|F(s)| \leq \varepsilon|s|^2 + \tilde{b}_2 |s|^{\bar{p}}(e^{4\pi\nu s^2} - 1), \ \forall s \in \mathbb{R}. \tag{4.3}$$

Taking the nonlinearity f having the critical exponential growth at infinity into account, the following Trudinger-Moser inequality found in [26, 50, 61] will play a crucial role in this section.

Lemma 4.1. *If $\alpha > 0$ and $u \in H^1(\mathbb{R}^2)$, then*

$$\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1) dx < +\infty.$$

Moreover, if $|\nabla u|_2^2 \leq 1$, $|u|_2^2 \leq M < +\infty$ and $\alpha < 4\pi$, then there exists $K_{\alpha,M} = K(M, \alpha)$ such that

$$\int_{\mathbb{R}^2} (e^{\alpha|u|^2} - 1) dx \leq K_{\alpha,M}. \tag{4.4}$$

Now, we are ready to exhibit the detailed proofs of Theorems 1.1 and 1.2. In order to show them clearly, we shall divide into two subsections.

4.1. **Proof of Theorem 1.1.**
In this subsection, due to Theorem 2.1, we will know that the minimization constant $\Upsilon_{\varepsilon,R}(a)$ defined in (2.5) can be attained by some nontrivial function in $H^1(\mathbb{R}^2)$ for every fixed $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$. In other words, there is a function $u_R \in H^1(\mathbb{R}^2)$ such that

$$u_R \in S(a) \text{ and } J_{\varepsilon,R}(u_R) = \Upsilon_{\varepsilon,R}(a),\ \forall R > R^*,\ \kappa \in (0,1),\ a > a^* \text{ and } \varepsilon \in (0,\varepsilon^*). \tag{4.5}$$

Moreover, there is a $\sigma_R < 0$ such that the couple (u_R, μ_R) is a solution of Problems (2.1)-(2.2) for all $R > R^*$, $\kappa \in (0,1)$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, where $u_R(x) > 0$ for all $x \in \mathbb{R}^2$.

According to the discussions in the Introduction, the reader could observe that if u_R in (4.5) satisfies $|u_R|_\infty \le R$, then u_R is in fact a solution of the original Eq. (1.1) with $\sigma = \sigma_R$, thereby it is available to arrive at the proof of Theorem 1.1. As a consequence, the foremost objection for us is to take the L^∞-estimate on u_R.

To the aim, we establish the uniform estimate on $|\nabla u_R|_2^2$ below.

Lemma 4.2. *Suppose that V satisfies (V) and f meets (1.3) with $(f_1)-(f_3)$. Let u_R be given by (4.5) for each $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, then there exists a $\kappa^* = \kappa^*(R) \in (0,1)$ such that if $\kappa \in (0,\kappa^*)$, there holds that $|\nabla u_R|_2^2 < \frac{2-\mu}{2(2+\mu)\nu^2}$ for every $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, where the constant $\nu > 1$ is appearing in (4.2) and (4.3).*

Proof. Since $u_R \in S(a)$, we borrow the calculations in Lemma 2.2 to obtain

$$J_{\varepsilon,R}(u_R) \ge \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx - \frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}} \left(\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx\right)^{q-\frac{4-\mu}{2}}.$$

Since $2 < q < \frac{6-\mu}{2}$, by means of the Young's inequality, there is $C_1 > 0$ independent of $R > R^*$ such that

$$\begin{aligned}
&\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}} \left(\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx\right)^{q-\frac{4-\mu}{2}} \\
&\le C_1 \left[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\right]^{\frac{2}{6-\mu-2q}} + \frac{1}{4}\int_{\mathbb{R}^2} |\nabla u_R|^2\,dx.
\end{aligned}$$

Thereby, for every $R > R^*$, $a > a^*$ and $\varepsilon \in (0,\varepsilon^*)$, it holds that

$$|\nabla u_R|_2^2 \le 4J_{\varepsilon,R}(u_R) + 4C_1 \left[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\right]^{\frac{2}{6-\mu-2q}}.$$

Assuming that

$$4C_1 \left[\frac{\kappa \mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}{2R^{(4-\mu)(q-1)} q^{4-\mu}}\right]^{\frac{2}{6-\mu-2q}} \le \frac{2-\mu}{2(2+\mu)\nu^2},$$

we arrive at

$$|\nabla u_R|_2^2 \le 4J_{\varepsilon,R}(u_R) + \frac{2-\mu}{2(2+\mu)\nu^2},\ \forall R > R^*,\ a > a^* \text{ and } \varepsilon \in (0,\varepsilon^*).$$

In light of $J_{\varepsilon,R}(u_R) = \Upsilon_{\varepsilon,R}(a) \le 0$ by Lemma 2.3 and (4.5), so it permits us to choose

$$\kappa^* = \kappa^*(R) \triangleq \min\left\{\left[\frac{2R^{(4-\mu)(q-1)} q^{4-\mu}}{\mathbb{C}^{\frac{4-\mu}{2}} C_\mu f^{4-\mu}(R) a^{4-\mu}}\right]\left[\frac{2-\mu}{8C_1(2+\mu)\nu^2}\right]^{\frac{6-\mu-2q}{2}}, 1\right\}$$

and then we can finish the proof of this lemma. □

With Lemma 4.2 in hands, we can derive the following result.

Lemma 4.3. *Suppose that V satisfies (V) and f meets (1.3) with $(f_1)-(f_3)$. Then, for every fixed $R>R^*$, $\kappa\in(0,\kappa^*)$, $a>a^*$ and $\varepsilon\in(0,\varepsilon^*)$, there exists a constant $\mathfrak{C}\in(0,+\infty)$ which is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$ such that*

$$\Gamma(x)\triangleq|x|^{-\mu}*F(u_R)\le\mathfrak{C},$$

where u_R comes from (4.5).

Proof. Since $u_R\in S(a)$, adopting Lemma 4.2 and (1.17), there is a constant $\mathbb{T}\in(0,+\infty)$ which is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$ such that

$$|u|_l^l\le\mathbb{T},\ \forall l\in(2,+\infty). \tag{4.6}$$

By (4.6), we find a constant $\mathbb{C}_0\in(0,+\infty)$ which is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$ such that

$$\begin{aligned}\int_{\mathbb{R}^2}\frac{|u_R(y)|^2}{|x-y|^\mu}dy&=\int_{|x-y|<1}\frac{|u_R(y)|^2}{|x-y|^\mu}dy+\int_{|x-y|\ge1}\frac{|u_R(y)|^2}{|x-y|^\mu}dy\\&\le\bar{\mathcal{C}}_\mu\left(\int_{\mathbb{R}^2}|u_R(y)|^{\frac{2(2+\mu)}{2-\mu}}dy\right)^{\frac{2-\mu}{2+\mu}}+a^2\le\mathbb{C}_0.\end{aligned}\tag{4.7}$$

Let us define $\bar u_R=\nu\sqrt{\frac{2(2+\mu)}{2-\mu}}u_R$, then $u_R\in S(a)$ and Lemma 4.2 give us that

$$|\bar u_R|_2^2=\frac{2\nu^2a^2(2+\mu)}{2-\mu}\text{ and }|\nabla\bar u_R|_2^2\le1$$

which together with (4.4) and $\nu>1$ implies that

$$\int_{\mathbb{R}^2}(e^{\frac{8\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2}-1)dy=\int_{\mathbb{R}^2}(e^{4\pi\nu^{-1}|\bar u_R(y)|^2}-1)dy\le K(a,\nu,\mu).\tag{4.8}$$

The above inequality shall determine a constant $\mathbb{C}_1\in(0,+\infty)$ which is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$ to reach

$$\begin{aligned}\int_{|x-y|<1}\frac{|u_R(y)|^{\bar p}(e^{4\pi\nu|u_R(y)|^2}-1)}{|x-y|^\mu}dy&\le\mathcal{C}_\mu\left(\int_{\mathbb{R}^2}|u_R(y)|^{\frac{\bar p(2+\mu)}{2-\mu}}(e^{\frac{4\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2}-1)dy\right)^{\frac{2-\mu}{2+\mu}}\\&\le\mathcal{C}_\mu\left(\int_{\mathbb{R}^2}|u_R(y)|^{\frac{2\bar p(2+\mu)}{2-\mu}}dy\right)^{\frac{2-\mu}{2(2+\mu)}}\left(\int_{\mathbb{R}^2}(e^{\frac{8\pi\nu(2+\mu)}{2-\mu}|u_R(y)|^2}-1)dy\right)^{\frac{2-\mu}{2(2+\mu)}}\le\mathbb{C}_1.\end{aligned}$$

Using some very similar calculations, we have that

$$\begin{aligned}\int_{|x-y|\ge1}\frac{|u_R(y)|^{\bar p}(e^{4\pi\nu|u_R(y)|^2}-1)}{|x-y|^\mu}dy&\le\int_{\mathbb{R}^2}|u_R(y)|^{\bar p}(e^{4\pi\nu|u_R(y)|^2}-1)dy\\&\le\left(\int_{\mathbb{R}^2}|u_R(y)|^{2\bar p}dy\right)^{\frac12}\left(\int_{\mathbb{R}^2}(e^{8\pi\nu|u_R(y)|^2}-1)dy\right)^{\frac12}\le\mathbb{C}_2.\end{aligned}$$

where $\mathbb{C}_2\in(0,+\infty)$ is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$. It follows from these two facts that

$$\int_{\mathbb{R}^2}\frac{|u_R(y)|^{\bar p}(e^{4\pi\nu|u_R(y)|^2}-1)}{|x-y|^\mu}dy\le\mathbb{C}_1+\mathbb{C}_2.\tag{4.9}$$

Recalling (4.3) with (4.7) and (4.9), the proof will be done by choosing $\mathfrak{C}=\mathbb{C}_0+\mathbb{C}_1+\mathbb{C}_2$. □

With the help of the study made above, we can get the estimate for $|u_R|_\infty$ as follows.

Lemma 4.4. *Suppose that V satisfies (V) and f meets (1.3) with $(f_1)-(f_3)$. Then, for every fixed $R>R^*$, $\kappa\in(0,\kappa^*)$, $a>a^*$ and $\varepsilon\in(0,\varepsilon^*)$, there exists a constant $M\in(0,+\infty)$ which is independent of $R>R^*$ and $\varepsilon\in(0,\varepsilon^*)$ such that $|u_R|_\infty\le M$, where u_R comes from (4.5).*

Proof. In view of the definition of f_R in (1.13), one has $f_R(s) \le f(s)$ and $F_R(s) \le F(s)$ for all $R > 0$ and $s \in \mathbb{R}$. Since (u_R, σ_R) with $u_R(x) > 0$ for all $x \in \mathbb{R}^2$ and $\sigma_R < 0$ is a couple of weak solution to Eq. (2.1), we then apply (V) and Lemma 4.3 to arrive at

$$-\Delta u_R + u_R \le \bar{f}(u_R) \triangleq u_R + \mathfrak{C} f(u_R) \text{ in } \mathbb{R}^2.$$

Proceeding as the very similar calculations in Lemma 4.3, we are able to prove that $|\bar{f}(u_R)|_2 \le K$, where $K \in (0, +\infty)$ is a constant which is independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$. It then follows from the Lax-Milgram theorem that there is a $w_R \in H^1(\mathbb{R}^2)$ such that

$$-\Delta w_R + w_R = \bar{f}(u_R) \text{ in } \mathbb{R}^2.$$

Moreover, it can choose w_R to be positive in $\mathbb{R}^2$. At this stage, we can follow the methods used in [9, 11, 12, 14, 56, 58] to finish the proof. For the completeness, we shall exhibit the details. To the end, we have the claim below.

Claim 4.5. *For all $R > R^*$, $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$, there holds*

$$0 < u_R(x) \le w_R(x), \ \forall x \in \mathbb{R}^2.$$

Actually, define the test function

$$\phi(x) \triangleq (u_R - w_R)^+(x) \in H^1(\mathbb{R}^2).$$

Muitiplying this function ϕ on both sides of $-\Delta(u_R - w_R) + (u_R - w_R) \le 0$ in $\mathbb{R}^2$, we shall get the following inequality

$$\int_{\mathbb{R}^2} [\nabla(u_R - w_R)\nabla\phi + (u_R - w_R)\phi] dx \le 0.$$

An elementary computation gives us that

$$\int_{\mathbb{R}^2} [|\nabla(u_R - w_R)^+|^2 + |(u_R - w_R)^+|^2] dx = 0$$

yielding the claim.

Owing to Claim 4.5, the proof of this lemma becomes available. It invokes from [25, Theorem 9.25] that there is a $K_2 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$\|w_R\|_{H^2} \le K_2 |f_R(u_R)|_2, \ \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

leading to

$$\|w_R\|_{H^2} \le K_3, \ \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*),$$

for some $K_3 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$. In view of the continuous embedding $H^2(\mathbb{R}^2) \hookrightarrow L^\infty(\mathbb{R}^2)$, there exists $K_4 > 0$ independent of $R > R^*$ and $\varepsilon \in (0, \varepsilon^*)$ such that

$$|w_R|_\infty \le K_4, \ \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

From which, we are derived from Claim 4.5 that

$$|u_R|_\infty \le M, \ \forall R > R^* \text{ and } \varepsilon \in (0, \varepsilon^*).$$

Consequently, the proof of this lemma is completed. □

Proof of Theorem 1.1. According to the above discussions, we can arrive at the first part of the proof of Theorem 1.1 by fixing $R > \{R^*, M\}$, because in this case the function $u_R \in S(a)$ is a positive solution of Eq. (1.1) with $\sigma = \sigma_R < 0$ for each $\kappa \in (0, \kappa^*)$, $a > a^*$ and $\varepsilon \in (0, \varepsilon^*)$. The remaining part follows Theorem 2.11 directly. The proof is completed. □

4.2. **Proof of Theorem 1.2.**

In this subsection, we shall show the proof of Theorem 1.2. Recalling Theorem 3.1, there is a couple $(u_R, \sigma_R) \in E_\lambda \times \mathbb{R}$ such that it is a weak solution to Eq. (3.1) with $\sigma = \sigma_R < 0$ for every $R > R_*$, $\kappa \in (0,1)$, $a > a_*$ and $\lambda > \lambda_*$, where $u_R(x) > 0$ for each $x \in \mathbb{R}^2$. Moreover, it holds that

$$u_R \in S(a) \text{ and } \mathcal{J}_{\lambda,R}(u_R) = \bar{\Upsilon}_{\varepsilon,R}(a),\ \forall R > R_*,\ \kappa \in (0,1),\ a > a_* \text{ and } \lambda > \lambda_*. \tag{4.10}$$

Proceeding as what we have done in Subsection 4.1, we are able to conclude the counterparts of Lemmas 4.2, 4.3 and 4.4 as follows. Because there are no essential differences, we just present them without the detailed proofs.

Lemma 4.6. *Suppose that V satisfies $(V_1)-(V_3)$ and f meets (1.3) with $(f_1)-(f_3)$. Let u_R be given by (4.10) for each $R > R_*$, $a > a_*$ and $\lambda > \lambda_*$, then there exists an $\kappa_* = \kappa_*(R) \in (0,1)$ such that if $\kappa \in (0,\kappa_*)$, it holds that $|\nabla u_R|_2^2 < \frac{2-\mu}{2(2+\mu)\nu^2}$ for every $R > R_*$, $a > a_*$ and $\lambda > \lambda_*$, where the constant $\nu > 1$ is appearing in (4.2) and (4.3).*

Lemma 4.7. *Suppose that V satisfies $(V_1)-(V_3)$ and f requires (1.3) with $(f_1)-(f_3)$. Then, for every fixed $R > R_*$, $\kappa \in (0,\kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$, there exists a constant $\bar{\mathfrak{C}} \in (0,+\infty)$ which is independent of $R > R_*$ and $\lambda > \lambda_*$ such that*

$$\bar{\Gamma}(x) \triangleq |x|^{-\mu} * F(u_R) \le \bar{\mathfrak{C}},$$

where u_R comes from (4.10).

Lemma 4.8. *Suppose that V satisfies $(V_1)-(V_3)$ and f requires (1.3) with $(f_1)-(f_3)$. Then, for all fixed $R > R_*$, $\kappa \in (0,\kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$, there is $\bar{M} \in (0,+\infty)$ which is independent of $R > R_*$ and $\lambda > \lambda_*$ such that $|u_R|_\infty \le \bar{M}$, where u_R comes from (4.10).*

Proof of Theorem 1.2. We are able to fix $R > \{R_*, M\}$, and Theorem 3.1 thereby indicates the first part of the proof of Theorem 1.2 to satisfy that $u_R \in S(a)$ is a positive solution of Eq. (1.1) with $\sigma = \sigma_R < 0$ for all $\kappa \in (0,\kappa_*)$, $a > a_*$ and $\lambda > \lambda_*$. As to the remaining part of the proof of Theorem 1.2, we refer to Theorem 3.8. The proof is completed. □

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