20 Minutes of Nonlinear Elasticity

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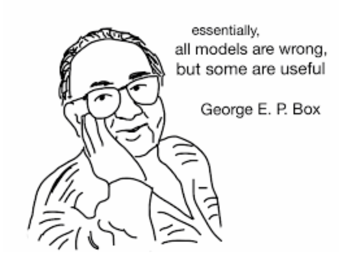
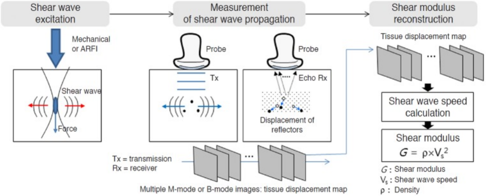
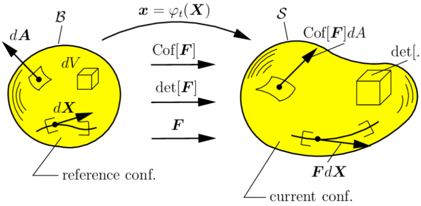


Elastography



Nonlinear elasticity

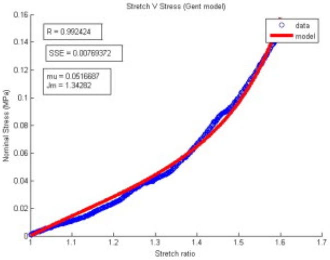


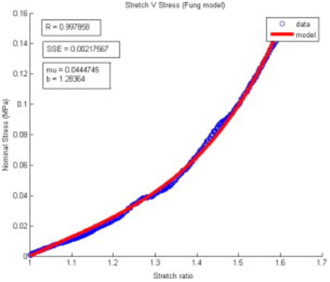
- Incompressible $\det \boldsymbol{F} = 1$, isotropic strain $\boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^T$
- Strain-Energy $W = W(I_1, I_2)$ where

$$I_1 = \operatorname{trace} \boldsymbol{B},\ I_2 = \operatorname{trace} \boldsymbol{B}^{-1}.$$

Data

Destrade et al. I.J.S.S., 46 (2009) 4322–4330



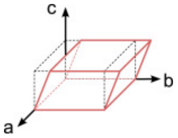


$$W = -\frac{1}{2}\mu J \ln\left(1 - \frac{I_1 - 3}{J}\right), \qquad W = \frac{\mu}{2b}\left(\exp[b(I_1 - 3)] - 1\right).$$

Generalized neo-Hookean materials

- $W = W(I_1)$.
- $W = -\frac{1}{2}\mu J \ln\left(1 - \frac{I_1 - 3}{J}\right) \quad J \to \infty \quad W = -\frac{1}{2}\mu(I_1 - 3)$.
- $W = \frac{\mu}{2b}\left(\exp[b(I_1 - 3)] - 1\right) \quad b \to 0 \quad W = -\frac{1}{2}\mu(I_1 - 3)$.
-
  $$W = \frac{1}{2}\mu\left[(I_1 - 3) + \frac{1}{2}\kappa(I_1 - 3)^2\right],$$

  Taylor expansion of Fung-Demiray or Gent when $\kappa = b$ and $\kappa = 1/J$ respectively.



$$x = X + KY, \quad y = Y, \quad z = Z$$

Cauchy stress

$$T_{11} = 2K^2 \frac{\partial W}{\partial I_1}, \quad T_{22} = -2K^2 \frac{\partial W}{\partial I_2}, \quad T_{12} = 2K \left(\frac{\partial W}{\partial I_1} + \frac{\partial W}{\partial I_2} \right)$$

and

$$T_{13} = T_{23} = T_{33} = 0.$$

- Simple shear the shear stress $\tau = T_{12}$

$$\tau = Q(K^2)K, \quad Q(K^2) = 2(W_1 + W_2).$$

- An experiment collect a certain number of data points (K_i, τ_i) for $i = 1, \ldots, n$.
- Linear Elasticity

$$\tau_i = \mu K_i$$

- Gent model we have that

$$\tau_i = \mu_G \frac{K_i}{1 - \frac{K_i^2}{J}},$$

- Weakly nonlinear model

$$\tau_i = \mu_{wnl}(1 + \kappa K_i^2)K_i.$$

ASSUME: The real material is a Gent material.
Then if you infer the real material with a linear model

$$\frac{\mu_G}{\mu} = 1 - \frac{K_i^2}{J}$$

and if you use a weakly nonlinear theory

$$\frac{\mu_G}{\mu_{wnl}} = \left(1 - \frac{K_i^2}{J}\right)(1 + \kappa K_i^2)$$

and if $\kappa = 1/J$ we obtain

$$\frac{\mu_G}{\mu_{WL}} = 1 - \frac{K_i^4}{J^2}.$$

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1.
$$x = X + f(Z), \quad y = Y, \quad z = Z$$

2.
$$\frac{d}{dZ}\left[2\left(\frac{\partial W}{\partial I_1}\right)\frac{df}{dZ}\right] = c,$$

3. Dirichlet BVP $f(-H) = 0, f(H) = 0$ via dimensionless variables

$$\mathcal{Q}\left(f_Z^2\right) f_Z = cZ - k_1, \quad f(1) = 0, \quad f(-1) = 0.$$

4. Invariance $Z \to -Z$ then $k_1 = 0$ and BVP turn to be equivalent to IVP

$$\underbrace{\left[\mathcal{Q}\left(f_Z^2\right) + 2\mathcal{Q}' f_Z^2\right]}_{W''} f_{ZZ} = c, \qquad f(-1) = 0,\ f_Z(-1) = f_1,$$

1. $E = \int_{-1}^{1} \left[W(f_Z^2) + cf \right] dZ.$
2. Let $v = f_Z$, $\mathcal{I}(v, f) = W(v^2) + cf$. Then if $W \in C^0(\mathbb{R})$ and:
   - H_1) $v \to \mathcal{I}$ is convex for every $f \in \mathbb{R}$;
   - H_2) there exist $p > q \geq 1$ and $\alpha_1 > 0$, $\alpha_2, \alpha_3 \in \mathbb{R}$ such that, for every $f \in \mathbb{R}$, we have

   $$\mathcal{I} \geq \alpha_1 |v|^p + \alpha_2 |f|^q + \alpha_3,$$

   then the minimizer f^* of the functional E exists in the Sobolev space $\{f \in \mathcal{W}^{1,p}(-1, 1),\ f(-1) = f(1) = 0\}$.
3. H_1 the more stringent condition $\partial^2 W / \partial v^2 > 0$, $\forall v \in \mathbb{R}$ and the condition

   $$|\partial W / \partial v| \leq \alpha_4 (1 + |v|^p),\ \forall v \in \mathbb{R},$$

   when $W \in C^\infty$ then the minimizer is in C^∞.

- $W = \frac{1}{4} f_Z^4, \quad \mathcal{Q} \equiv f_Z^2, \quad W'' = 3 f_Z^3$, being $W'' = 0$ if $v = 0$

$$f(Z) = \frac{3}{4} c^{1/3} \left(Z^{4/3} - 1 \right),$$

  in $Z = 0$ gets a cusp and the second derivative of $f(Z)$ blows up.
- When $\kappa < 0$ let us rewrite the first integral for our problem as

$$Z(v) = \frac{1}{c} [v - |\kappa| v^3]$$

  and choose $c > 0$. The $Z(v)$ has two extrema $v_\pm = \pm \sqrt{\frac{1}{3|\kappa|}}$ where

$$Z(v_\pm) = \pm \frac{2}{3c} \sqrt{\frac{1}{3|\kappa|}} \to \frac{2}{3c} \sqrt{\frac{1}{3|\kappa|}} \geq 1.$$

Rectilinear Motion

1.

$$\frac{\partial^2 f}{\partial t^2} = c_T \frac{\partial}{\partial Z}\left[\frac{\partial f}{\partial Z} + \kappa\left(\frac{\partial f}{\partial Z}\right)^3\right] + \nu \underbrace{\frac{\partial^3 f}{\partial Z^2 \partial t}}_{\text{dissipation}} + \alpha \underbrace{\frac{\partial^4 f}{\partial Z^2 \partial^2 t}}_{\text{dispersion}},$$

2.

$$F_{tt} = c_T\left[F + \kappa F^3\right]_{ZZ} + \nu F_{ZZt} + \alpha F_{ZZtt},$$

3. If we consider a prestrain this equation is modified in

$$F_{tt} = \left[\delta F + \beta F^2 + \gamma F^3\right]_{ZZ} + \nu F_{ZZt} + \alpha F_{ZZtt},$$

where δ, β, γ depends on c_T, κ and $\lambda_1, \lambda_2, \lambda_3$.

1. $F = F(\zeta)$ where $\zeta = Z - \omega t$

$$[\omega^2 - \delta - \beta F - \gamma F^2]F = -\omega\nu F' + \alpha\omega^2 F''.$$

is the Euler-Lagrange equation associated with the Lagrangian

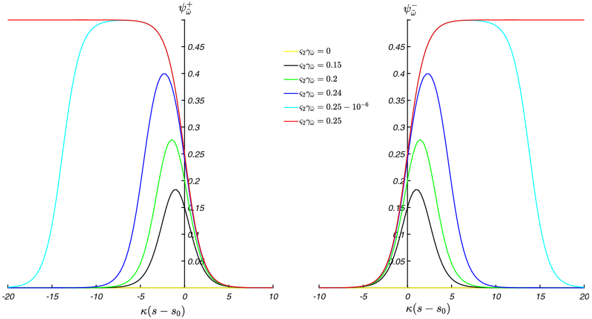
$$\mathcal{L} = \exp\left(-\frac{\nu}{\alpha\omega}\zeta\right)\left[\frac{\alpha\omega^2}{2}F'^2 + (\omega^2 - \delta)\frac{F^2}{2} - \beta\frac{F^3}{3} - \gamma\frac{F^4}{4}\right]$$

H. LE DRET and A. MOKRANE: *On Problems in the Calculus of Variations in Increasingly Elongated Domains* Chinese Annals of Mathematics Series B 2018.

2. For $\nu = 0$ similar to the Gardner equation

$$u_t + (u^2 - u^3)_x + u_{xxx} = 0 \to \frac{1}{2}u'^2 - \frac{\omega}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{4}u^4 = 0.$$

Flaton



(a) $\psi^+_{\tilde{\omega}}$ **(b)** $\psi^-_{\tilde{\omega}}$